Globally rigid frameworks and rigid tensegrity graphs in the plane

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Chapter 1

Introduction

The field of combinatorial rigidity is centered around geometric properties of straight line realizations of graphs in $\mathbb{R}^d$ that can be derived from the combinatorial properties of the graph when the placement of vertices is general enough.

One interesting question about realizations is the uniqueness of the distance of pairs of vertices given that the lengths of the edges remain the same. In other words, is there a realization with the same edge lengths but different distance of two designated vertices? The answer can be different for different realizations, but in certain cases the answer is always no when the edge lengths are generic enough (i.e. algebraically independent over $\mathbb{Q}$). The first part of the dissertation, which is based on [26, 37], characterizes this case for two important graph families and formulates a conjecture for the general case. This field has important applications in sensor network localization [15, 19].

If the distance of each vertex pair is determined by the edge lengths, then the realization is said to be unique, or globally rigid. It is known [25] when a graph with generic edge lengths is globally rigid in the plane. However, the problem becomes NP-hard [35], even for the one dimensional case, if the realization can be arbitrary. Moreover, there is no 'simple' sufficient condition for the global rigidity of a non-generic realization. Therefore the problem of algorithmically constructing a realization that is globally rigid is non-trivial even when we know that such a realization exists. The second part of the dissertation, which was published in [29], describes an algorithm for the construction of globally rigid realizations in the plane.

Another geometric property is rigidity, which means that the realization can not be continuously deformed with keeping the edge lengths constant other than by congruences of the whole space. Rigidity in the plane is a well understood problem, and depends only on the graph structure if the realization is not 'degenerate'. The problem becomes more interesting, however, if we allow certain edges to become longer and other edges to become shorter during the deformation. Tensegrity graphs are edge-labeled graphs that encode these restrictions. The third part of the dissertation, based on [28], focuses on the existence of rigid realizations of such tensegrity graphs in the plane.

A common theme in the proofs and algorithms presented in this work is that they all use some constructive characterization result of certain graph families. These results state that each member of the graph family can be constructed from a small graph using certain simple graph operations and that all graphs constructed this way belong to the graph family. Therefore one of the key elements in the proofs is that these operations on graphs or realizations preserve some rigidity property.

With the exception of one graph extension result on global rigidity that is stated generally for $\mathbb{R}^d$, all of the results in this work are for the $d = 2$ case. We note that for $d = 1$, almost all rigidity problems are trivial or easy, while for $d \geq 3$ they seem to be hopelessly hard.

Throughout this work, we will consider finite graphs without loops or multiple edges. The number of vertices of a graph $G = (V, E)$ will be denoted by n, and the number of its edges by m. We will assume that the vertices and edges are numbered $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$. The complete graph on n vertices is denoted by K_n.

In the rest of the introduction we will give the precise definition of rigidity concepts that are needed to formulate the main results of the dissertation.

1.1 Rigidity and infinitesimal rigidity of frameworks

A d-dimensional *framework* is a pair (G, p), where $G = (V, E)$ is a graph and p is a map from V to $\mathbb{R}^d$. This p is called a d-dimensional *realization* of G. An alternative way to look at p is as an n-tuple of d-dimensional vectors, or equivalently, an nd-dimensional vector, $p = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^{nd}$. This vector is also called a d-dimensional *configuration*. The dimension of the affine span of $\{p(v) \mid v \in V\}$ will be denoted by $\dim p$.

Given a framework (G, p) and an edge $e = uv \in E$, we define

$$l_e(p) = ||p(u) - p(v)||^2,$$

where $||.||$ denotes the Euclidean norm in $\mathbb{R}^d$. The *edge function* of G is a map from the set of all realizations, $\mathbb{R}^{nd}$ to $\mathbb{R}^m$, given by

$$f_G(p) = (l_{e_1}(p), l_{e_2}(p), \ldots, l_{e_m}(p)).$$

Observe that the edge function f_G is a polynomial map from $\mathbb{R}^{nd}$ to $\mathbb{R}^m$. Two frameworks (G, p) and (G, q) are *equivalent* if $f_G(p) = f_G(q)$, in other words, if $||p(u) - p(v)|| = ||q(u) - q(v)||$ holds for all pairs u, v with $uv \in E$. Frameworks (G, p) and (G, q) are *congruent* if $||p(u) - p(v)|| = ||q(u) - q(v)||$ holds for all pairs u, v with $u, v \in V$, or equivalently, if $f_{K_n}(p) = f_{K_n}(q)$.

A *flexing* of the framework (G, p) is a function $\pi : (-1, 1) \times V \to \mathbb{R}^d$, where $\pi(0) = p$ and the frameworks (G, p) and $(G, \pi(t))$ are equivalent for all $t \in (-1, 1)$. The flexing π is *trivial* if the frameworks (G, p) and $(G, \pi(t))$ are congruent for all $t \in (-1, 1)$. A framework is said to be *flexible* if it has a non-trivial continuous flexing. We say that a framework (G, p) is *rigid* if there exists an $\epsilon > 0$ such that if (G, q) is equivalent to (G, p) and $||p(v) - q(v)|| < \epsilon$ for all $v \in V$ then (G, q) is congruent to (G, p). The following result establishes the connection between the rigidity and flexibility of frameworks, and the equivalence of alternative definitions of flexibility.

Theorem 1.1.1. *[18, 1] Let (G, p) be a d-dimensional framework. The following are equivalent:*

(a) (G, p) is not rigid;

(b) (G, p) is flexible (i.e. (G, p) has a non-trivial continuous flexing);

(c) (G, p) has a non-trivial analytical flexing;

(d) (G, p) has an analytical flexing π such that (G, p) and $(G, \pi(t))$ are not congruent for all $t \in (-1, 1)$, $t \neq 0$.

Given a framework (G,p) and a smooth flexing π, the conservation of edge lengths can be written as

$$||\pi(t,u)-\pi(t,v)||^2 = ||p(u)-p(v)|| = \text{constant}.$$

A linearized version of this can be obtained by taking derivatives at $t=0$,

$$(p(u)-p(v))(\dot{\pi}(u)-\dot{\pi}(v)) = 0,$$

where $\dot{\pi}(v) = \frac{d}{dt}\pi(t,v)|_{t=0}, v \in V$ are called *infinitesimal velocities.* An *infinitesimal motion* of a framework (G,p) is an assignment of infinitesimal velocities to the vertices, $q : V \to \mathbb{R}^d$ satisfying

$$(p(u)-p(v))(q(u)-q(v)) = 0 \tag{1.1}$$

for all pairs u,v with $uv \in E$. The above argument shows that if π is a smooth flexing of (G,p), then $\dot{\pi}$ is an infinitesimal motion of (G,p). If we think of infinitesimal motions as nd-dimensional vectors, then the set of infinitesimal motions of a framework (G,p) is a linear subspace of $\mathbb{R}^{nd}$, given by the $|E|$ linear equations of the form (1.1). The matrix of this system of linear equations is the *rigidity matrix* of (G,p) and it is denoted by $R(G,p)$. This is a matrix of size $|E| \times nd$, where, for each edge $uv \in E$, in the row corresponding to uv, the entries in the d columns corresponding to vertices u and v contain the d coordinates of $(p(u)-p(v))$ and $(p(v)-p(u))$, respectively, and the remaining entries are zeros. With this notation, a vector $q \in \mathbb{R}^{nd}$ is an infinitesimal motion of (G,p) if and only if $R(G,p)q = 0$, in other words, the space of infinitesimal motions of (G,p) is the kernel of $R(G,p)$. It will be important to observe that the Jacobian of the edge function of G at a point $p \in \mathbb{R}^{nd}$ is twice the rigidity matrix of (G,p),

$$df_G|_p = 2R(G,p). \tag{1.2}$$

Let S be a $d \times d$ antisymmetric matrix and $t \in \mathbb{R}^d$. For a framework (G,p) consider the following infinitesimal motion:

$$q(v) = Sp(v) + t \tag{1.3}$$

for all $v \in V$. This is indeed an infinitesimal motion of (G,p), since

$$(p(u)-p(v))(q(u)-q(v)) = (p(u)-p(v))S(p(u)-p(v)) = 0$$

for all pairs $u, v \in V$. Since an infinitesimal motion of the form (1.3) is the derivative of a trivial smooth flexing of (G, p), it is called a *trivial infinitesimal motion.* A framework (G, p) is said to be *infinitesimally flexible* if it has a non-trivial infinitesimal motion, otherwise it is called *infinitesimally rigid.* Since the space of infinitesimal motions is the kernel of $R(G, p)$, its dimension is $nd - \operatorname{rank} R(G, p)$, and the framework is infinitesimally rigid if this dimension is equal to the dimension of the space of trivial infinitesimal motions, which depends only on the configuration p. The following two lemmas characterize the infinitesimal rigidity of frameworks in terms of the rank of their rigidity matrix, their proof can be found in [44]. First we will characterize the infinitesimal rigidity of frameworks on the complete graph.

Lemma 1.1.2. *Let p be a d-dimensional configuration on n vertices.*

(a) If $\dim p \geq d - 1$, then $\operatorname{rank} R_{K_n}(p) = nd - \binom{d+1}{2}$, and the space of trivial infinitesimal motions has dimension $\binom{d+1}{2}$.

(b) If $n < d$ and $\dim p = n - 1$, then $\operatorname{rank} R_{K_n}(p) = \binom{n}{2}$, and the space of trivial infinitesimal motions has dimension $nd - \binom{n}{2}$.

(c) (K_n, p) is infinitesimally rigid if and only if $\dim p \geq d - 1$ or $n < d$ and $\dim p = n - 1$.

For the rank of the rigidity matrix of the complete graph, it will be convenient to introduce the following notation:

$$S(n, d) = \begin{cases} nd - \dbinom{d+1}{2}, & \text{if } n \geq d \\ \dbinom{n}{2}, & \text{if } n < d \end{cases}$$

Lemma 1.1.3. *Let (G, p) be a d-dimensional framework on n vertices. Then*

(a) $\operatorname{rank} R_G(p) \leq S(n, d)$;

(b) (G, p) is infinitesimally rigid if and only if $\operatorname{rank} R(G, p) = S(n, d)$.

Next we will examine the relationship between the rigidity and the infinitesimal rigidity of frameworks. The following Lemma is from Gluck.

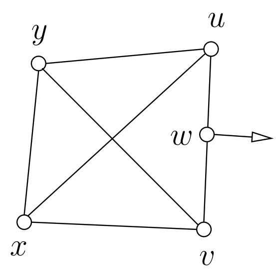

Figure 1.1: A framework (G, p), which is rigid, but not infinitesimally rigid. The infinitesimal motion, which is non-zero only on vertex w, is non-trivial.

Lemma 1.1.4. *[18] If the framework (G, p) is infinitesimally rigid, then it is rigid.*

The converse of this is not true, the framework on Figure 1.1 is rigid, but not infinitesimally rigid. However, if we exclude certain 'degenerate' configurations, rigidity and infinitesimal rigidity becomes equivalent. In order to establish this, let us recall some notions from differential topology. Given two smooth manifolds, M and N and a smooth map $f : M \to N$, we denote the derivative of f at some point $p \in M$ by $df|_p$, which is a linear map from T_pM – the tangent space of M at p – to $T_{f(p)}N$. Let k be the maximum rank of $df|_q$ over all $q \in M$. A point $p \in M$ is said to be a *regular point* of f, if $\operatorname{rank} df|_p = k$, and a *critical point*, if $\operatorname{rank} df|_p < k$. A point $f(p) \in N$ is a *regular value* of f, if the preimage $f^{-1}(f(p))$ contains only regular points, otherwise $f(p)$ is called a *critical value*.

We say that a framework (G, p) is *regular*, if p is a regular point of f_G. Using the inverse function theorem, it can be shown (see e.g. [1, Proposition 2]) that if (G, p) is a regular framework, then there is an U_p neighborhood of p, such that $f_G^{-1}(f_G(p)) \cap U_p$ is a manifold, whose tangent space at p is the kernel of df_p. This gives the converse of Lemma 1.1.4 for regular frameworks.

Theorem 1.1.5. *[1, 2] Let (G, p) be a regular framework. If (G, p) is infinitesimally flexible, then it is flexible. Furthermore, if q is a non-trivial infinitesimal motion of (G, p), then there is a non-trivial smooth flexing π of (G, p) such that $\dot{\pi} = q$.*

Corollary 1.1.6. *Let (G, p) be a regular framework. Then (G, p) is rigid if and only if it is infinitesimally rigid.*

Since the rank of the rigidity matrix for a given graph G is constant on the set of regular points of f_G and infinitesimal rigidity of a framework (G, p) depends only on the rank of $R(G,p)$ it follows that if a regular framework (G,p) is infinitesimally rigid, then all other regular frameworks (G,q) are infinitesimally rigid as well. Let k be the maximum rank of the rigidity matrix $R(G,p)$ over all configurations. Then the set of critical points of f_G can be described by a polynomial equation, namely, if Q denotes the sum of the squares of the determinants of the $k \times k$ submatrices of $R(G,p)$ in terms of the coordinates of p, then $p \in \mathbb{R}^{nd}$ is a critical point if and only if $Q(p) = 0$. This means that the set of regular points of f_G is an open dense subset of $\mathbb{R}^{nd}$, and for almost all configurations $p \in \mathbb{R}^{nd}$ (with respect to the nd-dimensional Lebesgue-measure), the framework (G,p) is regular. So infinitesimal rigidity and rigidity are 'generic' properties in the sense that the infinitesimal rigidity or rigidity of a framework (G,p) depends only on the graph G for almost all configurations.

1.2 The rigidity matroid

Let (G,p) be a d-dimensional realization of a graph $G = (V,E)$. The rigidity matrix of (G,p) defines the *rigidity matroid* $\mathcal{R}_d(G,p)$ of (G,p) on the ground set E by linear independence of the rows of the rigidity matrix. We say that a framework (G,p) is *strongly regular* if (H,p) is regular for all subgraphs H of G. Any two strongly regular frameworks (G,p) and (G,q) have the same rigidity matroid. We call this the *rigidity matroid* $\mathcal{R}_d(G) = (E,r)$ of the graph G. We denote the rank of $\mathcal{R}_d(G)$ by $r_d(G)$. From Lemma 1.1.2 it follows that $r_d(K_n) = S(n,d)$.

We say that the graph G is *rigid in* $\mathbb{R}^d$, if every (or equivalently, if some) regular d-dimensional framework (G,p) is rigid (or equivalently, infinitesimally rigid). The characterization of rigid graphs in terms of their rank follows from Lemma 1.1.3.

Theorem 1.2.1. *Let $G = (V,E)$ be a graph on n vertices. Then G is rigid in $\mathbb{R}^d$ if and only if $r_d(G) = S(n,d)$.*

We say that a graph $G = (V,E)$ is *M-independent in* $\mathbb{R}^d$ if E is independent in $\mathcal{R}_d(G)$. Knowing when subgraphs of G are M-independent allows us to determine the rank of G. We can easily get a necessary condition for the M-independence of a graph G by applying Lemma 1.1.3 on induced subgraphs. For $X \subseteq V$, let $E_G(X)$ denote the set, and $i_G(X)$ the number of edges in $G[X]$, that is, in the subgraph induced by X in G.

Theorem 1.2.2. *Let $G = (V, E)$ be a graph that is M-independent in $\mathbb{R}^d$. Then for every subset $X \subseteq V$ we have $i_G(X) \leq S(|X|, d)$.*

A graph $G = (V, E)$ is called *d-sparse* if it satisfies this necessary condition, that is, if $i_G(X) \leq S(|X|, d)$ for all $X \subseteq V$. As we will see later in this chapter, this condition is sufficient only for $d = 1, 2$.

A graph $G = (V, E)$ is *minimally rigid in* $\mathbb{R}^d$ if G is rigid in $\mathbb{R}^d$, but $G - e$ is not rigid for all $e \in E$. Theorem 1.2.1 implies that G is minimally rigid in $\mathbb{R}^d$ if and only if G is M-independent and $|E| = S(|V|, d)$. Note that, if G is rigid in $\mathbb{R}^d$, then the edge sets of the minimally rigid spanning subgraphs of G form the bases in the d-dimensional rigidity matroid of G.

1.3 Global rigidity

We say that a framework (G, p) is *globally rigid* if every framework (G, q) which is equivalent to (G, p) is congruent to (G, p). Unlike infinitesimal rigidity, which can be decided in polynomial time by checking the rank of the rigidity matrix, Saxe [35] has shown that it is NP-hard to decide if even a 1-dimensional framework is globally rigid. The problem becomes more tractable, however, if we assume that there are no algebraic dependencies between the coordinates of the points of the framework.

A framework (G, p) (or a configuration $p \in \mathbb{R}^{nd}$) is said to be *generic* if the set containing the coordinates of all its points is algebraically independent over $\mathbb{Q}$. A necessary condition for the global rigidity of a generic framework is from Hendrickson [21]. A graph $G = (V, E)$ is *redundantly rigid in* $\mathbb{R}^d$ (a framework (G, p) is redundantly rigid) if $G - e$ is rigid in $\mathbb{R}^d$ (the framework $(G - e, p)$ is infinitesimally rigid) for all $e \in E$.

