UNIT 7. THE FUNDAMENTAL EQUATIONS OF HYPERSURFACE THEORY

Gauss frame of a parameterized hypersurface, formulae for the partial derivatives of the Gauss frame vector fields, Christoffel symbols, Gauss and Codazzi-Mainardi equations, fundamental theorem of hypersurfaces, "Theorema Egregium", components of the curvature tensor, tensors in linear algebra, tensor fields over a hypersurface, curvature tensor.

Now we derive some formulae for hypersurfaces. Consider a regular parameterized hypersurface $\mathbf{r}:\Omega\longrightarrow\mathbb{R}^{n+1}$. The partial derivatives $\mathbf{r}_1,\dots,\mathbf{r}_n$ define a basis of the tangent space of the hypersurface at each point. If we add to these vectors the normal vector of the hypersurface, we get a basis of $\mathbb{R}^{n+1}$ at each point of the hypersurface. The system of the vector fields $\mathbf{r}_1,\dots,\mathbf{r}_n,N$ along $\mathbf{r}$ is called the <u>Gauss frame</u> of the hypersurface. Gauss frame plays similar role in the theory of hypersurfaces as Frenet frame does in curve theory. Similarity is not complete however, since a Gauss frame is much more dependent on the parameterization. Nevertheless, in the same way as for Frenet frames, it is important to know how the derivatives of the frame vector fields with respect to the parameters can be expressed as a linear combination of the frame vectors. For this we have to determine the coefficients Γ_{ij}^{k}, α_{ij}, β_j^k, γ_j in the expressions

$$\mathbf{r}_{ij} = \sum_k \Gamma_{ij}^{k}\,\mathbf{r}_k + \alpha_{ij}N, \qquad N_j = \sum_k \beta_j^k\,\mathbf{r}_k + \gamma_j\,N. \qquad (*)$$

Let us begin with the simple observation that since N_j is known to be tangential, and $N_j = -L(\mathbf{r}_j)$, where L is the Weingarten map,

$$\boxed{\gamma_j = 0 \text{ for all } j}$$

and $(-\beta_j^k)_{j,k=1}^{n}$ is the matrix $\mathcal{L} = \mathcal{B}\,\mathcal{G}^{-1}$ of the Weingarten map with respect to the basis $\mathbf{r}_1,\dots,\mathbf{r}_n$. Denote by g_{ij} and b_{ij} the entries of the first and second fundamental forms as usual, and denote by g^{ij} the components of the inverse matrix of the matrix of the first fundamental form. (Attention! Entries of $\mathcal{G}$ and $\mathcal{G}^{-1}$ are distinguished by the position of indices.) Then

$$\boxed{\beta_j^k = -\sum_i b_{ji}g^{ik}}\ .$$

Taking the dot product of the first equation of (*) with N we gain the equality $\langle \mathbf{r}_{ij}, N\rangle = \alpha_{ij}$ and since $\langle \mathbf{r}_{ij}, N\rangle = b_{ij}$,

$$\boxed{\alpha_{ij} = b_{ij} \quad \text{for all } i,j}\ .$$

There is only one question left: what are the coefficients Γ_{ij}^{k} equal to? Let us take the dot product of the first equation of (*) with $\mathbf{r}_\ell$

$$\langle \mathbf{r}_{ij}, \mathbf{r}_\ell\rangle = \sum_k \Gamma_{ij}^{k} \langle \mathbf{r}_k, \mathbf{r}_\ell\rangle = \sum_k \Gamma_{ij}^{k} g_{k\ell} \quad ,$$

or denoting the dot product $\langle \mathbf{r}_{ij}, \mathbf{r}_\ell\rangle$ shortly by $\Gamma_{ij\ell}$,