Theorem 1.3.1. *[21] Let (G, p) be a d-dimensional generic framework. If (G, p) is globally rigid then either G is a complete graph with at most $d + 1$ vertices, or G is $(d + 1)$-connected and (G, p) is redundantly rigid.*

Technically, what Hendrickson proved was a slightly weaker version of this theorem. He proved the necessary condition for those frameworks (G, p), where $f_G(p)$ is a regular value of f_G, and he showed using Sard's Lemma that this is the case for almost all configurations $p \in \mathbb{R}^{nd}$. It was Connelly [10] who proved that if (G, p) is a generic framework, then $f_G(p)$ is a regular value.

There are two sufficient conditions for the global rigidity of frameworks due to Connelly, both in terms of stresses and stress matrices.

Let (G, p) be a framework on a graph $G = (V, E)$. A *stress* of the framework (G, p) is a map $\omega : E \to \mathbb{R}$, satisfying

$$\sum_{uv \in E} \omega_{uv}(p(u) - p(v)) = 0 \tag{1.4}$$

for each vertex $v \in V$. The stress is *non-zero* (*nowhere-zero*), if $w_{uv} \neq 0$ for at least one (resp., for all) $uv \in E$. The *stress matrix* Ω associated with a stress ω is an n-by-n symmetric matrix defined by

$$\Omega_{uv} = \begin{cases} \sum\limits_{uw \in E} \omega_{uw} & \text{if } u = v \\ -\omega_{uv} & \text{if } u \neq v \text{ and } uv \in E \\ 0 & \text{if } u \neq v \text{ and } uv \notin E \end{cases}$$

It is easy to see, that if $\dim p = d$ and Ω is a stress matrix associated with a stress ω, then $\operatorname{rank}(\Omega) \leq |V| - d - 1$.

The first sufficient condition for global rigidity is for generic frameworks.

Theorem 1.3.2. *[10] Let (G, p) be a d-dimensional generic framework on n vertices for which there is a stress ω, such that the rank of the associated stress matrix Ω is $n - d - 1$. Then (G, p) is globally rigid.*

If we leave the condition that the framework is generic, the rank of the stress matrix alone is not enough to guarantee global rigidity, even if the framework is infinitesimally rigid. As an example, consider the framework on Figure 1.2, which is not globally rigid, since it can 'fold' around the diagonal. In Section 3.1, by adding two additional conditions, one for the framework, and the other for the stress matrix we will derive another sufficient condition for global rigidity.

Both Hendrickson and Connelly conjectured that their conditions for the global rigidity of generic frameworks were both necessary and sufficient. However, Connelly [9] gave examples showing that $(d + 1)$-connectivity and redundant rigidity are not sufficient for a generic framework to be globally rigid when $d \geq 3$. These examples

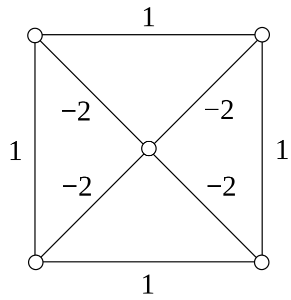

Figure 1.2: An infinitesimally rigid framework together with a stress. The stress matrix has rank $n-3$, but the framework is not globally rigid in the plane.

were complete bipartite graphs $K_{a,b}$, where $a+b=\binom{d+2}{2}$ and $a,b \geq d+2$. The only known example in $\mathbb{R}^3$ is the complete bipartite graph $K_{5,5}$. In a recent paper [17], Frank and Jiang gave further counterexamples to Hendrickson's conjecture for $d \geq 4$, among them an infinite class of graphs for $d=5$.

Another interesting question is whether global rigidity is a generic property of a graph G in the sense that if a generic framework (G,p) is globally rigid, is it true that every other generic framework (G,q) is globally rigid as well? A positive answer to this question was given by Gortler, Healy and Thurston [20], by showing that Connelly's condition for the global rigidity of generic frameworks is also necessary.

Theorem 1.3.3. *[20] Let (G,p) be a d-dimensional generic framework on $n \geq d+2$ vertices. If (G,p) is globally rigid, then there is a stress ω, such that the rank of the associated stress matrix Ω is $n-d-1$.*

Putting Theorems 1.3.2 and 1.3.3 together with the observation that if there is a stress matrix with maximum rank (i.e. $n-d-1$) for one generic framework, then there is one for all other generic frameworks [22, Theorem 2.5], we get that if a generic framework is globally rigid, then every other generic framework for the same graph is globally rigid as well. Therefore we can say that a graph G is *globally rigid* in $\mathbb{R}^d$ if every (or equivalently, if some) generic realization of G in $\mathbb{R}^d$ is globally rigid. A necessary and sufficient condition for the global rigidity of graphs can thus be stated as follow.

Theorem 1.3.4. *[20] Let $G=(V,E)$ be a graph with $|V| \geq d+2$. G is globally rigid in $\mathbb{R}^d$, if and only if there is a generic framework (G,p) and a stress ω, such*

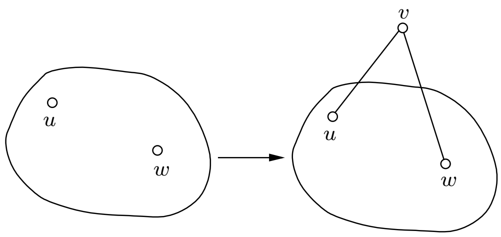

Figure 1.3: The 2-dimensional 0-extension Henneberg operation.

that the rank of the associated stress matrix Ω *is* $|V| - d - 1$.

In [14], Connelly and Whiteley showed that it is enough to demand that (G, p) is infinitesimally rigid in Theorem 1.3.4. Thus one can give a polynomial-size certificate for the global rigidity of a graph G in the form of an infinitesimally rigid framework (G, p) and a stress ω. However, it is not guaranteed that the certificate itself is globally rigid. See the example in Figure 1.2.

1.4 Graph operations

This section is dedicated to the overview of the most commonly used graph operations that can be used to construct rigid (globally rigid) graphs from smaller rigid (globally rigid) graphs.

1.4.1 Henneberg operations

First we will introduce the two graph operations that – together with the simple edge addition – are used in the inductive construction of all rigid and globally rigid graphs in the plane. These operations, named after Henneberg [23], are key elements in the combinatorial characterization of rigid and globally rigid graphs in the plane. The main idea is that we have a necessary condition, like Theorem 1.2.2 or Theorem 1.3.1, and a constructive characterization theorem that generates every graph that meets the necessary condition. If the first element in the construction sequence is rigid (globally rigid), and each operation in the construction preserves rigidity (global rigidity), then the resulting graph will be rigid (globally rigid) as well.

Given a graph $G = (V, E)$ and distinct vertices $x_1, x_2, \ldots, x_d \in V$, a *d-dimensional*

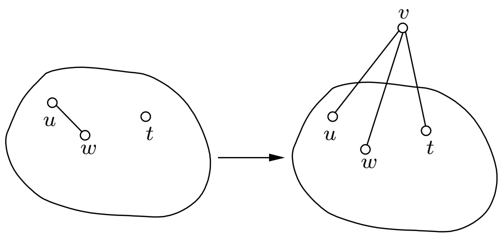

Figure 1.4: The 2-dimensional 1-extension Henneberg operation

0-*extension* of G is a graph obtained from G by adding a new vertex z and new edges $zx_1, zx_2, \ldots, zx_d$. See Figure 1.3 for $d = 2$.

Lemma 1.4.1. *[40, 44] Let (G, p) be an infinitesimally rigid d-dimensional framework and let $x_1, x_2, \ldots, x_d \in V$ be distinct vertices, such that $p(x_1), p(x_2), \ldots, p(x_d)$ are affine independent. Suppose that H is obtained from G by a d-dimensional 0-extension on $x_1, \ldots, x_d$ and p^* is an extension of p with the new vertex z such that $p^*(z)$ is not in the affine span of $p(x_1), \ldots, p(x_d)$. Then (H, p^*) is infinitesimally rigid.*

Corollary 1.4.2. *Let G be a graph that is rigid in $\mathbb{R}^d$. Suppose that H is obtained from G by a d-dimensional 0-extension. Then H is rigid in $\mathbb{R}^d$.*

Given a graph $G = (V, E)$ and distinct vertices $x_1, x_2, \ldots, x_{d+1} \in V$ with $x_1x_2 \in E$, a *d-dimensional* 1-*extension* of G is a graph obtained from G by deleting the edge x_1x_2 and adding a new vertex z and new edges $zx_1, zx_2, \ldots, zx_{d+1}$. See Figure 1.4 for $d = 2$.

Theorem 1.4.3. *[40, 44] Let G be a graph that is rigid in $\mathbb{R}^d$. Suppose that H is obtained from G by a d-dimensional 1-extension. Then H is rigid in $\mathbb{R}^d$.*

The corresponding result for frameworks is the following.

Theorem 1.4.4. *[40, 44] Let (G, p) and (H, p^*) be two frameworks where H is a d-dimensional 1-extension of G on the edge uw and additional vertices $t_1, \ldots, t_{d-1}$ and p^* is the extension of p with the new vertex v such that $p^*(v)$ is on the line $p(u)p(w)$ (excluding $p(u), p(w)$). Suppose that (G, p) is infinitesimally rigid and $p(v) - p(u), p(v) - p(t_1), \ldots, p(v) - p(t_{d-1})$ are linearly independent. Then (H, p^*) is infinitesimally rigid.*

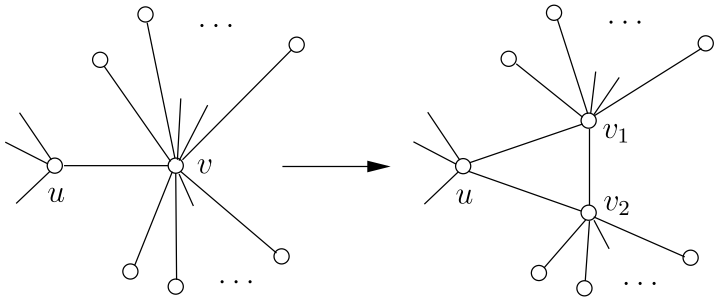

Figure 1.5: The 2-dimensional vertex splitting operation on edge uv and vertex v.

The following theorem about global rigidity and 1-extensions is a corollary to a more general extension result that we will discuss in the next chapter.

Theorem 1.4.5. *Let G be a graph with at least $d+2$ vertices that is globally rigid in $\mathbb{R}^d$. Suppose that H is obtained from G by a d-dimensional 1-extension. Then H is globally rigid in $\mathbb{R}^d$.*

We note that for the combinatorial characterization of globally rigid graphs in the plane, the following slightly weaker form of this theorem is sufficient, which is a result of Connelly.

Theorem 1.4.6. *[10] Suppose that a graph G can be obtained from K_{d+2} by a sequence of d-dimensional 1-extensions and edge additions. Then every generic realization of G in $\mathbb{R}^d$ is globally rigid.*

1.4.2 Vertex splitting

Given a graph $G = (V, E)$, $d-1$ edges $u_1v, \ldots, u_{d-1}v \in E$, and a bipartition F_1, F_2 of the edges incident to v, but different from $u_1v, \ldots, u_{d-1}v$, the *d-dimensional vertex splitting* operation *on edges $u_1v, \ldots, u_{d-1}v$ at vertex v* replaces vertex v by two new vertices v_1 and v_2, replaces the edges $u_1v, \ldots, u_{d-1}v$ by $2d-1$ new edges $u_1v_1, u_1v_2, \ldots, u_{d-1}v_1, u_{d-1}v_2, v_1v_2$, and replaces each edge $wv \in F_i$ by an edge wv_i, $i = 1, 2$, see Figure 1.5 for the 2-dimensional case. The vertex splitting operation is said to be *non-trivial* if F_1, F_2 are both non-empty, or equivalently, if each of the split vertices v_1, v_2 has degree at least $d+1$.

It is known that vertex splitting preserves the rigidity of graphs in all dimensions.

Theorem 1.4.7. *[43, 44] If G is a rigid graph in $\mathbb{R}^d$ and G' is obtained from G by a d-dimensional vertex splitting operation, then G' is rigid in $\mathbb{R}^d$.*

An analogous statement for global rigidity, with the additional condition that the vertex split is non-trivial was conjectured by Cheung and Whiteley.

Conjecture 1.4.8. *[6] If G is a globally rigid graph in $\mathbb{R}^d$ and G' is obtained from G by a non-trivial d-dimensional vertex splitting operation, then G' is globally rigid in $\mathbb{R}^d$.*

The 2-dimensional version of Conjecture 1.4.8 was verified in [29] by showing that 2-dimensional vertex splitting preserves both 3-connectivity and redundant rigidity when applied to 3-connected and redundantly rigid graphs.

1.4.3 Gluing and 2-sums

A simple graph operation that takes two graphs and produces a graph of which both input graphs are subgraphs is called gluing. Let $G = (V, E)$ be a graph and let $V_1, V_2 \subseteq V$ be two vertex sets such that $V_1 \cup V_2 = V$ and $V_1 \cap V_2 \neq \emptyset$. Let E_1 and E_2 be the edge sets induced by V_1 and V_2. We say that G is the result of gluing the subgraphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ together along the vertex set $V_1 \cap V_2$.

The following well known gluing lemma is about gluing together two rigid graphs.

Theorem 1.4.9. *[44] Let G be a graph that is obtained by gluing together two subgraphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$. Suppose that H_1 and H_2 are both rigid in $\mathbb{R}^d$ and that $|V_1 \cap V_2| \geq d$. Then G is rigid in $\mathbb{R}^d$.*

The analogous result for globally rigid graphs follows from the observation that if two congruences of $\mathbb{R}^d$ are the same on $d + 1$ points in general position, then they are identical.

Theorem 1.4.10. *Let G be a graph that is obtained by gluing together two subgraphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$. Suppose that H_1 and H_2 are both globally rigid in $\mathbb{R}^d$ and that $|V_1 \cap V_2| \geq d + 1$. Then G is globally rigid in $\mathbb{R}^d$.*

The following operation, which is related to gluing, will often be used as an inductive construction throughout this work. Given two graphs $H_1 = (V_1, E_1)$ and

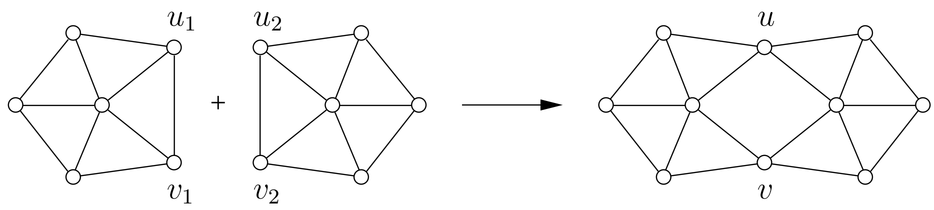

Figure 1.6: The 2-sum of two graphs along the pair u_1v_1, u_2v_2.

$H_2 = (V_2, E_2)$ with $V_1 \cap V_2 = \emptyset$ and two designated edges $u_1v_1 \in E_1$ and $u_2v_2 \in E_2$, the 2-*sum* of H_1 and H_2 (along the edge pair u_1v_1, u_2v_2) is the graph obtained from $H_1 - u_1v_1$ and $H_2 - u_2v_2$ by identifying u_1 with u_2 and v_1 with v_2, see Figure 1.6. We denote a 2-sum of H_1 and H_2 by $H_1 \oplus_2 H_2$.

1.4.4 Coning

All the previous graph operations are used to construct (globally) rigid graphs in a given dimension. In contrast, the coning operation is used to transfer (global) rigidity results between dimensions. Given a graph $G = (V, E)$, the *coning* operation adds a new vertex v, and new edges vw for all $w \in V$. The resulting cone graph is denoted by $G * \{v\}$. It is known that coning a graph that is rigid in $\mathbb{R}^d$, results in graph that is rigid in $\mathbb{R}^{d+1}$ [42]. In a recent paper, Connelly and Whiteley extended this result for global rigidity, proving a conjecture of Cheung and Whiteley [6].

Theorem 1.4.11. *[14] A graph G is globally rigid in $\mathbb{R}^d$ if and only if the cone graph $G * \{v\}$ is globally rigid in $\mathbb{R}^{d+1}$.*

1.5 Combinatorial characterizations of rigidity and global rigidity

The problem of characterizing when a graph is rigid in $\mathbb{R}^d$ has been solved for $d = 1, 2$. A graph G is rigid in $\mathbb{R}$ if and only if it is connected. For the $d = 2$ case, we begin by stating the purely combinatorial constructive characterization theorem that is needed to characterize minimally rigid and M-independent graphs.

Theorem 1.5.1. *[25] Let $G = (V, E)$ be a 2-sparse graph with $|E| = 2|V| - 3$. Then*

G can be obtained from an edge by a sequence of (2-dimensional) 0-extensions and 1-extensions.

Theorem 1.2.2, Corollary 1.4.2, Theorem 1.4.3 and Theorem 1.5.1 together give the characterization of minimally rigid graphs in the plane.

Theorem 1.5.2. *[30, 25] A graph $G = (V, E)$ is minimally rigid in $\mathbb{R}^2$ if and only if G is 2-sparse and $|E| = 2|V| - 3$.*

For M-independence, this gives Laman's Theorem.