$$\Gamma_{ij\ell} = \sum_k \Gamma_{ij}^{k} g_{k\ell} \quad .$$

The coefficients Γ_{ij}^{k} and Γ_{ijk} are called the Christoffel symbols of first and second type respectively. The last equation shows how to express Christoffel symbols of second type with the help of Christoffel symbols of first type. It can also be used to express Christoffel symbols of first type in terms of secondary Christoffel symbols. Indeed, multiplying the equation with $g^{\ell s}$, taking sum for ℓ and using $\sum_\ell g_{k\ell} g^{\ell s} = \delta_k^\ell$ ($\leftarrow$ Kronecker delta), we get

$$\sum_\ell \Gamma_{ij\ell} g^{\ell s} = \sum_\ell \sum_k \Gamma_{ij}^{k} g_{k\ell} g^{\ell s} = \sum_k \Gamma_{ij}^{k} \delta_k^s = \Gamma_{ij}^{s} \ .$$

Now let us try to determine Christoffel symbols of second type. Differentiating the equality $g_{ij} = \langle \mathbf{r}_i, \mathbf{r}_j\rangle$ with respect to the k-th variable and then permuting the role of indices i,j,k we get the equalities

$$\begin{aligned} g_{ij,k} &= \langle \mathbf{r}_{ik}, \mathbf{r}_j\rangle + \langle \mathbf{r}_i, \mathbf{r}_{jk}\rangle \\ g_{jk,i} &= \langle \mathbf{r}_{ji}, \mathbf{r}_k\rangle + \langle \mathbf{r}_j, \mathbf{r}_{ki}\rangle \\ g_{ki,j} &= \langle \mathbf{r}_{kj}, \mathbf{r}_i\rangle + \langle \mathbf{r}_k, \mathbf{r}_{ij}\rangle \ . \end{aligned}$$

Solving this linear system of equations for the secondary Christoffel symbols standing on the right hand side, we obtain

$$\Gamma_{ijk} = \langle \mathbf{r}_{ij}, \mathbf{r}_k\rangle = \frac{1}{2}\left(g_{ik,j} + g_{jk,i} - g_{ij,k}\right)$$

and

$$\boxed{\Gamma_{ij}^{k} = \sum_\ell \Gamma_{ij\ell} g^{\ell k} = \sum_\ell \frac{1}{2}\left(g_{i\ell,j} + g_{j\ell,i} - g_{ij,\ell}\right) g^{\ell k}} \quad .$$

Observe that the Christoffel symbols *depend only on the first fundamental form* of the hypersurface.

Now we ask the following question. Suppose we are given $2n^2$ smooth functions g_{ij}, b_{ij} i,j=1,2,...,n on an open domain Ω of $\mathbb{R}^{n+1}$. When can we find a parameterized hypersurface $\mathbf{r}:\Omega \longrightarrow \mathbb{R}^{n+1}$ with fundamental forms $\mathcal{G} = (g_{ij})$ and $\mathcal{B} = (b_{ij})$. We have some obvious restrictions on the functions g_{ij} and b_{ij}. First, $g_{ij} = g_{ji}$, $b_{ij} = b_{ji}$, and since $\mathcal{G}$ is the matrix of a positive definite bilinear form, the determinants of the corner submatrices $(g_{ij})_{i,j=1}^{k}$

must be positive for $k = 1,\dots,n$. However, the examples we have show that these conditions are not enough to guarantee the existence of a hypersurface. For example, if $\mathcal{G}$ is the identity matrix everywhere, while $\mathcal{B} = f\,\mathcal{G}$ for some function on Ω, then the hypersurface (if exists) consists of umbilics. We know however that if a surface consists of umbilics, then the principal curvatures are constant, so although our choice of $\mathcal{B}$ and $\mathcal{G}$ satisfies all the conditions we have listed so far, it does not correspond to a hypersurface unless f is constant. So there must be some further relations between the components of $\mathcal{B}$ and $\mathcal{G}$. Our plan to find some of these correlations is the following. Let us express $\mathbf{r}_{ijk}$ and $\mathbf{r}_{ikj}$ as a linear combination of the Gauss frame vectors. The coefficients we get are functions of the entries of the first and second fundamental forms. For $\mathbf{r}_{ijk} = \mathbf{r}_{ikj}$, the corresponding coefficients in the expressions for these vectors must be equal and it can be hoped that this way we arrive at further non-trivial relations between $\mathcal{G}$ and $\mathcal{B}$. This was the philosophy, and now let us get down to work.

$$\mathbf{r}_{ijk} = \Big(\sum_\ell \Gamma_{ij}^{\ell}\, \mathbf{r}_\ell + b_{ij}N \Big)_{,k} = \sum_\ell \big(\Gamma_{ij,k}^{\ell}\mathbf{r}_\ell + \Gamma_{ij}^{\ell}\, \mathbf{r}_{\ell k} \big) + b_{ij,k}N + b_{ij}N_k =$$

$$= \sum_\ell \Big(\Gamma_{ij,k}^{\ell}\mathbf{r}_\ell + \Gamma_{ij}^{\ell} \big(\sum_s \Gamma_{\ell k}^{s}\, \mathbf{r}_s + b_{\ell k}N\big)\Big) + b_{ij,k}N - b_{ij} \sum_{\ell s}\sum b_{ks}g^{s\ell}\mathbf{r}_\ell =$$

$$= \sum_\ell \Big(\Gamma_{ij,k}^{\ell} + \sum_s \Gamma_{ij}^{s}\, \Gamma_{sk}^{\ell} - b_{ij} \sum_s b_{ks}g^{s\ell}\Big)\, \mathbf{r}_\ell + \Big(b_{ij,k} + \sum_\ell \Gamma_{ij}^{\ell}\, b_{\ell k}\Big)N \,.$$

Comparing the coefficient of $\mathbf{r}_\ell$ in $\mathbf{r}_{ijk}$ and $\mathbf{r}_{ikj}$, we obtain

$$\boxed{\Gamma_{ij,k}^{\ell} - \Gamma_{ik,j}^{\ell} + \sum_s \big(\Gamma_{ij}^{s}\, \Gamma_{sk}^{\ell} - \Gamma_{ik}^{s}\, \Gamma_{sj}^{\ell}\big) = \sum_s \big(b_{ij}\, b_{ks} - b_{ik}\, b_{js}\big)\, g^{s\ell}}$$

while comparison of the coefficient of N gives

$$\boxed{b_{ij,k} - b_{ik,j} = \sum_\ell \Gamma_{ik}^{\ell}\, b_{\ell j} - \sum_\ell \Gamma_{ij}^{\ell}\, b_{\ell k}}\,.$$

The first n^4 equations (we have an equation for all i,j,k,ℓ), are the <u>Gauss equations</u> for the hypersurface. The second family of n^3 equations are the <u>Codazzi-Mainardi equations</u>.

<u>Exercise</u>. Express the second order derivatives N_{ij} and N_{ji} as a linear combination of the Gauss frame vectors. Compare the corresponding coefficients and prove that their equality follows from the Gauss and Codazzi-Mainardi equations.

The exercise points out that a similar try to derive new relations between $\mathcal{G}$ and $\mathcal{B}$ does not lead to really new results. This is no wonder, since the Gauss and Codazzi-Mainardi equations together with the previously listed obvious conditions on $\mathcal{G}$ and $\mathcal{B}$ form a complete system of necessary and sufficient conditions for the existence of a hypersurface with fundamental forms $\mathcal{G}$ and $\mathcal{B}$.

Theorem. (Fundamental theorem of hypersurfaces). Let $\Omega \subset \mathbb{R}^n$ be an open connected and simply connected subset of $\mathbb{R}^n$ (e.g. an open ball or cube), and suppose that we are given two smooth n by n matrix valued functions $\mathcal{G}$ and $\mathcal{B}$ on Ω such that $\mathcal{G} = (g_{ij})$ and $\mathcal{B} = (b_{ij})$ assign to every point a symmetric matrix, $\mathcal{G}$ gives the matrix of a positive definite bilinear form. In this case, if the functions Γ_{ij}^k derived from the components of $\mathcal{G}$ according to the above formulae satisfy the Gauss and Codazzi-Mainardi equations, then there exists a regular parameterized hypersurface $\mathbf{r}:\Omega \longrightarrow \mathbb{R}^{n+1}$ for which the matrix representations of the first and second fundamental forms are $\mathcal{G}$ and $\mathcal{B}$ respectively. Furthermore, this hypersurface is unique up to rigid motions of the whole space. Namely, if $\mathbf{r}_1$ and $\mathbf{r}_2$ are two such hypersurfaces, then there exists an isometry (=distance preserving bijection) $\Phi:\mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ for which $\mathbf{r}_2 = \Phi \circ \mathbf{r}_1$.