Theorem 1.5.3. *[30] A graph $G = (V, E)$ is M-independent in $\mathbb{R}^2$ if and only if G is 2-sparse, that is $i_G(X) \leq 2|X| - 3$ for all $X \subseteq V$ with $|X| \geq 2$.*

We will need the following lemma about the inverse operation of 1-extension on minimally rigid graphs.

Lemma 1.5.4. *[25, Lemma 2.8(b)] Let $G = (V, E)$ be a minimally rigid graph in $\mathbb{R}^2$ and let $v \in V$ be a vertex with $d(v) = 3$. Then there are $u, w \in N_G(v)$, $uw \notin E$ such that the graph $H = G - v + uw$ is minimally rigid in $\mathbb{R}^2$.*

For global rigidity a similar situation holds: the problem of characterizing when a generic framework is globally rigid in $\mathbb{R}^d$ has also been solved for $d = 1, 2$. A 1-dimensional generic framework (G, p) is globally rigid if and only if either G is the complete graph on two vertices or G is 2-connected. The characterization for $d = 2$ follows from the following constructive characterization result.

Theorem 1.5.5. *[25, Theorem 6.15] Let G be a 3-connected graph which is redundantly rigid in $\mathbb{R}^2$. Then G can be obtained from K_4 by a sequence of (2-dimensional) 1-extensions and edge additions.*

By observing that complete graphs are globally rigid and using Theorem 1.4.6, we obtain a complete characterization for globally rigid generic frameworks in $\mathbb{R}^2$.

Theorem 1.5.6. *[10, 25] Let (G, p) be a 2-dimensional generic framework. Then (G, p) is globally rigid if and only if either G is a complete graph on two or three vertices, or G is 3-connected and redundantly rigid in $\mathbb{R}^2$.*

Though the characterization of rigid and globally rigid graphs in higher dimensions is an open problem, there is a special class of graphs for which both of these problems have been solved. Let $H = (V, E)$ be a multigraph with minimum degree at least one. As in [13], we define the *body-bar graph induced by H*, denoted by G_H, as the graph obtained from H by replacing each vertex $v \in V$ by a complete graph B_v (a 'body') on $d_H(v)$ vertices and replacing each edge uv by an edge (a 'bar') between B_u nd B_v in such a way that the bars are pairwise disjoint. In [38, 39], Tay has shown that rigidity and redundant rigidity of body-bar graphs in $\mathbb{R}^d$ can be determined in polynomial time for all $d \geq 1$. In [13], Connelly, Jordán and Whiteley solved the characterization of globally rigid body-bar graphs by showing that a body-bar graph is globally rigid in $\mathbb{R}^d$ if and only if it is redundantly rigid in $\mathbb{R}^d$.

1.6 Tensegrity graphs and frameworks

Tensegrity frameworks are defined on a set of points in $\mathbb{R}^d$ and consist of bars, cables, and struts, which provide upper and/or lower bounds for the distance between their endpoints.

A *tensegrity graph* $T = (V; B, C, S)$ is a simple graph on the vertex set $V = \{v_1, v_2 \ldots, v_n\}$ whose edge set is partitioned into three pairwise disjoint sets B, C, and S, called *bars, cables*, and *struts*, respectively. The elements of $E = B \cup C \cup S$ are the *members* of T. A tensegrity graph containing no bars is called a *cable-strut tensegrity graph*, denoted by $T = (V; C, S)$. The *underlying graph* of T is the (unlabeled) graph $\overline{T} = (V; E)$. A d-dimensional *tensegrity framework* is a pair (T, p), where T is a tensegrity graph and p is a map from V to $\mathbb{R}^d$. (T, p) is also called a *realization* of T. If T has neither cables nor struts then we call it a *bar graph* and a realization (T, p) is said to be a *bar framework*.

An *infinitesimal motion* of a tensegrity framework (T, p) is an assignment of infinitesimal velocities $q : V \to \mathbb{R}^d$ to the vertices, such that

$$
\begin{aligned}
(p(u) - p(v))(q(u) - q(v)) &= 0 \quad \text{for all } uv \in B,\\
(p(u) - p(v))(q(u) - q(v)) &\leq 0 \quad \text{for all } uv \in C,\\
(p(u) - p(v))(q(u) - q(v)) &\geq 0 \quad \text{for all } uv \in S.
\end{aligned}
$$

An infinitesimal motion is *trivial* if it can be obtained as the derivative of a rigid

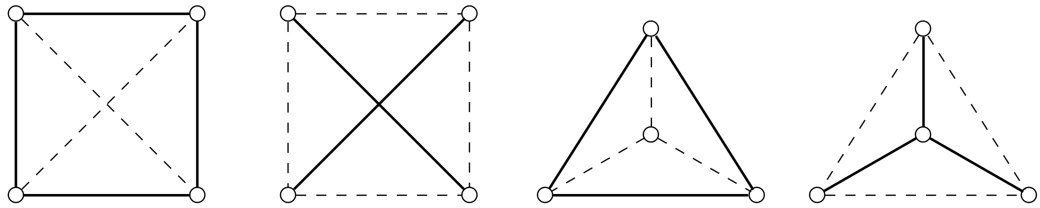

Figure 1.7: Infinitesimally rigid realizations of the four rigid tensegrity graphs on K_4. (In this work we use solid (dashed) lines to denote struts (resp. cables).)

congruence of all of $\mathbb{R}^d$ restricted to the vertices of (T, p). The tensegrity framework (T, p) is *infinitesimally rigid* in $\mathbb{R}^d$ if all of its infinitesimal motions are trivial. A tensegrity graph T is said to be *rigid* in $\mathbb{R}^d$ if it has an infinitesimally rigid realization (T, p) in $\mathbb{R}^d$.

A *stress* of a tensegrity framework (T, p) is an assignment of scalars ω_{ij} to the members ij of T satisfying $\omega_{ij} \leq 0$ for cables, $\omega_{ij} \geq 0$ for struts and

$$\sum_{uv \in E} \omega_{uv}(p(u) - p(v)) = 0 \quad \text{for each } v \in V.$$

Following [34] we say that $\omega = (\dots, \omega_{ij}, \dots) \in \mathbb{R}^E$ is a *proper stress*, if $\omega_{ij} \neq 0$ for all $ij \in C \cup S$.

The following basic results on infinitesimally rigid tensegrity frameworks are due to Roth and Whiteley, see [34, Theorem 5.2(c), Corollary 5.3].

Theorem 1.6.1. *[34] Suppose that (T, p) is a tensegrity framework in $\mathbb{R}^d$. Then*
(i) (T, p) is infinitesimally rigid in $\mathbb{R}^d$ if and only if the bar framework $(\overline{T}, p)$ is infinitesimally rigid in $\mathbb{R}^d$ and there exists a proper stress of (T, p);
(ii) if (T, p) is infinitesimally rigid then the bar framework obtained by deleting any cable or strut of T and replacing the remaining members of T by bars is infinitesimally rigid in $\mathbb{R}^d$.

We define the *rigidity matrix* of a tensegrity framework (T, p), denoted by $R(T, p)$, as the rigidity matrix of $(\overline{T}, p)$. Note that a stress of (T, p) corresponds to a row dependency of $R(T, p)$.

A configuration $p \in \mathbb{R}^{dn}$ is a *regular point* of T if

$$\operatorname{rank} R(T, p) = \max\{\operatorname{rank} R(T, q) : q \in \mathbb{R}^{dn}\}.$$

It is *strongly regular* if it also gives rise to a regular point for all non-empty edge-induced subgraphs of T. Note that the regular (strongly regular) points of T form an open and dense subset of $\mathbb{R}^{dn}$. We also say that a realization (T, p) is regular (strongly regular) if p is a regular (strongly regular, respectively) point of T. A member e of T is *redundant* in (T, p) if $\operatorname{rank} R(T, p) = \operatorname{rank} R(T - e, p)$. The proof of the next lemma is implicit in the proof of [34, Theorem 5.4].

Theorem 1.6.2. *[34] Let (T, p) be a regular realization of tensegrity graph T in $\mathbb{R}^d$, let ω be a proper stress of (T, p), and let e be a redundant member in (T, p). Then the set of configurations $q \in \mathbb{R}^{dn}$, for which (T, q) is a regular realization of T, which has a proper stress ω' with $\omega'(e) = \omega(e)$, is open in $\mathbb{R}^{dn}$.*

Clearly, G is the underlying graph of a rigid tensegrity graph if and only if G is rigid - simply label each edge as a bar. If bars are not allowed then we may use Theorem 1.6.1 to deduce the following characterization.

Theorem 1.6.3. *A graph $G = (V, E)$ is the underlying graph of a rigid cable-strut tensegrity graph in $\mathbb{R}^d$ if and only if G is redundantly rigid in $\mathbb{R}^d$.*

Proof: Necessity follows from Theorem 1.6.1(ii). To prove sufficiency consider a strongly regular realization (G, p) of G in $\mathbb{R}^d$ as a bar framework. Since G is redundantly rigid, (G, p) is infinitesimally rigid and each edge of G is redundant in (G, p). This implies that for each edge $e \in E$ there is a set of edges $C_e \subseteq E$ with $e \in C_e$ for which there is a proper stress ω_e in the subframework consisting of the bars of C_e. By using the fact that the stresses of the bar framework (G, p) form a linear subspace of $\mathbb{R}^E$, we can deduce that there exist non-zero coefficients $\lambda_e \in \mathbb{R}$, $e \in E$, for which $\omega = \sum_{e \in E} \lambda_e \omega_e$ is a proper stress for (G, p). Let $C = \{ij \in E \mid \omega_{ij} < 0\}$ and $S = \{ij \in E \mid \omega_{ij} > 0\}$. This labeling gives rise to a rigid cable-strut tensegrity graph T with underlying graph G by Theorem 1.6.1(i). •

Chapter 2

Globally Linked Pairs of Vertices

In this chapter we will consider properties of generic frameworks which are weaker than global rigidity. A pair of vertices $\{u, v\}$ in a framework (G, p) is *globally linked* in (G, p) if, in all equivalent frameworks (G, q), we have $||p(u) - p(v)|| = ||q(u) - q(v)||$. The pair $\{u, v\}$ is *globally linked* in G in $\mathbb{R}^d$ if it is globally linked in all d-dimensional generic frameworks (G, p). Thus G is globally rigid in $\mathbb{R}^d$ if and only if all pairs of vertices of G are globally linked in $\mathbb{R}^d$. Unlike global rigidity, however, 'global linkedness' is not a generic property even in $\mathbb{R}^2$. Figures 2.1 and 2.2 give an example of a pair of vertices in a rigid graph G which is globally linked in one generic realization, but not in another. Note that if $d = 1$ then global linkedness is a generic property: $\{u, v\}$ is globally linked in G in $\mathbb{R}$ if and only if G has two openly disjoint uv-paths.

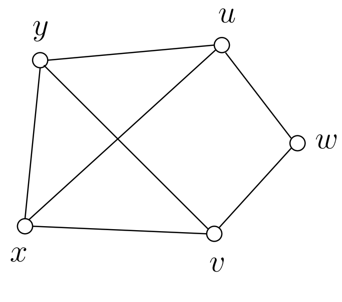

Figure 2.1: A realization (G, p) of a rigid graph G in $\mathbb{R}^2$. The pair $\{u, v\}$ is globally linked in (G, p).

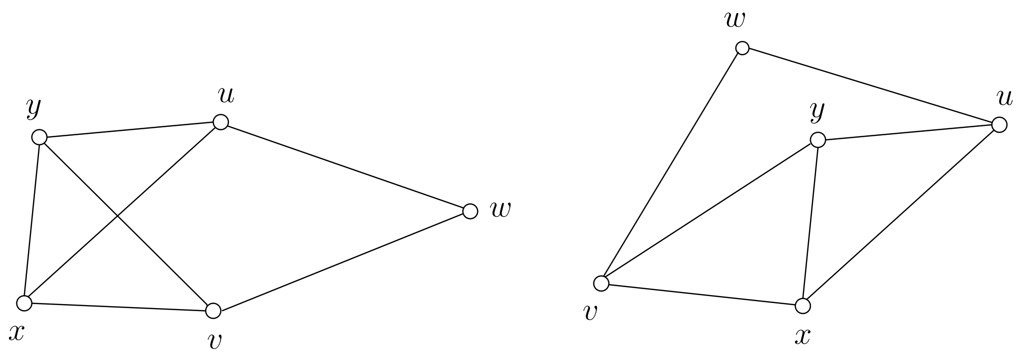

Figure 2.2: Two equivalent realizations of the rigid graph G of Figure 2.1, which show that the pair $\{u, v\}$ is not globally linked in G in $\mathbb{R}^2$.

We first show that global linkedness is preserved by the 1-extension operation. By using Theorem 1.3.1, we deduce that global rigidity is preserved by the 1-extension operation. This immediately gives Theorem 1.4.6 and hence simplifies the proof of Theorem 1.5.6. (Connelly deduces Theorem 1.4.6 from a sufficient condition for the global rigidity of a d-dimensional framework in terms of the rank of its stress matrix, [10, Theorem 1.5]. His proof of this sufficient condition uses powerful results from geometry, differential topology, and the elimination theory of semi-algebraic sets.)

In the remainder of the chapter we consider the following problems for a 2-dimensional generic realization (G, p) of a graph $G = (V, E)$.

(a) Given $\{u, v\} \subset V$, when is $\{u, v\}$ globally linked in (G, p)?

(b) Given $v \in V$ and $U \subset V$, when is v uniquely localizable with respect to U, that is to say, when is it true that every realization (G, q) which is equivalent to (G, p) and satisfies $p(u) = q(u)$ for all $u \in U$, must also satisfy $p(v) = q(v)$?

(c) Given $\{u, v\} \subset V$, when is $\{u, v\}$ globally loose in G, that is to say, when is it true that for all generic realizations (G, p), there exists an equivalent realization (G, q) which satisfies $||p(u) - p(v)|| \neq ||q(u) - q(v)||$?

(d) How many different realizations of G are there which are each equivalent to (G, p)?

We use our result on 1-extensions to solve each of these problems for M-connected graphs in $\mathbb{R}^2$, an important family of rigid graphs. Our results imply that the answer to each of the problems described in (a), (b) and (d) is generic when G is M-connected, in the sense that the answer is the same for all generic realizations of G.

2.1 Generic points and polynomial maps

In this section we prove some basic results on algebraically independent sets, field extensions and polynomial maps. They will be needed to handle 1-extensions of generic frameworks. A point $\mathbf{x} \in \mathbb{R}^n$ is *generic* if its components form an algebraically independent set over $\mathbb{Q}$.

Lemma 2.1.1. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ by $f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x}))$, where $f_i(\mathbf{x})$ is a polynomial with integer coefficients for all $1 \leq i \leq m$. Suppose that $\max_{\mathbf{x}\in\mathbb{R}^n}\{rank\ df|_{\mathbf{x}}\} = m$. If $\mathbf{p}$ is a generic point in $\mathbb{R}^n$, then $f(\mathbf{p})$ is a generic point in $\mathbb{R}^m$.*

Proof: Since $\mathbf{p}$ is generic, we have rank $df|_{\mathbf{p}} = m$. Relabeling if necessary, we may suppose that the first m columns of $df|_{\mathbf{p}}$ are linearly independent. Let $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. Define $f' : \mathbb{R}^m \to \mathbb{R}^m$ by

$$f'(x_1, x_2, \ldots, x_m) = f(x_1, x_2, \ldots, x_m, p_{m+1}, \ldots, p_n).$$

Let $\mathbf{p}' = (p_1, p_2, \ldots, p_m)$. Then $f'(\mathbf{p}') = f(\mathbf{p})$ and rank $df'|_{\mathbf{p}'} = m$.

Let $f'(\mathbf{p}') = (\beta_1, \beta_2, \ldots, \beta_m)$. Suppose that $g(\beta_1, \beta_2, \ldots, \beta_m) = 0$ for some polynomial g with integer coefficients. Then $g(f_1(\mathbf{p}), f_2(\mathbf{p}), \ldots, f_m(\mathbf{p})) = 0$. Since $\mathbf{p}$ is generic, we must have $g(f'(\mathbf{x})) = 0$ for all $\mathbf{x} \in \mathbb{R}^m$. By the inverse function theorem f' maps a sufficiently small open neighborhood U of $\mathbf{p}'$ diffeomorphically onto $f'(U)$. Thus $g(\mathbf{y}) = g(f'(\mathbf{x})) = 0$ for all $\mathbf{y} \in f'(U)$. Since g is a polynomial map and $f'(U)$ is an open subset of $\mathbb{R}^m$, we have $g \equiv 0$. Hence $f'(\mathbf{p}') = f(\mathbf{p})$ is generic. •

Given a point $\mathbf{p} \in \mathbb{R}^n$ we use $\mathbb{Q}(\mathbf{p})$ to denote the field extension of $\mathbb{Q}$ by the coordinates of $\mathbf{p}$. Given fields $K \subseteq L$ with L a finitely generated field extension of K, the *transcendence degree* of L over K, $td[L : K]$, is the size of a largest subset of L which is algebraically independent over K, see [36, Section 18.1]. (It follows from the Steinitz exchange axiom, see [36, Lemma 18.4] and [32, Section 6.7], that this definition gives rise to a matroid on L, where the rank of a subset S of L is $td[K(S):K]$.) We use $\overline{K}$ to denote the algebraic closure of K. Note that each element of $\overline{K}$ is a loop in the above mentioned matroid and hence $td[\overline{K}:K]=0$.

Lemma 2.1.2. *Let $f:\mathbb{R}^n \to \mathbb{R}^n$ by $f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_n(\mathbf{x}))$, where $f_i(\mathbf{x})$ is a polynomial with integer coefficients for all $1 \le i \le n$. Suppose that $f(\mathbf{p})$ is a generic point in $\mathbb{R}^n$. Let $L = \mathbb{Q}(\mathbf{p})$ and $K = \mathbb{Q}(f(\mathbf{p}))$. Then $\overline{K} = \overline{L}$.*

Proof: Since $f_i(\mathbf{x})$ is a polynomial with integer coefficients, we have $f_i(\mathbf{p}) \in L$ for all $1 \le i \le n$. Thus $K \subseteq L$. Since $f(\mathbf{p})$ is generic we have $td[K:\mathbb{Q}] = n$. Since $K \subseteq L$ and $L = \mathbb{Q}(\mathbf{p})$ we have $td[L:\mathbb{Q}] = n$. Thus $\overline{K} \subseteq \overline{L}$ and $td[\overline{K}:\mathbb{Q}] = n = td[\overline{L}:\mathbb{Q}]$. Suppose $\overline{K} \neq \overline{L}$, and choose $\gamma \in \overline{L} - \overline{K}$. Then γ is not algebraic over K so $S = \{\gamma, f_1(\mathbf{p}), f_2(\mathbf{p}), \dots, f_n(\mathbf{p})\}$ is algebraically independent over $\mathbb{Q}$. This contradicts the facts that $S \subseteq \overline{L}$ and $td[\overline{L}:\mathbb{Q}] = n$. •

Lemma 2.1.3. *Let K be a countable subfield of $\mathbb{R}$ and let $f \in \mathbb{R}[x_1, \dots, x_n]$ be an irreducible polynomial. If the cardinality of the root-set of f is continuum, and all roots of f are non-generic over K, then there is $\lambda \neq 0$, such that $\lambda f \in K[x_1, \dots, x_n]$.*

Proof: For all $p \in \mathbb{R}^n$, which is a root of f, p is also the root of a non-zero irreducible polynomial in $K[x_1, \dots, x_n]$. Since the cardinality of $K[x_1, \dots, x_n]$ is countable, there is a non-zero irreducible polynomial $g \in K[x_1, \dots, x_n]$, such that the cardinality of the set of common roots of f and g is continuum. From Bezout's theorem, we get that f and g cannot be relative primes, and since both f and g are irreducible, there must be a $\lambda \neq 0$, such that $\lambda f = g$. •

2.2 Quasi-generic frameworks

In this section we prove some preliminary results on generic frameworks which we will use in our proof that 1-extensions preserve global linkedness. We say that a d-dimensional configuration $p = (p_1, \dots, p_n) \in \mathbb{R}^{nd}$ is in *standard position* if p_1 is the origin and $p_i \in \langle \mathbf{e_1}, \dots, \mathbf{e_{i-1}} \rangle$ for all $2 \le i \le \min(n,d)$, where $\mathbf{e_1}, \dots, \mathbf{e_d}$ is the standard basis of $\mathbb{R}^d$. If p is in standard position, then $nd - S(n,d)$ coordinates of p are zero. Let $Z_{n,d} \subset \mathbb{R}^{nd}$ be the set of configurations that are in standard position and let $\pi_{n,d} : Z_{n,d} \to \mathbb{R}^{S(n,d)}$ be the projection on those coordinates that are not zeroed out in standard position. Let $\iota_{n,d} : \mathbb{R}^{S(n,d)} \to Z_{n,d}$ be the inverse of $\pi_{n,d}$. A framework (G,p) is in *standard position* if the realization p is in standard position.

A small problem with the definition of generic frameworks is that it is not invariant under the congruences of $\mathbb{R}^d$. For every generic framework (G,q) there is a framework (G,p) in standard position that is congruent to (G,q), but, since p has zero coordinates, the framework (G,p) is not generic. To overcome this difficulty, we define a framework (G,p) (or a configuration $p \in \mathbb{R}^{nd}$) to be *quasi-generic* if it is congruent to a generic framework (G,q) (or a generic configuration q).

Lemma 2.2.1. *If (G,p) is a quasi-generic framework and G is M-independent then $f_G(p)$ is generic.*

Proof: Choose a generic framework (G,q) congruent to (G,p). Since G is M-independent, $\operatorname{rank} df_G|_q = |E|$. Hence Lemma 2.1.1 implies that $f_G(q)$ is generic. The lemma now follows since $f_G(p) = f_G(q)$. •

Lemma 2.2.2. *Suppose that (G,p) is in standard position, G is minimally rigid and $f_G(p)$ is generic. Let $L = \mathbb{Q}(p)$ and $K = \mathbb{Q}(f_G(p))$. Then $\pi_{n,d}(p)$ is generic and $\overline{K} = \overline{L}$.*

Proof: Let $p' = \pi_{n,d}(p)$ and define $f : \mathbb{R}^{S(n,d)} \to \mathbb{R}^{S(n,d)}$ by $f(x) = f_G(\iota_{n,d}(x))$. Then $f(p') = f_G(p)$ is generic. We have $L = \mathbb{Q}(p')$ and $K = \mathbb{Q}(f(p'))$. By Lemma 2.1.2, we have $\overline{K} = \overline{L}$. Furthermore, $S(n,d) = td[\overline{K}, \mathbb{Q}] = td[\overline{L}, \mathbb{Q}]$. Thus p' is a generic point in $\mathbb{R}^{S(n,d)}$. •

Lemma 2.2.3. *A configuration $p \in \mathbb{R}^{nd}$ is quasi-generic if and only if there is a configuration q in standard position that is congruent to p and $\pi_{n,d}(q)$ is generic.*

Proof: Suppose p is quasi-generic. Let $G = (V,E)$ be a minimally rigid graph, and consider the quasi-generic framework (G,p). By Lemma 2.2.1, $f_G(p)$ is a generic point in $\mathbb{R}^{S(n,d)}$. Choose a framework (G, q) in standard position that is congruent to (G, p). Then $f_G(q) = f_G(p)$ is generic, and by Lemma 2.2.2, $\pi_{n,d}(q)$ is generic.

We next suppose that there is a configuration $q = (q_1, \ldots, q_n)$ in standard position that is congruent to p and $q' = \pi_{n,d}(q)$ is generic. We have to show that there is a generic configuration $s = (s_1, \ldots, s_n)$ that is congruent to q. Let $A \in \mathbb{R}^{d\times d}$ be an orthogonal matrix and $b \in \mathbb{R}^d$ a vector such that the set

$$C = \{a_{ij} \,|\, 1 \le j < \min(n, d); j < i \le d\} \cup \{b_i \,|\, 1 \le i \le d\} \cup \{q'_i \,|\, 1 \le i \le S(n, d)\}$$

is algebraically independent over $\mathbb{Q}$. Such an A matrix exists because the lower triangular entries of an orthogonal matrix can be chosen arbitrarily, while all the other entries are determined by the orthogonality equations. Let $s_i = Aq_i + b$ for all $1 \le i \le n$. Since A is orthogonal, s is congruent to q. Let $K = \mathbb{Q}(C)$, and $L = \mathbb{Q}(s)$. We show that $\overline{K} \subseteq \overline{L}$. It suffices to show that the elements of C are all algebraic over L. Since $b = s_1$, the coordinates of b are in L. Let $k = \min(n, d)$. Since q is in standard position, we have

$$\begin{aligned} q_1 &= (0, 0, \ldots, 0) \\ q_2 &= (q_{21}, 0, \ldots, 0) \\ &\vdots \\ q_k &= (q_{k1}, \ldots, q_{k,k-1}, \ldots, 0) \end{aligned}$$

Using the $\binom{k}{2}$ equations $||q_i - q_j||^2 = ||s_i - s_j||^2$ for all $1 \le i < j \le k$, the $\binom{k}{2}$ coordinates $q_{21}, \ldots, q_{k,k-1}$ can be expressed as algebraic expressions of the coordinates of s, and therefore $q_{21}, \ldots, q_{k,k-1} \in \overline{L}$. Let a_j denote the jth column vector of A. Then we have

$$\begin{aligned} a_1 &= \frac{1}{q_{21}}(s_2 - b) \\ &\vdots \\ a_{k-1} &= \frac{1}{q_{k,k-1}}(s_k - b - q_{k1}a_1 - \ldots - q_{k,k-2}a_{k-2}) \end{aligned}$$

These equations show that the coordinates of $a_1, \ldots, a_{k-1}$ are all algebraic over L. Moreover, if $n \ge d$, then from the orthogonality equations we get that $A \in \overline{L}^{d\times d}$. This implies that $A^{-1} \in \overline{L}^{d\times d}$ and thus $q_i = A^{-1}(s_i - b) \in \overline{L}^d$ for all $d < i \le n$.

Hence $\overline{K} \subseteq \overline{L}$ and $td[L:\mathbb{Q}] \geq td[K:\mathbb{Q}] = nd$. Thus s is a generic configuration that is congruent to p, so p is quasi-generic. •

Corollary 2.2.4. *Suppose that (G,p) is a rigid generic framework and that (G,q) is equivalent to (G,p). Then (G,q) is quasi-generic.*

Proof: Let H be a minimally rigid spanning subgraph of G. Choose isometries of $\mathbb{R}^d$ which map (H,p) and (H,q) to two frameworks (H,p') and (H,q') in standard position. By Lemma 2.2.1, $f_H(p)$ is generic. Thus $f_H(q') = f_H(p') = f_H(p)$ is generic. By Lemmas 2.2.2 and 2.2.3, (H,q') is quasi-generic. Hence (H,q) and (G,q) are quasi-generic. •

2.3 1-extensions and globally linked pairs

Let (G,p) be a d-dimensional framework and $u,v \in V$. Recall that $\{u,v\}$ is *globally linked* in (G,p) if, in all equivalent frameworks (G,q), we have $||p(u)-p(v)|| = ||q(u)-q(v)||$. The pair $\{u,v\}$ is *globally linked* in G in $\mathbb{R}^d$ if it is globally linked in all generic frameworks (G,p). Note that, if G is a rigid graph in $\mathbb{R}^d$, then Corollary 2.2.4 implies that a pair of vertices $\{u,v\}$ is globally linked in G in $\mathbb{R}^d$ if and only if we have $||p(u)-p(v)|| = ||q(u)-q(v)||$ for all equivalent pairs of quasi-generic frameworks (G,p) and (G,q). In this section we show that global linkedness and global rigidity is preserved by the d-dimensional 1-extension operation. By relabeling the vertices if necessary we may suppose that all 1-extensions will be on edge v_1v_2 and vertices $v_1,\ldots,v_{d+1}$. For $v \in V(G)$ let $N_G(v)$ denote the set of vertices adjacent to vertex v in graph G.

Lemma 2.3.1. *Let G be a graph, and $v \in V(G)$ with $N_G(v) = \{v_1,\ldots,v_{d+1}\}$. If $G-v$ is rigid in $\mathbb{R}^d$ then $\{v_1,v_2\}$ is globally linked in G in $\mathbb{R}^d$.*

Proof: Let (G,p^*) and (G,q^*) be equivalent quasi-generic frameworks. By Lemma

2.2.3, (G, p^*) is congruent to a framework (G, p), where

$$\begin{aligned} p(v_1) &= (0, 0, \dots, 0) \\ p(v_2) &= (p_{21}, 0, \dots, 0) \\ &\vdots \\ p(v_{d+1}) &= (p_{d+1,1}, \dots, p_{d+1,d}) \\ p(v) &= (p_{n1}, \dots, p_{nd}) \end{aligned}$$

and $\{p_{21}, \dots, p_{d+1,d}, \dots, p_{n1}, \dots, p_{nd}\}$ is algebraically independent over $\mathbb{Q}$. Similarly (G, q^*) is congruent to a framework (G, q), where

$$\begin{aligned} q(v_1) &= (0, 0, \dots, 0) \\ q(v_2) &= (q_{21}, 0, \dots, 0) \\ &\vdots \\ q(v_{d+1}) &= (q_{d+1,1}, \dots, q_{d+1,d}) \\ q(v) &= (q_{n1}, \dots, q_{nd}) \end{aligned}$$

and $\{q_{21}, \dots, q_{d+1,d}, \dots, q_{n1}, \dots, q_{nd}\}$ is algebraically independent over $\mathbb{Q}$. Then

$$||p^*(v_1)-p^*(v_2)||^2-||q^*(v_1)-q^*(v_2)||^2 = ||p(v_1)-p(v_2)||^2-||q(v_1)-q(v_2)||^2 = p_{21}^2-q_{21}^2.$$

Hence it will suffice to show that $p_{21}^2 - q_{21}^2 = 0$. By symmetry we may suppose that $p_{21}^2 - q_{21}^2 \geq 0$.

Let $p' = p|_{V-v}$ and $q' = q|_{V-v}$. Consider the equivalent quasi-generic frameworks $(G - v, p')$ and $(G - v, q')$. Applying Lemmas 2.2.1 and 2.2.2 to a minimally rigid spanning subgraph of $G - v$, we have $\overline{K} = \overline{L}$ where $K = \mathbb{Q}(p')$ and $L = \mathbb{Q}(q')$. Thus $q_{21}, \dots, q_{d+1,d} \in \overline{K}$. Since (G, q) is equivalent to (G, p), we have the following equations.

$$\begin{aligned} p_{n1}^2 + p_{n2}^2 + \ldots + p_{nd}^2 &= q_{n1}^2 + q_{n2}^2 + \ldots + q_{nd}^2 \\ (p_{n1} - p_{21})^2 + p_{n2}^2 + \ldots + p_{nd}^2 &= (q_{n1} - q_{21})^2 + q_{n2}^2 + \ldots + q_{nd}^2 \\ &\vdots \\ (p_{n1} - p_{d+1,1})^2 + \ldots + (p_{nd} - p_{d+1,d})^2 &= (q_{n1} - q_{d+1,1})^2 + \ldots + (q_{nd} - q_{d+1,d})^2 \end{aligned}$$

Subtracting from each equation the first one, we get

$$\begin{aligned}
q_{n1} &= \frac{p_{21}}{q_{21}}p_{n1} + \frac{q_{21}^2 - p_{21}^2}{2q_{21}} \\
q_{n2} &= \frac{p_{31}}{q_{32}}p_{n1} + \frac{p_{32}}{q_{32}}p_{n2} - \frac{q_{31}}{q_{32}}q_{n1} + \frac{q_{31}^2 - p_{31}^2 + q_{32}^2 - p_{32}^2}{2q_{32}} \\
&\vdots \\
q_{nd} &= \sum_{i=1}^{d} \frac{p_{d+1,i}}{q_{d+1,d}}p_{ni} - \sum_{i=1}^{d-1} \frac{q_{d+1,i}}{q_{d+1,d}}q_{ni} + \sum_{i=1}^{d} \frac{q_{d+1,i}^2 - p_{d+1,i}^2}{q_{d+1,d}}
\end{aligned}$$

The variables $q_{n1}, \ldots, q_{n,d}$ can be recursively eliminated from the right hand side of these equations, giving the following simple formula

$$q_n = Ap_n + b$$

where $A \in \overline{K}^{d\times d}$ is a lower triangular matrix and $b \in \overline{K}^d$. Using the equation $q_n^\top q_n = p_n^\top p_n$, we obtain

$$p_n^\top (A^\top A - I)p_n + 2b^\top Ap_n + b^\top b = 0.$$

This means that there is a polynomial

$$f = z^\top (A^\top A - I)z + 2b^\top Az + b^\top b \in \overline{K}[z_1, \ldots, z_d]$$

such that $f(p_{n1}, \ldots, p_{nd}) = 0$. Since $\{p_{21}, \ldots, p_{d+1,d}, \ldots, p_{n1}, \ldots, p_{nd}\}$ is algebraically independent over $\mathbb{Q}$, $\{p_{n1}, \ldots, p_{nd}\}$ is algebraically independent over $\overline{K}$. Thus $f \equiv 0$. In particular, the coefficient of z_1^2 is zero, giving us

$$a_{11}^2 + a_{21}^2 + \ldots + a_{d1}^2 - 1 = 0$$

. Since $a_{11}^2 = p_{21}^2/q_{21}^2 \geq 1$, we must have $a_{11}^2 = 1$ and thus $p_{21}^2 - q_{21}^2 = 0$. •

Theorem 2.3.2. *Let $G = (V, E)$ be a graph, $v_1, v_2, \ldots, v_{d+1}, v \in V$, with $N_G(v) = \{v_1, v_2, \ldots, v_{d+1}\}$, $v_1v_2 \notin E$ and $H = G - v + v_1v_2$. Suppose that $H - v_1v_2$ is rigid in $\mathbb{R}^d$ and that $\{x, y\}$ is globally linked in H in $\mathbb{R}^d$. Then $\{x, y\}$ is globally linked in G in $\mathbb{R}^d$.*

Proof: Suppose (G,p) is a generic framework and that (G,q) is equivalent to (G,p). Let $p' = p|_{V-v}$ and $q' = q|_{V-v}$. Since $G - v = H - v_1v_2$ is rigid, Lemma 2.3.1 implies that $\{v_1, v_2\}$ is globally linked in G. Thus

$$||p'(v_1) - p'(v_2)|| = ||p(v_1) - p(v_2)|| = ||q(v_1) - q(v_2)|| = ||q'(v_1) - q'(v_2)||.$$

Hence (H, p') and (H, q') are equivalent. Since $\{x, y\}$ is globally linked in H, we have

$$||p(x) - p(y)|| = ||p'(x) - p'(y)|| = ||q'(x) - q'(y)|| = ||q(x) - q(y)||.$$

Thus $\{x, y\}$ is globally linked in G. •

Corollary 2.3.3. *Suppose that H is globally rigid with $|V(H)| \geq d+2$ and G is obtained from H by a d-dimensional 1-extension. Then G is globally rigid.*

Proof: Let $H = G - v + v_1v_2$. Since H is globally rigid, $H - e$ is rigid for all edges e of H by Theorem 1.3.1. Hence $H - v_1v_2$ is rigid. Theorem 2.3.2 and the fact that H is globally rigid now imply that all pairs $\{x, y\} \subseteq V - v$ are globally linked in G. Suppose (G,p) is a generic framework and that (G,q) is equivalent to (G,p). Let $p' = p|_{V-v}$ and $q' = q|_{V-v}$. Since all pairs $\{x, y\} \subseteq V - v$ are globally linked in G, $(G - v, p')$ is congruent to $(G - v, q')$. Since (G,p) is generic and v has $d+1$ neighbors in G, this congruence extends to a congruence between (G,p) and (G,q). •

2.4 Globally linked pairs in M-connected graphs

In the rest of the chapter we will consider the $d = 2$ case.