Let us denote the expressions standing on the left hand sides of the Gauss equations by

$$\boxed{R_{ijk}^{\ell} := \Gamma_{ij,k}^{\ell} - \Gamma_{ik,j}^{\ell} + \sum_s \left(\Gamma_{ij}^{s}\Gamma_{sk}^{\ell} - \Gamma_{ik}^{s}\Gamma_{sj}^{\ell}\right)}\,.$$

Then Gauss equations can be abbreviated writing

$$R_{ijk}^{\ell} = \sum_s \left(b_{ij}\, b_{ks} - b_{ik}\, b_{js}\right) g^{s\ell}.$$

Let us multiply this equation by $g_{\ell m}$ and take a sum for ℓ

$$\sum_\ell R_{ijk}^{\ell}\, g_{\ell m} = \sum_\ell \sum_s \left(b_{ij}\, b_{ks} - b_{ik}\, b_{js}\right) g^{s\ell}\, g_{\ell m} =$$
$$= \sum_s \left(b_{ij}\, b_{ks} - b_{ik}\, b_{js}\right) \delta_m^s = \left(b_{ij}\, b_{km} - b_{ik}\, b_{jm}\right).$$

Introducing the functions $\boxed{R_{imjk} := \sum_\ell R_{ijk}^{\ell}\, g_{\ell m}}$, we may write

$$\boxed{R_{imjk} = \left(b_{ij}\, b_{km} - b_{ik}\, b_{jm}\right)}\,.$$

Let us observe, that the functions R_{imjk} can be expressed in terms of the first fundamental form $\mathcal{G}$.

Corollary. (Theorema Egregium) The Gaussian curvature of a regular parameterized surface in $\mathbb{R}^3$ can be expressed in terms of the first fundamental form as follows

$$K = \frac{R_{1212}}{\det \mathcal{G}}\,.$$

Theorema Egregium is one of those theorems of Gauss he was very proud of.

The surprising fact is not the actual form of this formula but the mere existence of a formula that expresses the Gaussian curvature in terms of the first fundamental form. The geometrical meaning of the existence of such a formula is that the *Gaussian curvature does not change when we bend the surface* (although principal curvatures do change in general!).

Definition. Let $\mathbf{r}:\Omega \longrightarrow \mathbb{R}^{n+1}$ be a hypersurface. Consider the mapping R that assigns four tangential vector fields $\mathbf{X} = \sum_i X^i \mathbf{r}_i$, $\mathbf{Y} = \sum_i Y^i \mathbf{r}_i$, $\mathbf{Z} = \sum_i Z^i \mathbf{r}_i$, $\mathbf{W} = \sum_i W^i \mathbf{r}_i$ a function according to the formula

$$R(X,Y;Z,W) = \sum_i \sum_m \sum_j \sum_k R_{imjk} X^i Y^m Z^j W^k.$$

We shall call R the curvature tensor of the hypersurface, the functions R_{imjk} the components of the curvature tensor.

Let us briefly recall some definition from linear algebra, concerning tensors.

Let V be a vector space (over $\mathbb{R}$). The set V^* of linear functions form a vector space with respect to the operations

$$(\ell_1+\ell_2)(\mathbf{v}) := \ell_1(\mathbf{v}) + \ell_2(\mathbf{v}), \qquad (\lambda\,\ell)(\mathbf{v}) := \lambda\big(\ell(\mathbf{v})\big).$$

The vector space V^* of linear functions on V is called the dual space of V. If V is finite dimensional and $\mathbf{e}_1,\dots,\mathbf{e}_n$ is a basis of V, then we may consider the linear functions $\mathbf{e}^1,\dots,\mathbf{e}^n \in V^*$ defined by $\mathbf{e}^i(\mathbf{e}_j) = \delta^i_j$. It is not difficult to prove that these linear functions form a basis of V^* called the dual basis of the basis $\mathbf{e}_1,\dots,\mathbf{e}_n$. As a consequence we get that dim V = dim V^* for finite dimensional vector spaces. A tensor of valency (/order /type) (k,ℓ) over V is a multilinear function

$$T : V^* \times \dots \times V^* \times V \times \dots \times V \longrightarrow \mathbb{R}$$

defined on the Cartesian product of k copies of V^* and ℓ copies of V. "Multilinear" means that fixing all but one variables, we obtain a linear function of the free variable. Denote by $T^{(k,\ell)}V$ the set of tensors of valency (k,ℓ). The sum of two tensors of order (k,ℓ) and the scalar multiple of a tensor are tensors of the same order, hence the set of tensors of a given valency form a vector space. If $\mathbf{e}_1,\dots,\mathbf{e}_n$ is a basis of V, then every tensor T is uniquely determined by its values on basis vector combinations, i.e. by the numbers

$$T^{i_1\dots i_k}_{j_1\dots j_\ell} = T(\mathbf{e}^{i_1},\dots,\mathbf{e}^{i_k};\mathbf{e}_{j_1},\dots,\mathbf{e}_{j_\ell}),$$

which are called the components of the tensor T with respect to the basis

$\mathbf{e}_1,\dots,\mathbf{e}_n$. Since any $(\dim V)^{(k+\ell)}$ numbers $T^{i_1\dots i_k}_{j_1\dots j_\ell}$ correspond to a tensor, $\dim T^{(k,\ell)}V = (\dim V)^{(k+\ell)}$.

Now let us consider a regular parameterized hypersurface M, $\underline{r}:\Omega \longrightarrow \mathbb{R}^{n+1}$. A <u>tensor field of valency</u> (k,ℓ) over M is a mapping T that assigns to every point $\underline{u} \in \Omega$ a tensor of valency (k,ℓ) over the tangent space of M at $\underline{r}(\underline{u})$. $T(\underline{u})$ is uniquely determined by its components $T^{i_1\dots i_k}_{j_1\dots j_\ell}(\underline{u})$ with respect to the basis $\mathbf{r}_1(\underline{u}),\dots,\mathbf{r}_n(\underline{u})$. The functions $\underline{u} \longmapsto T^{i_1\dots i_k}_{j_1\dots j_\ell}(\underline{u})$ are called the <u>components of the tensor field</u> T. T is said to be a <u>smooth tensor field</u> if its components are smooth.

Examples.

- Function on M are tensor fields of valency (0,0).
- Tangential vector fields are tensor fields of valency (1,0) (V is isomorphic to V^{**} in a natural way).
- The first and second fundamental forms of a hypersurface are tensor fields of valency (0,2).
- The mapping that assigns to every point of a hypersurface the Weingarten map at that point is a tensor of valency (1,1). (The linear space of $V \longrightarrow V$ linear mappings is isomorphic to $T^{(1,1)}V$ in a natural way.)
- Let f be a smooth function on M. Consider the tensor field of valency (0,1) defined on a tangent vector X to be the derivative of f in the direction X. This tensor field is the differential of f.
- The curvature tensor is a tensor field of valency (0,4).

The curvature tensor is one of the most basic objects of study in differential geometry. In the previous computations the curvature tensor came across like a rabbit from a cylinder. To understand its real meaning, we shall introduce the curvature tensor in a more natural way in a more general framework, in the framework of Riemannian manifolds. For this purpose, we have to get acquainted with some fundamental definitions and constructions. This will be the goal of the following units.