Given a graph $G = (V, E)$, a subgraph $H = (W, C)$ is said to be an *M-circuit* in G if C is a circuit (i.e. a minimal dependent set) in $\mathcal{R}(G)$. In particular, G is an *M-circuit* if E is a circuit in $\mathcal{R}(G)$. For example, K_4, $K_{3,3}$ plus an edge, and $K_{3,4}$ are all M-circuits. Using Theorem 1.5.3 we may deduce that G is an M-circuit if and only if $|E| = 2|V| - 2$ and $G - e$ is minimally rigid for all $e \in E$. Recall that a graph G is *redundantly rigid* if $G - e$ is rigid for all $e \in E$. Note also that a graph G is redundantly rigid if and only if G is rigid and each edge of G belongs to a circuit in $\mathcal{R}(G)$ i.e. an M-circuit of G.

Any two maximal redundantly rigid subgraphs of a graph $G = (V, E)$ can have at most one vertex in common, and hence are edge-disjoint (see [25]). Defining a *redundantly rigid component* of G to be either a maximal redundantly rigid subgraph of G, or a subgraph induced by an edge which belongs to no M-circuit of G, we deduce that the redundantly rigid components of G partition E. Since each redundantly rigid component is rigid, this partition is a refinement of the partition of E given by the rigid components of G. Note that the redundantly rigid components of G are induced subgraphs of G.

Given a matroid $\mathcal{M} = (E, \mathcal{I})$, we define a relation on E by saying that $e, f \in E$ are related if $e = f$ or if there is a circuit C in $\mathcal{M}$ with $e, f \in C$. It is well-known that this is an equivalence relation. The equivalence classes are called the *components* of $\mathcal{M}$. If $\mathcal{M}$ has at least two elements and only one component then $\mathcal{M}$ is said to be *connected*.

We say that a graph $G = (V, E)$ is *M-connected* if $\mathcal{R}(G)$ is connected. Thus M-circuits are special M-connected graphs. Another example is the complete bipartite graph $K_{3,m}$, which is M-connected for all $m \geq 4$. The *M-components* of G are the subgraphs of G induced by the components of $\mathcal{R}(G)$. Note that the M-components of G are induced subgraphs. For more examples and basic properties of M-circuits and M-connected graphs see [4, 25]. In this chapter we will need the following lemmas.

We say that a graph G is *nearly 3-connected* if G can be made 3-connected by adding at most one new edge. We need the following result on M-connected graphs. The first part appears as [25, Lemma 3.1]. The second part was proved in [25, Theorem 3.2] for redundantly rigid graphs. The same proof goes through under the weaker hypothesis that each edge of G is in an M-circuit.

Theorem 2.4.1. *[25] (a) If G is M-connected then G is redundantly rigid.*
(b) If G is nearly 3-connected and each edge of G is in an M-circuit then G is M-connected.

Note that Theorems 1.5.6 and 2.4.1 imply that globally rigid graphs are M-connected and 3-connected M-connected graphs are globally rigid.

Next we will state two preliminary results about 2-sums that will be needed in the characterization of globally linked pairs in M-connected graphs.

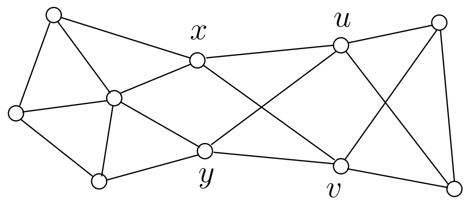

Figure 2.3: An M-circuit G obtained from a 'wheel' on six vertices and two copies of K_4 by taking 2-sums. The identified pairs of vertices, $\{u, v\}$ and $\{x, y\}$, are globally linked in G.

Lemma 2.4.2. *Suppose G_1 and G_2 are graphs and $G = G_1 \oplus_2 G_2$.*
(a) [4, Lemma 4.1] If G_1 and G_2 are M-circuits then G is an M-circuit.
(b) [25, Lemma 3.3] If G_1 and G_2 are M-connected then G is M-connected.

A *j-separation* of a graph $H = (V, E)$ is a pair (H_1, H_2) of edge-disjoint subgraphs of H each with at least $j+1$ vertices such that $H = H_1 \cup H_2$ and $|V(H_1) \cap V(H_2)| = j$. Note that H is 3-connected if and only if H has at least 4 vertices and has no j-separation for all $0 \leq j \leq 2$. If (H_1, H_2) is a 2-separation of H, then we say that $V(H_1) \cap V(H_2)$ is a *2-separator* of H.

Let $G = (V, E)$ be a 2-connected graph and suppose that (H_1, H_2) is a 2-separation of G with $V(H_1) \cap V(H_2) = \{u, v\}$. For $1 \leq i \leq 2$, let $H_i' = H_i + uv$ if $uv \notin E(H_i)$ and otherwise put $H_i' = H_i$. We say that H_1', H_2' are the *cleavage graphs* obtained by *cleaving G along $\{u, v\}$*.

Lemma 2.4.3. *Suppose G is a 2-connected graph and G_1 and G_2 are cleavage graphs obtained by cleaving G along a 2-separator $\{u, v\}$.*
(a) [4, Lemmas 2.4(c), 4.2] If G is an M-circuit then $uv \notin E(G)$, and G_1 and G_2 are both M-circuits.
(b) [25, Lemma 3.4] If G is M-connected then G_1 and G_2 are also M-connected.

We can use Theorem 2.3.2 to characterize globally linked pairs in M-connected graphs. First we need some preliminary lemmas, illustrated by Figure 2.3.

Lemma 2.4.4. *Let G_1, G_2 be M-circuits such that G_1 is 3-connected. Let $G = G_1 \oplus_2 G_2$, where the pair of identified vertices is $\{x, y\}$. Then $\{x, y\}$ is globally linked in G.*

Proof: We will use induction on $|V(G_1)|$. Suppose that the 2-sum was obtained along the edges $x_i, y_i \in E(G_i)$, $1 \le i \le 2$. If $G_1 = K_4$, with $V(G_1) = \{v, t, x_1, y_1\}$, then $G - v = G_2 - x_2y_2 + t + \{tx_2, ty_2\}$. Since G_2 is redundantly rigid by Lemma 2.4.1(a), $G_2 - x_2y_2$, and hence also $G - v$, are rigid. By Lemma 2.3.1, $\{x, y\}$ is globally linked in G. Thus we may suppose that $|V(G_1)| \ge 5$.

By [4, Theorem 5.9] there is $v \in V(G_1) - \{x_1, y_1\}$, with $N(v) = \{u, w, t\}$, such that $G_1^v = G - v + uw$ is a 3-connected M-circuit. Let $H = G_1^v \oplus_2 G_2$ be the 2-sum along the edge pair x_1y_1, x_2y_2. Then H is an M-circuit by Lemma 2.4.2(a), and hence, by induction, $\{x, y\}$ is globally linked in H. Since H is an M-circuit, $H - uw$ is rigid. Hence by Theorem 2.3.2, $\{x, y\}$ is globally linked in G. •

Corollary 2.4.5. *Let G be an M-circuit and $\{u, v\}$ be a 2-separator of G. Then $\{u, v\}$ is globally linked in G.*

Proof: We will use induction on $|V(G)|$. Since G is an M-circuit and is not 3-connected, we can choose a 2-separator $\{x, y\}$ in G and express G as $G = G_1 \oplus_2 G_2$, where the pair of identified vertices is $\{x, y\}$. Suppose that this 2-sum was obtained along the edges $x_i, y_i \in E(G_i)$, $1 \le i \le 2$. By Lemma 2.4.3(a), $xy \notin E(G)$ and G_1, G_2 are M-circuits. By choosing $\{x, y\}$ so that G_1 is minimal, we may also ensure that G_1 is 3-connected. By Lemma 2.4.4, $\{x, y\}$ is globally linked in G. Thus we may suppose that $\{u, v\} \ne \{x, y\}$. Since G_1 is 3-connected, $\{u, v\}$ is a 2-separator of G_2. By induction, $\{u, v\}$ is globally linked in G_2. Since $\{x, y\}$ is globally linked in G and $(G_2 - x_2y_2) \subseteq G$, it follows that $\{u, v\}$ is also globally linked in G. •

Let $H = (V, E)$ be a graph and $x, y \in V$. We will use $\kappa_H(x, y)$ to denote the maximum number of pairwise openly disjoint xy-paths in H. If $xy \notin E$ then, by Menger's theorem, $\kappa_H(x, y)$ is equal to the size of a smallest set $S \subseteq V(H) - \{x, y\}$ for which there is no xy-path in $H - S$.

Lemma 2.4.6. *Let (G, p) be a generic framework, $x, y \in V(G)$, $xy \notin E(G)$, and suppose that $\kappa_G(x, y) \le 2$. Then $\{x, y\}$ is not globally linked in (G, p).*

Proof: Since there do not exist three pairwise openly disjoint xy-paths in G, it follows from Menger's theorem that there exists $u, v \in V(G)$ such that x and y belong to different components of $G - \{u, v\}$. Let H be the component of $G - \{u, v\}$

which contains x. Construct (G,q) from (G,p) by reflecting $p(V(H))$ in the line through $p(u),p(v)$. Then (G,p) is equivalent to (G,q). Furthermore $||p(x)-p(y)|| \neq ||q(x)-q(y)||$, since $p(y)=q(y)$ and, since (G,p) is generic, $p(y)$ does not lie on the line through $p(u),p(v)$. Thus $\{x,y\}$ is not globally linked in (G,p). •

Theorem 2.4.7. *Let $G=(V,E)$ be an M-connected graph and $x,y \in V$. Then $\{x,y\}$ is globally linked in G if and only if $\kappa_G(x,y) \geq 3$.*

Proof: We first prove necessity. Suppose that $\{x,y\}$ is globally linked. If $xy \notin E$ then the existence of three openly disjoint xy-paths follows from Lemma 2.4.6. If $xy \in E$ then, since G is M-connected, $G-xy$ is rigid by Theorem 2.4.1(a). Since rigid graphs are 2-connected, we have two openly disjoint xy-paths in $G-xy$. Thus we have three openly disjoint xy-paths in G.

We next prove sufficiency. Suppose that there exist three pairwise openly disjoint xy-paths in G. We will use induction on $|V(G)|$ to show that $\{x,y\}$ is globally linked in G. If G is 3-connected then G is globally rigid by Theorems 1.5.6 and 2.4.1(a), and hence $\{x,y\}$ is globally linked in G. Thus we may suppose that $G-\{u,v\}$ is disconnected for some $u,v \in V$. Choose two vertices w,z belonging to different components of $G-\{u,v\}$. Since G is M-connected, there exists an M-circuit H in G with $w,x \in V(H)$. Then $\{u,v\}$ is a 2-separator of H. By Corollary 2.4.5, $\{u,v\}$ is globally linked in H. Thus $\{u,v\}$ is globally linked in G.

Let G_1,G_2 be the cleavage graphs obtained by cleaving G along the 2-separator $\{u,v\}$. The graphs G_1,G_2 are both M-connected by Lemma 2.4.3(b). Using the fact that there are three pairwise openly disjoint xy-paths in G, and relabeling if necessary, we have $x,y \in V(G_1)$. It is easy to see that there are three pairwise openly disjoint xy-paths in G_1. By induction $\{x,y\}$ is globally linked in G_1. Since $\{u,v\}$ is globally linked in G and $(G_1-u_1v_1) \subseteq G$, $\{x,y\}$ is also globally linked in G. •

2.5 Globally loose pairs

We say that a pair of vertices $\{u,v\}$ is *globally loose* in a graph G if for every generic framework (G,p) there exists an equivalent framework (G,q) such that $||p(u)-$

$p(v)|| \neq ||q(u) - q(v)||$. It follows from Lemma 2.4.6 and Theorem 2.4.7 that if G is M-connected then each pair $\{u, v\}$ is either globally linked or globally loose in G, and that $\{u, v\}$ is globally loose if and only if $\kappa_G(u, v) = 2$. On the other hand, the pair $\{u, v\}$ in the rigid graph given in Figure 2.1 is neither globally linked nor globally loose.

We will obtain a sufficient condition for a pair $\{u, v\}$ to be globally loose in a graph G. An edge e of a globally rigid graph H is *critical* if $H - e$ is not globally rigid.

Theorem 2.5.1. *Let $G = (V, E)$ be a graph and $u, v \in V$. Suppose that $uv \notin E$, and that G has a globally rigid supergraph H in which uv is a critical edge. Then $\{u, v\}$ is globally loose in G.*

Proof: Let (G, p) be a generic framework and let H be a globally rigid supergraph of G in which uv is critical. Since uv is critical in H, it follows that $(H - uv, p)$ is not globally rigid. Thus there is an equivalent, but not congruent realization $(H - uv, q)$. Clearly, $||p(u) - p(v)|| \neq ||q(u) - q(v)||$ must hold. Now G is a subgraph of $H - uv$, and hence the framework (G, q) verifies that $\{u, v\}$ is globally loose in G. •

We call a minimally rigid graph G *special* if every proper rigid subgraph H of G is complete (and hence is a complete graph on two or three vertices). The graphs $K_{3,3}$ and the prism are both special, as well as all graphs which can be obtained from $K_{3,3}$ by the following operation: replace two incident edges ab, bc by six edges $aa', a'b, bc', c'c, ac', a'c$, where a', c' are new vertices. Thus this family is infinite. It is easy to show that special graphs are 3-connected. It follows from the definition that if G is special and $uv \notin E(G)$ then $G + uv$ is a 3-connected M-circuit. Thus $G + uv$ is globally rigid by Lemma 2.4.1(a) and Theorem 1.5.6, and uv is critical in $G + uv$. Hence Theorem 2.5.1 implies that each pair of vertices in a special graph is either globally linked or globally loose:

Theorem 2.5.2. *Let $G = (V, E)$ be special and suppose that $u, v \in V$. Then $\{u, v\}$ is globally loose in G if and only if $uv \notin E$.*

2.6 Globally linked pairs in minimally rigid graphs

Theorem 2.5.2 implies that a pair $\{u, v\}$ is globally linked in a special graph $G = (V, E)$ if and only if $uv \in E$. In this section we will prove that this holds for all minimally rigid graphs. As a first step we show that the non-globally-linked relation is preserved by the 0-extension operation.

Lemma 2.6.1. *Let $H = (V, E)$ be a graph and $u, v \in V$. Suppose that $\{u, v\}$ is not globally linked in H and that G is a 0-extension of H. Then $\{u, v\}$ is not globally linked in G.*

Proof: Since $\{u, v\}$ is not globally linked in H, there exists a generic framework (H, p), and an equivalent framework (H, q), such that $||p(u)-p(v)|| \neq ||q(u)-q(v)||$. Let G be obtained from H by adding vertex w and edges wx, wy. Let α_1, α_2 be two real numbers such that the set containing α_1, α_2, and the edge lengths in $f_H(p)$ is algebraically independent over $\mathbb{Q}$, and such that $\alpha_1 + \alpha_2$ is large enough and $\alpha_1 - \alpha_2$ is small enough. (Note that $f_H(p)$ is generic by Lemma 2.2.1.) Now we may choose a pair of points r_p, r_q in $\mathbb{R}^2$ such that $||r_p - p(x)|| = \alpha_1 = ||r_q - q(x)||$ and $||r_p - p(y)|| = \alpha_2 = ||r_q - q(y)||$. Thus extending (H, p) by $p(w) = r_p$ and (H, q) by $q(w) = r_q$ gives a pair of equivalent frameworks on G such that $||p(u) - p(v)|| \neq ||q(u) - q(v)||$ holds. Note that (the extended) p is generic by Lemmas 2.2.2 and 2.2.3. •

To prove an analogous result for 1-extensions, we need the following key lemma about extending equivalent frameworks with a vertex of degree three.

Lemma 2.6.2. *Let $G = (V, E)$ be a graph and $v \in V$ with $N_G(v) = \{u, w, z\}$. Suppose that $(G-v, p)$ and $(G-v, q)$ are equivalent frameworks, where p is generic, $q(u)$, $q(w)$ and $q(z)$ are not collinear, and $||q(u)-q(w)|| \notin \overline{\mathbb{Q}(p)}$. Then there are equivalent frameworks (G, p^*) and (G, q^*) where p^* is generic, $p^*|_{V-v} = p$ and $q^*|_{V-v} = q$.*

Proof: Let $K = \overline{\mathbb{Q}(p)}$. The extension of the generic configuration p with a point $p_v \in \mathbb{R}^2$ is generic, if and only if p_v is a generic point over K. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be an isometry that takes $p(u)$ to the origin and $p(w)$ to a point on the first coordinate axis. This isometry has the form $Tz = Az + b$, where $A \in K^{2\times 2}$ and $b \in K^2$. With an argument similar to that in Lemma 2.1.1, it can be shown that p_v is generic over K if and only if $T(p_v)$ is generic over K.

Let p' be a configuration that is congruent to p and $p'(u) = (0, 0)$, $p'(w) = (p_3, 0)$ and $p'(z) = (p_4, p_5)$. Similarly, let q' be a configuration that is congruent to q and $q'(u) = (0, 0)$, $q'(w) = (q_3, 0)$ and $q'(z) = (q_4, q_5)$. Since $q(u)$, $q(w)$ and $q(z)$ are not collinear, we have that $q_5 \neq 0$. Moreover $q_3 = ||q(u) - q(w)|| \notin K$. By reflecting the configuration q' on the first coordinate axis, if necessary, we may assume that $q_5 \neq p_5$.

We call a point $(p_1, p_2) \in \mathbb{R}^2$ *feasible*, if there exists a point $(q_1, q_2) \in \mathbb{R}^2$, such that the extended frameworks (G, p') and (G, q') are equivalent, where $p'(v) = (p_1, p_2)$ and $q'(v) = (q_1, q_2)$. To prove the lemma, we have to show that there exists a feasible point that is generic over K.

The set of feasible points can be described by the following equations:

$$q_1^2 + q_2^2 = p_1^2 + p_2^2 \tag{2.1}$$

$$(q_1 - q_3)^2 + q_2^2 = (p_1 - p_3)^2 + p_2^2 \tag{2.2}$$

$$(q_1 - q_4)^2 + (q_2 - q_5)^2 = (p_1 - p_4)^2 + (p_2 - p_5)^2 \tag{2.3}$$

From equation (2.1) and (2.2) we get that

$$q_1 = \frac{q_3^2 - p_3^2 + 2p_1p_3}{2q_3},$$

and using equation (2.3) we get that

$$q_2 = \frac{q_4^2 + q_5^2 - p_4^2 - p_5^2 + 2p_1p_4 + 2p_2p_5 - q_4(\frac{q_3^2 - p_3^2 + 2p_1p_3}{q_3})}{2q_5}$$

And this (q_1, q_2) is a solution to the above equation system, if and only if equation (2.1) also holds, that is

$$4q_3^2q_5^2(q_1^2 + q_2^2 - p_1^2 - p_2^2) = a_{11}p_1^2 + a_{22}p_2^2 + a_{12}p_1p_2 + a_1p_1 + a_2p_2 + a_0 = 0.$$

This means, that there is an

$$f = a_{11}x_1^2 + a_{22}x_2^2 + a_{12}x_1x_2 + a_1x_1 + a_2x_2 + a_0 \in \mathbb{R}[x_1, x_2]$$

polynomial, such that (p_1, p_2) is feasible if and only if $f(p_1, p_2) = 0$. In other words, the feasible points are on a second-degree algebraic curve.

Let $r = q_4^2 - p_4^2$, and $s = q_5^2 - p_5^2$. The coefficients of f are the following:

$$\begin{aligned}
a_{11} &= 4q_5^2(p_3^2 - q_3^2) + 4(q_3p_4 - p_3q_4)^2\\
a_{22} &= 4q_3^2(p_5^2 - q_5^2)\\
a_{12} &= 8p_5q_3(q_3p_4 - p_3q_4)\\
a_1 &= 4(q_3p_4 - p_3q_4)(q_3(r+s) - q_4(q_3^2 - p_3^2)) + 4p_3q_5^2(q_3^2 - p_3^2)\\
a_2 &= 4p_5q_3(q_3(r+s) - q_4(q_3^2 - p_3^2))\\
a_0 &= (q_3(r+s) - q_4(q_3^2 - p_3^2))^2 + q_5^2(q_3^2 - p_3^2)^2
\end{aligned}$$

Since $q_3^2 \neq p_3^2$ and $q_5 \neq 0$ it follows that $a_0 > 0$.

Claim 2.6.3. *The algebraic curve defined by $f(x_1, x_2) = 0$ is not empty and it is not a single point.*

Proof: If $q_5^2 \neq p_5^2$, then the following two points $A, B \in \mathbb{R}^2$ are both on the curve.

$$A = \left(\frac{p_3 + q_3}{2}, \frac{r + s - (p_3 + q_3)(p_4 - q_4)}{2(p_5 - q_5)} \right)$$

$$B = \left(\frac{p_3 - q_3}{2}, \frac{r + s - (p_3 - q_3)(p_4 + q_4)}{2(p_5 + q_5)} \right)$$

Since $A \neq B$, in this case the proof is complete, so we may suppose that $q_5^2 = p_5^2$. Since $q_5 \neq p_5$, the point A belongs to the curve, so the curve is not empty, and since $a_{22} = 0$, it can not be a single point either. •

Let us suppose indirectly, that there is no point on the $f = 0$ curve that is generic over K. Since this curve is not empty, and is not a single point, it can be either an ellipse, a parabola, a hyperbola or the union of two lines. In either case the cardinality of the root-set of each irreducible component of f is continuum. Applying Lemma 2.1.3 to the irreducible components of f we get that there is $\lambda \neq 0$, such that $\lambda f \in K[x_1, x_2]$.

Claim 2.6.4. $q_5^2 = p_5^2$.

Proof: Consider the following two polynomials:

$$\begin{aligned} F(x) = {} & a_{12}^2(p_3^2 a_{12}^2 - 4a_2^2)x^3 + \\ & 8p_5^2 a_{22}\left[p_3^2 a_{11} a_{12}^2 + 2a_2^2(a_{22} - a_{11}) - 2a_{12}^2 a_0\right]x^2 + \\ & 16p_5^4 a_{22}^2\left[4(a_{22} - a_{11})a_0 + p_3^2 a_{11}^2 + a_2^2\right]x + \\ & 64p_5^6 a_{22}^3 a_0, \end{aligned}$$

$$G(x) = \left[4p_5^2 q_5^2 a_{22}^2 - 4sp_5^2 a_{11} a_{22} - s^2 a_{12}^2\right]x - 4p_3^2 p_5^2 q_5^2 a_{22}^2.$$

If $a_{22} \neq 0$ then the constant term of both F and G are non-zero and substituting all coefficients with their appropriate expressions we get that $F(q_5^2 - p_5^2) = 0$ and $G(q_3^2) = 0$. Since $\lambda f \in K[x_1, x_2]$ it follows that $\lambda^4 F \in K[x]$, $q_5 \in K$, $\lambda^2 G \in K[x]$ and finally $q_3 \in K$, which is a contradiction. This means that a_{22} must be zero, and thus $q_5^2 = p_5^2$. •

Claim 2.6.5. $a_{12} \neq 0$.

Proof: Consider the polynomial

$$F(x) = p_4(p_3 - p_4)a_{11}x - p_3^2 p_5 a_2.$$

If $q_3 p_4 - p_3 p_4 = 0$, then $a_{11} = 4p_5^2(p_3^2 - q_3^2) \neq 0$ and $F(q_3^2) = 0$. Since $\lambda F \in K[x]$ and $F \neq 0$ we get that $q_3 \in K$, which is a contradiction, so $q_3 p_4 - p_3 q_4 \neq 0$ and thus $a_{12} \neq 0$. •

Claim 2.6.6. *Either $q_4 \in K$ or there is $\mu \in K$ such that $q_4 = \mu q_3$.*

Proof: Since

$$\left[2p_5(a_1 + p_3 a_{11}) - p_4(2a_2 + p_3 a_{12})\right]q_3 + p_3(2a_2 + p_3 a_{12})q_4 = 0,$$

if $2a_2 + p_3 a_{12} \neq 0$, then there is $\mu \in K$ such that $q_4 = \mu q_3$. If, on the other hand

$$2a_2 + p_3 a_{23} = 8p_5 q_3(q_4^2 - q_3 q_4 - p_4^2 + p_3 p_4) = 0,$$

then $q_4^2 - q_3q_4 = p_4^2 - p_3p_4$. In this case q_4^2 is the root of the following polynomial:

$$\begin{aligned} F(x) = \ & \left[((p_3 - p_4)^2 - p_5^2)a_{12} + 2p_5(p_3 - p_4)a_{11}\right] x^2 + \\ & \left[(p_4(p_3 - p_4)(p_4^2 - p_5^2 - 2p_3p_4) + p_3^2p_5^2)a_{12} + 2p_4p_5(p_3 - p_4)^2 a_{11}\right] x + \\ & p_4^2(p_3 - p_4)^2((p_4^2 - p_5^2)a_{12} - 2p_4p_5a_{11}). \end{aligned}$$

Since $\lambda F \in K$, in order to show that $q_4 \in K$, we have to prove that $F \neq 0$. Let us suppose indirectly, that $F = 0$. In this case

$$\begin{aligned} ((p_3 - p_4)^2 - p_5^2)a_{12} + 2p_5(p_3 - p_4)a_{11} = 0, \\ (p_4^2 - p_5^2)a_{12} - 2p_4p_5a_{11} = 0. \end{aligned}$$

But since $a_{12} \neq 0$, the determinant of this linear equation, which is a non-zero polynomial of p, is zero, which is a contradiction. •

Now consider the following polynomial:

$$F(x, y) = \left[(p_4^2 - p_5^2)a_{12} - 2p_4p_5a_{11}\right] x^2 + 2p_3(p_5a_{11} - p_4a_{12})xy + p_3^2a_{12}y^2 + p_3^3p_5^2a_{12}.$$

Since $F(q_3, q_4) = 0$, if $q_4 \in K$, then q_3 is the root of $\lambda F(x, q_4) \in K[x]$, and the constant term of this polynomial is $p_3^2a_{12}(q_4^2 + p_5^2) \neq 0$. And if $q_3 = \mu q_4$ for some $\mu \in K$, then q_3 is the root of $\lambda F(x, \mu x) \in K[x]$, where the constant term is now $p_3^2p_5^2a_{12} \neq 0$. Either way we get that $q_3 \in K$, which is a contradiction that completes the proof. •

We can use this framework extension result to prove the converse of Lemma 2.3.1 for $d = 2$.

Theorem 2.6.7. *Let $H = (V, E)$ be a rigid graph and let G be a 1-extension of H on some edge $uw \in E$. Then $\{u, w\}$ is globally linked in G if and only if $H - uw$ is rigid.*

Proof: The "if" direction follows from Lemma 2.3.1, so we may suppose that $H - uw$ is not rigid.

Let (H, p) be a generic framework. Since (H, p) is infinitesimally rigid, but $(H - uw, p)$ is not infinitesimally rigid, there is an infinitesimal motion q of $(H - uw, p)$, such that

$$(p(u) - p(w))(q(u) - q(w)) \neq 0.$$

By Theorem 1.1.5 there is a smooth flexing $\pi : [-1, 1] \times V \to \mathbb{R}^2$ of the framework $(H - uw, p)$, such that $\dot{\pi} = q$.

Suppose that G is the 1-extension of H with a new vertex v such that $N_G(v) = \{u, w, z\}$. Since p is generic, $p(u)$, $p(w)$ and $p(z)$ are not collinear, and since π is continuous, $\pi(t, u)$, $\pi(t, w)$ and $\pi(t, z)$ are not collinear for all $0 < t < \varepsilon$, where ε is sufficiently small. Let

$$d(t) = ||\pi(t, u) - \pi(t, w)||^2.$$

Since $d'(0) \neq 0$, there is $0 < \mu < \varepsilon$ such that $d(\mu) \notin \overline{\mathbb{Q}(p)}$. Particulary,

$$||\pi(\mu, u) - \pi(\mu, w)|| \neq ||p(u) - p(w)||.$$

Applying Lemma 2.6.2 to $(G - v, p)$ and $(G - v, \pi(\mu))$ we can find equivalent frameworks (G, p^*) and (G, q^*) such that p^* is generic, $p^*|_{V-v} = p$ and $q^*|_{V-v} = \pi(\mu)$. From this we get that

$$||q^*(u) - q^*(w)|| = ||\pi(\mu, u) - \pi(\mu, w)|| \neq ||p(u) - p(w)|| = ||p^*(u) - p^*(w)||.$$

This means that $\{u, w\}$ is not globally linked in G. •

Theorem 2.6.8. *Let $H = (V, E)$ be a rigid graph and let G be a 1-extension of H on some edge $uw \in E$. Suppose that $H - uw$ is not rigid and that $\{x, y\}$ is not globally linked in H for some $x, y \in V$. Then $\{x, y\}$ is not globally linked in G.*

Proof: Since $\{x, y\}$ is not globally linked in H, there are equivalent frameworks (H, p_1) and (H, p_2) such that p_1 is generic and

$$||p_1(x) - p_1(y)|| \neq ||p_2(x) - p_2(y)||.$$

From Corollary 2.2.4 we get that the framework (H, p_2) is quasi-generic and therefore regular. Since (H, p_2) is infinitesimally rigid, but $(H - uw, p_2)$ is not infinitesimally rigid, there is an infinitesimal motion q of $(H - uw, p_2)$, such that

$$(p_2(u) - p_2(w))(q(u) - q(w)) \neq 0.$$

By Theorem 1.1.5 there is a smooth flexing $\pi : [-1, 1] \times V \to \mathbb{R}^2$ of the framework $(H - uw, p_2)$, such that $\dot{\pi} = q$.

Suppose that G is the 1-extension of H with a new vertex v such that $N_G(v) = \{u, w, z\}$. Since p_2 is quasi-generic, $p_2(u)$, $p_2(w)$ and $p_2(z)$ are not collinear, and since π is continuous, $\pi(t, u)$, $\pi(t, w)$ and $\pi(t, z)$ are not collinear for all $0 < t < \varepsilon$, where ε is sufficiently small. Also, there is $\nu < \varepsilon$ such that

$$||\pi(t, x) - \pi(t, y)|| \neq ||p_1(x) - p_1(y)||$$

for all $0 < t < \nu$. Let

$$d(t) = ||\pi(t, u) - \pi(t, w)||^2.$$

Since $d'(0) \neq 0$, there is $0 < \mu < \nu$ such that $d(\mu) \notin \overline{\mathbb{Q}(p_1)}$. Applying Lemma 2.6.2 to $(G-v, p_1)$ and $(G-v, \pi(\mu))$ we can find equivalent frameworks (G, p^*) and (G, q^*) such that p^* is generic, $p^*|_{V-v} = p_1$ and $q^*|_{V-v} = \pi(\mu)$. Therefore

$$||q^*(x) - q^*(y)|| = ||\pi(\mu, x) - \pi(\mu, y)|| \neq ||p_1(x) - p_1(y)|| = ||p^*(x) - p^*(y)||.$$

This means that $\{x, y\}$ is not globally linked in G. •

Theorem 2.6.9. *Let $G = (V, E)$ be a minimally rigid graph and suppose that $xy \notin E$. Then $\{x, y\}$ is not globally linked.*

Proof: The proof is by induction on $|V|$. The theorem is trivially true for $|V| \leq 3$, so we may assume that $|V| \geq 4$ and that the theorem holds for all minimally rigid graphs with at most $|V| - 1$ vertices. Since G is minimally rigid, it has either a vertex of degree two, or it has at least six vertices of degree three.

First suppose that G has a vertex v of degree two. If $v \in \{x, y\}$ then $\kappa_G(x, y) = 2$ and hence $\{x, y\}$ is not globally linked by Lemma 2.4.6. So suppose $v \neq x, y$ and consider $H = G - v$. H is also minimally rigid and $xy \notin E(H)$. By induction this implies that $\{x, y\}$ is not globally linked in H. Since G is a 0-extension of H, the theorem follows from Lemma 2.6.1.

If G has no vertex of degree two, then it has a vertex v of degree three, such that $v \neq x, y$. By Lemma 1.5.4 there are $u, w \in N_G(v)$, $uw \notin E$, such that $H = G - v + uw$ is also minimally rigid. If $xy \notin E(H)$, then we get by induction that $\{x, y\}$ is not globally linked in H. Since H is minimally rigid, $H - uw$ is not rigid and the theorem follows from Theorem 2.6.8. If $xy \in E(H)$, then $xy = uw$ and G is a 1-extension of H on xy. Since $H - xy$ is not rigid, the theorem follows from Theorem 2.6.7. •

Corollary 2.6.10. *Let G be a minimally rigid graph. Then $\{u, v\}$ is globally linked in G if and only if $uv \in E(G)$.*

2.7 Conjectures for the general case

Theorem 2.4.7 has the following immediate corollary.

Corollary 2.7.1. *Let $G = (V, E)$ be a graph and $x, y \in V$. If either $xy \in E$, or there is an M-component H of G with $\{x, y\} \subseteq V(H)$ and $\kappa_H(x, y) \geq 3$, then $\{x, y\}$ is globally linked in G.*

We conjecture that the converse is also true.

Conjecture 2.7.2. *The pair $\{x, y\}$ is globally linked in a graph $G = (V, E)$ if and only if either $xy \in E$ or there is an M-component H of G with $\{x, y\} \subseteq V(H)$ and $\kappa_H(x, y) \geq 3$.*

We have attempted to prove Conjecture 2.7.2 by considering two other conjectures on globally linked pairs which together are equivalent to Conjecture 2.7.2.

Conjecture 2.7.3. *Suppose that $\{x, y\}$ is a globally linked pair in a graph G. Then there is a redundantly rigid component R of G with $\{x, y\} \subseteq V(R)$.*

Conjecture 2.7.4. *Let G be a graph. Suppose that there is a redundantly rigid component R of G with $\{x, y\} \subseteq V(R)$ and $\{x, y\}$ is globally linked in G. Then $\{x, y\}$ is globally linked in R.*

It follows from Theorem 2.4.1(a) that Conjecture 2.7.2 implies both Conjectures 2.7.3 and 2.7.4.

The 'if' direction of Conjecture 2.7.2 follows from Corollary 2.7.1. We will prove that the 'only if' direction follows from Conjectures 2.7.3 and 2.7.4.

Proof: (of the 'only if' part of Conjecture 2.7.2 by assuming Conjectures 2.7.3 and 2.7.4 are true.) Suppose that $\{x, y\}$ is globally linked in $G = (V, E)$. We use induction on $|V|$ to show that either $xy \in E$ or there is an M-component H of G with $\{x, y\} \subseteq V(H)$ and $\kappa_H(x, y) \geq 3$. Since the statement is trivially true if $|V| \leq 3$, we may assume that $|V| \geq 4$ and that $xy \notin E$. It follows from the truth of Conjectures 2.7.3 and 2.7.4 that there is a redundantly rigid component R of G with $\{x, y\} \subseteq V(R)$ and such that $\{x, y\}$ is globally linked in R. This implies that $\kappa_R(x, y) \geq 3$ by Lemma 2.4.6. If R is 3-connected then R is M-connected by Theorem 2.4.1(b), and we are done by choosing $H = R$.

Now suppose that there is a 2-separator $\{u, v\}$ of R and let R_1, R_2 be the cleavage graphs obtained by cleaving R along $\{u, v\}$. Since $\kappa_R(x, y) \geq 3$, we may assume, without loss of generality, that $x, y \in V(R_1)$. Let us also suppose that the 2-separator has been chosen so that R_2 is inclusionwise minimal. This implies that R_2 is 3-connected. (Note that $|V(R_2)| \geq 4$, since R is redundantly rigid.)

Claim 2.7.5. *There is an M-circuit C in R_2 with $uv \in E(C)$.*

Proof: Since R is redundantly rigid, every edge $e \in E(R)$ belongs to an M-circuit C_e. Each M-circuit C' is a 2-connected subgraph of R. This fact and Lemma 2.4.3(a) imply that, if $C_e \not\subseteq R_2$ for some $e \in E(R_2) - uv$, then the claim will follow by choosing $C = (C_e \cap R_2) + uv$. Thus we may suppose that $C_e \subset R_2 - uv$ for all $e \in E(R_2) - uv$. Since R_2 is 3-connected, Theorem 2.4.1(b) implies that $R_2 - uv$ is M-connected, and hence rigid. Thus there is an M-circuit C in R_2 with $uv \in E(C)$. •

Since $\{x, y\}$ is globally linked in R, $\{u, v\}$ is a 2-seperation of R and $uv \in E(R_1)$, it follows that $\{x, y\}$ is globally linked in R_1. By induction, there is an M-connected subgraph H' of R_1 with $x, y \in V(H')$ and $\kappa_{H'}(x, y) \geq 3$. If $uv \notin E(H')$ then let H be an M-component of G containing H'. Thus we may suppose that $uv \in E(H')$. By Lemma 2.4.2(b), $H'' = H' \oplus_2 C$ is an M-connected subgraph of G containing x, y with $\kappa_{H''}(x, y) \geq 3$. The conjecture now follows by choosing an M-component H of G containing H''. •

Since the only M-components of a minimally rigid graph are subgraphs containing single edges, Theorem 2.6.9 implies that Conjecture 2.7.1 holds for minimally rigid graphs.

We close this section by noting that the M-components, and hence also the maximal globally rigid subgraphs, of a graph $G = (V, E)$ can be found in polynomial time, see [3] for details. Theorem 2.4.7 implies that one can identify even larger globally linked sets of vertices in G. A *globally rigid cluster* of G is a maximal subset of V in which all pairs of vertices are globally linked in G. By Theorem 2.7.1, the vertex sets of the 'cleavage units' (c.f. [25, Section 3]) of the M-components of G are globally linked sets in G. The truth of Conjecture 2.7.2 would imply that the vertex sets of these cleavage units are precisely the globally rigid clusters of G. For example, the maximal globally rigid subgraphs of the graph G in Figure 2.3 are the six copies of K_3 and the remaining four copies of K_2. On the other hand, G has three cleavage units, the copy of the wheel on six vertices and the two copies of K_4. The globally rigid clusters of G are precisely the vertex sets of these three cleavage units.

2.8 Uniquely localizable vertices

The theory of globally rigid graphs can be applied in localization problems of sensor networks, see for example [15]. In this section we consider another generalization of global rigidity, unique localizability, which also has direct applications in sensor network localization, see [19].

Let (G, p) be a generic framework with a designated set $P \subseteq V(G)$ of vertices. We say that a vertex $v \in V(G)$ is *uniquely localizable* in (G, p) with respect to P if whenever (G, q) is equivalent to (G, p) and $p(b) = q(b)$ for all vertices $b \in P$, then we also have $p(v) = q(v)$. We can think of P as the set of *pinned vertices* (or *anchor nodes* in a sensor network). Vertices in P are clearly uniquely localizable. It is easy to observe that if $v \in V - P$ is uniquely localizable then $|P| \geq 3$ and there exist three openly disjoint paths from v to P (c.f. Lemma 2.4.6). Note that unique localizablity is not a generic property. Consider the graph given in Figures 1 and 2. If we pin the set $P = \{u, x, y\}$ in the framework of Figure 1, then u is uniquely localizable with respect to P. This is not the case if we pin the same set in Figure 2. Thus the unique localizablity of v with respect to P depends on the lengths of the edges incident with w.

We call a vertex v *uniquely localizable* in graph G, with respect to $P \subseteq V(G)$, if v is uniquely localizable with respect to P in all generic frameworks (G, p). For a graph G and a set $P \subseteq V(G)$ let $G + K(P)$ denote the graph obtained from G by adding all edges bb' for which $bb' \notin E$ and $b, b' \in P$. The following lemma is easy to prove.

Lemma 2.8.1. *Let $G = (V, E)$ be a graph, $P \subseteq V$ and $v \in V - P$. Then v is uniquely localizable in G with respect to P if and only if $|P| \geq 3$ and $\{v, b\}$ is globally linked in $G + K(P)$ for all (or equivalently, for at least three) vertices $b \in P$.*

Lemma 2.8.1 and Theorem 2.4.7 imply the following characterization of uniquely localizable vertices when $G + K(P)$ is M-connected.

Corollary 2.8.2. *Let $G = (V, E)$ be a graph, $P \subseteq V$ and $v \in V - P$. Suppose that $G + K(P)$is M-connected. Then v is uniquely localizable in G with respect to P if and only if $|P| \geq 3$ and $\kappa(v, b) \geq 3$ for all $b \in P$.*

We note that for the problem of finding a vertex set $P \subseteq V$ with minimum number of vertices (or minimum total cost), such that $G + K(P)$ is M-connected, Jordán gave a $\frac{3}{2}$-approximation algorithm [27].

The truth of Conjecture 2.7.2 and Lemma 2.8.1 would imply the following characterization of uniquely localizable vertices in an arbitrary graph.

Conjecture 2.8.3. *Let $G = (V, E)$ be a graph, $P \subseteq V$ and $v \in V - P$. Then v is uniquely localizable in G with respect to P if and only if $|P| \geq 3$ and there is an M-component H of $G + K(P)$ with $P + v \subseteq V(H)$ and $\kappa_H(v, b) \geq 3$ for all $b \in P$.*

As noted in the previous section, the M-components of a graph can be found in polynomial time. More precisely, [3] gives an algorithm which determines the M-components of a graph $G = (V, E)$ in $O(|V|^2)$ time. We can also determine whether two vertices of G are joined by three openly disjoint paths in $O(|V| + |E|)$ time, see [31].

2.9 The number of equivalent realizations

The following folklore result is known to hold in $\mathbb{R}^d$. We include a proof for the 2-dimensional case for the sake of completeness.

Theorem 2.9.1. *Suppose that (G, p) is a rigid generic framework. Then the number of distinct congruence classes of frameworks which are equivalent to (G, p) is finite.*

Proof: Let $D = \sum_{uv \in E} ||p(u) - p(v)||$ and $B = \{x \in \mathbb{R}^{2n} : ||x|| \leq D\}$, where $n = |V|$. By Lemma 2.2.3 and Corollary 2.2.4, we can choose a representative

(G, q_i) for each congruence class, such that (G, q_i) is in standard position. Since G is connected, $q_i \in B$.

Suppose that there are infinitely many distinct congruence classes of (G, p). Since B is compact, we may choose a sequence of representatives (G, q_i) converging to a limit (G, q). Then (G, q) is equivalent to (G, p) and hence, by Corollary 2.2.4, (G, q) is quasi-generic. This contradicts the fact that (G, p), and hence (G, q), is rigid since the frameworks (G, q_i) are pairwise non-congruent. •

Given a rigid generic framework (G, p), let $h(G, p)$ denote the number of distinct congruence classes of frameworks which are equivalent to (G, p). Given a rigid graph G, let $h(G) = \max\{h(G, p)\}$, where the maximum is taken over all generic frameworks (G, p). The graph of Figure 1 shows that $h(G, p)$ need not be the same for all generic realizations (G, p) of a rigid graph G.

Borcea and Streinu [5] investigated the number of realizations of minimally rigid frameworks (G, p) with generic edge lengths. (Note that, by Lemmas 2.2.2 and 2.2.3, the edge lengths of (G, p) are generic if and only if there is a generic realization (G, q) with the same edge lengths as (G, p).) They counted the number of realizations up to rigid motions i.e. combinations of translations and rotations of the plane. This number is twice as large as $h(G, p)$ since reflections of the plane are not allowed. Their results imply that $h(G) \leq 4^n$ for all rigid graphs G. They also construct an infinite family of generic minimally rigid frameworks (G, p) for which $h(G, p)$ has order $12^{\frac{n}{3}} \sim (2.28)^n$.

We will determine the exact value of $h(G, p)$ for all generic realizations (G, p) of an M-connected graph $G = (V, E)$. For $u, v \in V$, let $b(u, v)$ denote the number of components of $G - \{u, v\}$ and put $c(G) = \sum_{u,v \in V}(b(u, v) - 1)$.

Theorem 2.9.2. *Let G be an M-connected graph. Then $h(G, p) = 2^{c(G)}$ for all generic realizations (G, p) of G.*

Proof: Choose a generic framework (G, p). We use induction on $c(G)$. If $c(G) = 0$ then G is 3-connected. It follows from Lemma 2.4.1(a) and Theorem 1.5.6 that G is globally rigid, and hence $h(G, p) = 1 = 2^{c(G)}$. Hence we may assume that there exists a 2-separation (G_1, G_2) in G with $V(G_1) \cap V(G_2) = \{u, v\}$. Let G_1 and G_2 denote the cleavage graphs obtained by cleaving G along $\{u, v\}$. Note that $uv \in E(G_i)$ and by, Lemma 2.4.3(b), G_i is M-connected, for $1 \leq i \leq 2$. Choosing the 2-separation so that G_1 is minimal, we also have that G_1 is 3-connected (c.f. [4, Lemma 2.8]) and, by [25, Lemma 3.6], $c(G_2) = c(G) - 1$.

By Theorem 2.4.7, $\{u, v\}$ is globally linked in G. Since G_1 is globally rigid by Theorem 1.5.6, each congruence class of (G, p) contains a unique framework (G, q) with $p(x) = q(x)$ for all $x \in V(G_1)$. Letting $p' = p|_{V(G_2)}$ and $q' = q|_{V(G_2)}$, we may deduce that the number of distinct congruence classes of (G, p) is equal to the number of distinct frameworks (G_2, q') which are equivalent to (G_2, p') and satisfy $q'(u) = p'(u)$ and $q'(v) = p'(v)$. The number of such frameworks is $2h(G_2, p')$, since each congruence class of (G_2, p') contains exactly two such frameworks (which can be obtained from each other by a reflection in the line through $p'(u), p'(v)$). By induction $h(G_2, p') = 2^{c(G)-1}$. Thus $h(G, p) = 2^{c(G)}$. •

It follows from the proof of the above theorem that, if (G, p) is a generic realization of an M-connected graph G, then we can obtain a representative of each distinct congruence class of frameworks which are equivalent to (G, p) by iteratively applying the following operation to (G, p): choose a 2-separation $\{u, v\}$ of G and reflect some, but not all, of the components of $G - \{u, v\}$ in the line through the points $p(u), p(v)$.

Theorem 2.9.2 implies that $h(G, p)$ is the same for all generic realizations of an M-connected graph G. Note that this statement becomes false if we replace the hypothesis that G is M-connected by the weaker hypothesis that G is redundantly rigid. An example is the redundantly rigid graph G obtained from the graph in Figure 2.1 by replacing each edge by a copy of K_4.

Theorem 2.9.2 also implies that $h(G) \leq 2^{\frac{n-2}{2}-1}$ for all M-connected graphs G. A family of graphs attaining this bound is a collection of K_4's joined along a common edge.

Chapter 3

Globally Rigid Frameworks

In this chapter we are concerned with the following algorithmic problem: given a graph G, how to create, in polynomial time, a globally rigid realization (G, p) in $\mathbb{R}^d$, if such a realization exists? We will develop an algorithm for the case when $d = 2$ and G is globally rigid.

One of the difficulties is due to the fact that the output of the algorithm, which is a realization of G with rational coordinates, is non-generic. However, there is no 'simple' sufficient condition for the global rigidity of a non-generic framework. As an additional illustration, consider the problem of constructing a rigid realization of a rigid graph G in $\mathbb{R}^d$. In this case infinitesimal rigidity turns out to be a 'simple' sufficient condition that is essentially expressed by the rank of the rigidity matrix. Based on this fact, it was shown that a rigid realization, even with integer coordinates in a small grid, can be found in polynomial time, see [16].

Another issue is the level of degeneracy of the framework (G, p) output by the algorithm. Since rather degenerate frameworks may be globally rigid (for example, if G is connected and all vertices are mapped to the same point), it is natural to impose certain additional requirements. It is known, see e.g. [14], that if (G, p) is a globally rigid and infinitesimally rigid framework then there exists an $\epsilon > 0$ such that if $||p(v) - q(v)|| < \epsilon$ for all $v \in V$ then (G, q) is also globally rigid. Thus infinitesimal rigidity makes the framework 'stable' in terms of global rigidity. Therefore it is natural to try to make (G, p) infinitesimally rigid as well. In [14], a framework is called *strongly rigid*, if it is both infinitesimally rigid and globally rigid.

If G is triangle-reducible, which is a subfamily of globally rigid graphs that includes Cauchy graphs as well as Grünbaum graphs, the constructed realization will also be infinitesimally rigid. Our algorithm is based on a sufficient condition for global rigidity which is based on stress matrices as well as an inductive construction of globally rigid graphs which uses the 1-extension operation.

3.1 Sufficient conditions for global rigidity of frameworks

The sufficient conditions known for the global rigidity of frameworks are in terms of stresses. Let $G = (V, E)$ be a graph, where V is the set of vertices labeled $1, 2, ..., n$, and let (G, p) be a framework. Recall that a *stress* of (G, p) is a map $\omega : E \to \mathbb{R}$ that satisfies (1.4) for all $v \in V$, and the *stress matrix* Ω associated with a stress ω is an n-by-n symmetric matrix defined by

$$\Omega_{ij} = \begin{cases} \sum\limits_{ki \in E} \omega_{ki} & \text{if } i = j \\ -\omega_{ij} & \text{if } i \neq j \text{ and } ij \in E \\ 0 & \text{if } i \neq j \text{ and } ij \notin E \end{cases}$$

It is easy to see that Ω is the stress matrix of a stress of framework (G, p) if and only if Ω is symmetric, $\Omega_{ij} = 0$ whenever $ij \notin E$ $(i \neq j)$, and $P\Omega = 0$, where

$$P = \begin{bmatrix} p_{11} & p_{21} & \cdots & p_{n1} \\ p_{12} & p_{22} & \cdots & p_{n2} \\ 1 & 1 & \dots & 1 \end{bmatrix}$$

is the *augmented configuration matrix* of p.

For completeness, we provide a proof of the following theorem, which can be extracted from [8] and [41]. We say that a framework (G, p) is *bidirectional* if there exist vectors $v_1, v_2 \in \mathbb{R}^2$ such that for each $ij \in E$ either $p_i - p_j = \lambda v_1$ or $p_i - p_j = \lambda v_2$ holds for some $\lambda \in \mathbb{R}$. Otherwise (G, p) is said to be *multidirectional*.

Theorem 3.1.1. *Let (G, p) be a multidirectional framework on n vertices for which there is a stress ω, such that the associated stress matrix Ω is positive semi-definite and has rank $n - 3$. Then (G, p) is globally rigid.*

Proof: Let $q = (q_{11}, q_{21}, \ldots, q_{n1}, q_{12}, q_{22}, \ldots, q_{n2}) \in \mathbb{R}^{2n}$ and let

$$H(q) = \sum_{ij \in E} \omega_{ij}(q_i - q_j)^2 = q\widehat{\Omega}q^\top$$

be a quadratic form, where $q_i = (q_{i1}, q_{i2})$ and

$$\widehat{\Omega} = \begin{bmatrix} \Omega & 0 \\ 0 & \Omega \end{bmatrix}$$

Since Ω is positive semi-definite, so is $\widehat{\Omega}$. Thus $H(q) \geq 0$ for all $q \in \mathbb{R}^{2n}$.

Claim 3.1.2. *If $\nabla H(q) = 0$, then $H(q) = 0$.*

Proof: Let $g(t) = H(tq) = t^2 H(q)$. Then $g'(t) = \nabla H(tq)q = t\nabla H(q)q = 0$. Hence $g(t)$ is constant and $H(q) = g(1) = g(0) = 0$. •

The gradient of this form at a point q can be written as

$$\nabla H(q) = 2\left(\sum_{1j \in E} \omega_{1j}(q_1 - q_j), \ldots, \sum_{nj \in E} \omega_{nj}(q_n - q_j)\right) = 2\widehat{\Omega}q.$$

Since ω is a stress for p, we have $\nabla H(p) = 0$. Thus $H(p) = 0$ by Claim 3.1.2.

Consider a framework (G, p') that is equivalent to (G, p). First we show that p' is an affine image of p. By definition, $H(p') = H(p) = 0$. Thus, since $H(q) \geq 0$, the point p' is a local minimum of H, and hence $\nabla H(p') = 2\widehat{\Omega}p' = 0$. Let us define the following two subspaces of $\mathbb{R}^{2n}$:

$$S_1 = \{q \in \mathbb{R}^{2n} \mid q_i = Ap_i + b,\ 1 \leq i \leq n,\ A \in \mathbb{R}^{2\times 2}, b \in \mathbb{R}^2\}$$

$$S_2 = \{q \in \mathbb{R}^{2n} \mid \widehat{\Omega}q = 0\} = \ker \widehat{\Omega}$$

It is clear that $\dim S_1 = 6$. Since $\operatorname{rank} \widehat{\Omega} = 2 \operatorname{rank} \Omega = 2n - 6$, this implies $\dim S_2 = \dim \ker \widehat{\Omega} = 6$. To prove that $S_1 = S_2$, it is enough to show that $S_1 \subseteq S_2$. To see this suppose that $q \in S_1$. Then

$$(\widehat{\Omega}q)_i = \sum_{ij \in E} \omega_{ij}(q_i - q_j) = \sum_{ij \in E} \omega_{ij}(Ap_i - Ap_j) = A\sum_{ij \in E} \omega_{ij}(p_i - p_j) = 0,$$

which gives $q \in S_2$. Since $p' \in S_2$, there exist $A \in \mathbb{R}^{2\times 2}$ and $b \in \mathbb{R}^2$, such that $p'_i = Ap_i + b$ for each $1 \leq i \leq n$. Thus p' is an affine image of p, as claimed.

Next we show that the affine map $x \mapsto Ax+b$ is a congruence. Let $C = I - A^\top A$. Since (G, p') is equivalent to (G, p), we have

$$(p'_i-p'_j)^2 = (p'_i-p'_j)^\top(p'_i-p'_j) = (p_i-p_j)^\top A^\top A(p_i-p_j) = (p_i-p_j)^\top(p_i-p_j) = (p_i-p_j)^2$$

for each $ij \in E$. Hence $(p_i - p_j)^\top C(p_i - p_j) = 0$ for each $ij \in E$. Thus either the set $\{x \in \mathbb{R}^2 \,|\, x^\top Cx = 0\}$ is the union of two lines, or $C = 0$. In the former case (G, p) would be bidirectional, contradicting a hypothesis of the theorem. Hence we must have $C = 0$ and $A^\top A = I$, which implies that A is orthogonal and p' is congruent to p. •

We note that if a framework (G, p) satisfies the conditions of Theorem 3.1.1 then it is in fact *universally globally rigid*, which means that it is globally rigid in $\mathbb{R}^d$ for all $d \geq 2$. The proof of this fact can be found in unpublished work of Connelly [12]. Since we use Theorem 3.1.1 to verify the global rigidity of the frameworks output by our algorithm, it follows that the constructed frameworks are also universally globally rigid.

3.1.1 Gale transforms

Let (G, p) be a framework and suppose that the points in p affinely span $\mathbb{R}^2$. Let A be an $(n-3) \times n$ matrix with linearly independent rows, satisfying $AP^\top = 0$. Then we say that the columns of A, treated as points $a_1, \ldots, a_n \in \mathbb{R}^{n-3}$, form the *Gale transform* of the original points $p_1, \ldots, p_n \in \mathbb{R}^2$ [41]. We say that the four-tuple (G, p, ω, A) is a *Gale-framework* if (G, p) is a framework, $\omega : E \to \mathbb{R}$ is a map and $A = (a_1, \ldots, a_n) \in \mathbb{R}^{(n-3)\times n}$ is a Gale transform of p satisfying $a_i^\top a_j = -\omega_{ij}$ for all $ij \in E$ and $a_i^\top a_j = 0$ for all $i, j \in V$, $i \neq j$, $ij \notin E$. As we will see, $A^\top A$ is the positive semi-definite stress matrix of rank $n-3$ that we can use to certify the global rigidity of the framework (G, p).

For example, the following is a multidirectional Gale framework on K_4, given by its augmented configuration matrix P, A, and a stress ω, see Figure 3.2. (Note that ω is nowhere-zero and the framework is infinitesimally rigid and is in general position.)

$$P = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}$$

$$\omega_{12} = \omega_{23} = \omega_{34} = \omega_{14} = 1$$

$$\omega_{13} = \omega_{24} = -1$$

Lemma 3.1.3. *Let (G, p, ω, A) be a Gale-framework on n vertices. Then ω is a stress for (G, p) with a positive semi-definite stress matrix of rank $n - 3$.*

Proof: Let $\Omega = A^\top A$. By the definition of Gale-frameworks and the fact that $\Omega P^\top = A^\top A P^\top = A^\top 0 = 0$ we get that Ω is the stress matrix of ω and ω is a stress for (G, p). Ω has rank $n - 3$ since A has $n - 3$ independent rows and it is positive semi-definite since $q^\top \Omega q = q^\top A^\top A q = (Aq)^\top (Aq) \geq 0$ for all $q \in \mathbb{R}^n$. •

A Gale-framework is *multidirectional* if (G, p) is multidirectional. By Theorem 3.1.1 and Lemma 3.1.3 we obtain:

Theorem 3.1.4. *Let (G, p, ω, A) be a multidirectional Gale-framework. Then (G, p) is globally rigid.*

3.2 Extension of frameworks and Gale frameworks

In this section we describe the 'parameterized' version of the 1-extension operation which works on (Gale) frameworks and prove that if the parameters are chosen appropriately, they take a multidirectional Gale framework to a multidirectional Gale framework. We will also prove that when the 1-extension happens on the edge and the third vertex of a triangle, the parameters can be chosen so that the operation preserves infinitesimal rigidity, too.

Let (G, p) be a framework, let $uw \in E(G)$, $t \in V(G) - \{u, w\}$, and let $\alpha_u, \alpha_w, \alpha_t$ be real numbers with $\alpha_u + \alpha_w + \alpha_t = 1$. The *1-extension* operation on edge uw and vertex t *with parameters* $\alpha_u, \alpha_w, \alpha_t$ consists of performing a 1-extension on G

which adds a new vertex v, as well as extending the realization p by letting $p(v) = \alpha_u p(u) + \alpha_w p(w) + \alpha_t p(t)$.

Lemma 3.2.1. *Let (G, p) be a multidirectional framework and (G^*, p^*) its 1-extension with parameters $\alpha_u, \alpha_w, \alpha_t$. If $\alpha_t = 0$ or $\alpha_u\alpha_w \neq 0$, then (G^*, p^*) is multidirectional.*

Proof: If p_u, p_w, p_t are collinear or $\alpha_t = 0$, then the set of edge directions of (G^*, p^*) are the same as that of (G, p). Otherwise, p_u, p_w, p_t are affinely independent and $\alpha_u, \alpha_w, \alpha_t \neq 0$. In this case the edges vu, vw, vt define three independent directions, so (G^*, p^*) is multidirectional. •

Let (G, p, ω, A) be a Gale framework and let $\beta \neq 0$ be a real number. The *1-extension* operation on edge uw and vertex t *with parameters* $\alpha_u, \alpha_w, \alpha_t, \beta$ consists of performing a 1-extension of (G, p) with parameters $\alpha_u, \alpha_w, \alpha_t$ as defined above, as well as replacing ω and A by ω^* and A^* by letting

$$\omega^*_{ij} = \begin{cases} \omega_{ij} & \text{if } \ ij \in E - \{uw, ut, wt\} \\ \omega_{ij} - \beta^2\alpha_i\alpha_j & \text{if } \ ij \in \{ut, wt\} \\ \beta^2\alpha_j & \text{if } \ i = v \text{ and } j \in \{u, w, t\} \end{cases}$$

$$A^* = \begin{bmatrix} a_1 & \dots & a_u & a_w & a_t & \dots & a_n & 0 \\ 0 & \dots & \beta\alpha_u & \beta\alpha_w & \beta\alpha_t & \dots & 0 & -\beta \end{bmatrix}$$

Lemma 3.2.2. *Let (G, p, ω, A) be a Gale-framework and let $(G^*, p^*, \omega^*, A^*)$ be its 1-extension with parameters $\alpha_u, \alpha_w, \alpha_t, \beta$. If $\alpha_u\alpha_w = \omega_{uw}/\beta^2$, and if $\alpha_t = 0$ whenever $\{ut, wt\} \nsubseteq E$, then $(G^*, p^*, \omega^*, A^*)$ is a Gale-framework.*

Proof: Let a^*_i denote the columns of A^*, $1 \leq i \leq n+1$. It is easy to check that A^* is a Gale-transform of p^* and $a^{*\top}_i a^*_j = -\omega^*_{ij}$ if $ij \in E^*$. Let us suppose now that $ij \notin E^*$ for some $i, j \in V^*$, $i \neq j$. Then either $i = v$ and $j \notin V - \{u, w, t\}$, or $i \in \{u, w, t\}$ and $j \in V - \{u, w, t\}$ and $ij \notin E$, or $ij \in E - \{ut, wt\}$, or $ij = uw$. In the first case $a^{*\top}_i a^*_j = 0^\top a_j + \beta 0 = 0$. In the second case $a^{*\top}_i a^*_j = a_i^\top a_j + \beta\alpha_i 0 = a_i^\top a_j = 0$. In the third case $a^{*\top}_i a^*_j = a_i^\top a_t + \beta^2\alpha_i\alpha_t = 0$. In the last case $a^{*\top}_i a^*_j = a_u^\top a_w + \beta^2\alpha_u\alpha_w = -\omega_{uw} + \omega_{uw} = 0$. •

Our algorithm will create a globally rigid realization of G by iteratively constructing a multidirectional Gale framework on each graph in an inductive construction of G using edge additions and 1-extensions. We next describe how the parameters are defined when the algorithm applies edge addition or 1-extension to a Gale framework (G, p, ω, A).

- **Edge addition** In this case G^* is obtained from G by adding an edge uw. We define $(G^*, p^*, \omega^*, A^*)$ by letting $p^* = p$, $A^* = A$, $\omega^*_{ij} = \omega_{ij}$ if $ij \in E$ and $\omega^*_{uw} = 0$.

- **1-Extension**

  In this case G^* is obtained from G by a 1-extension on edge uw and vertex t. We define $(G^*, p^*, \omega^*, A^*)$ by defining the parameters $\beta, \alpha_u, \alpha_w, \alpha_t$. This will also determine $p(v)$. We consider three cases.

  Case 1 $\omega_{uw} = 0$.

  Let $\beta = 1$, $\alpha_t = 0$ and $\alpha_u = 0$ or $\alpha_w = 0$.

  (Note: It means that in this case $p(v) = p(w)$ or $p(v) = p(u)$.)

  Case 2 $\omega_{uw} \neq 0$ and $\{ut, wt\} \nsubseteq E$.

  Let $\alpha_t = 0$ and let α_u, α_w be chosen so that $\alpha_u\alpha_w$ has the same sign as ω_{uw}. Let $\beta^2 = \frac{\omega_{uw}}{\alpha_u\alpha_w}$.

  (Note: now $p(v) = \alpha_u p(u) + \alpha_w p(w)$ lies on the line of $p(u)p(w)$ or $p(v) = p(u) = p(w)$. If $\omega_{uw} > 0$ then $p(v)$ lies on the segment $[p(u), p(w)]$ (eg. $\alpha_u = \alpha_w = \frac{1}{2}$) and if $\omega_{uw} < 0$ then it lies in its complement (eg. $\alpha_u = 2$, $\alpha_w = -1$).)

  Case 3 $\omega_{uw} \neq 0$ and $\{ut, wt\} \subseteq E$.

  Let $\alpha_u, \alpha_w, \alpha_t$ be chosen so that $\alpha_u\alpha_w$ has the same sign as ω_{uw}, and so that $\alpha_t \notin \{0, \frac{\omega_{ut}}{\omega_{uw}}\alpha_w, \frac{\omega_{wt}}{\omega_{uw}}\alpha_u\}$. Let $\beta^2 = \frac{\omega_{uw}}{\alpha_u\alpha_w}$, see Figure 3.2.

  (Note: For example, if $\omega_{uw} > 0$ we can define $\alpha_u = \alpha_w = \alpha_t = \frac{1}{3}$ and if $\omega_{uw} < 0$ then let $\alpha_u = 3$, $\alpha_w = \alpha_t = -1$. Consider the case when $p(u), p(w), p(t)$ are not collinear. If $\omega_{uw} > 0$ then $p(v)$ can be placed anywhere in the angle $p(u)p(t)p(w)$ and in its mirror image to $p(t)$ (the lines $p(u)p(t)$ and $p(w)p(t)$

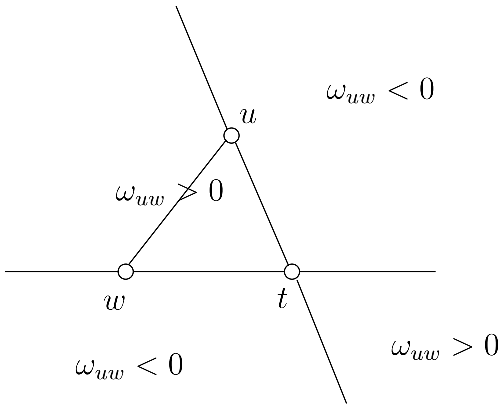

Figure 3.1: The possible placements of p_v in Case 3.

are excluded). Otherwise $p(v)$ can be in the two other angles defined by the lines through $p(u)p(t)$ and $p(w)p(t)$, but not on the lines themselves (see Figure 3.1). By excluding the three values for α_t we have excluded three lines.

Lemma 3.2.3. *Suppose that (G, p, ω, A) is a multidirectional Gale framework for which (G, p) is infinitesimally rigid, ω is nowhere-zero, and the points $p(v)$, $v \in V$, are in general position. Let $(G^*, p^*, \omega^*, A^*)$ be obtained from (G, p, ω, A) by a 1-extension as described in Case 3. Then $(G^*, p^*, \omega^*, A^*)$ is a multidirectional Gale framework, for which (G^*, p^*) is infinitesimally rigid and ω^* is nowhere-zero.*

Proof: By Lemmas 3.2.1 and 3.2.2 $(G^*, p^*, \omega^*, A^*)$ is a multidirectional Gale-framework. Since ω is nowhere-zero, we have $\omega_{uw} \neq 0$. Thus we must have $\alpha_u \neq 0$ and $\alpha_w \neq 0$. Hence $\omega^*_{vi} = \beta^2\alpha_i \neq 0$ for $i \in \{u, w, t\}$. Furthermore, the choice of α_t implies that $\omega^*_{ut} = \omega_{ut} - \beta^2\alpha_u\alpha_t = \omega_{ut} - \omega_{uw}\alpha_t/\alpha_w \neq 0$. Similarly, $\omega^*_{wt} \neq 0$. Thus ω^* is a nowhere-zero stress.

To show that (G^*, p^*) is infinitesimally rigid first observe that $(G - uw, p)$ is infinitesimally rigid, since ω is nowhere-zero. Moreover, the addition of the new point $p(v)$ preserves infinitesimal rigidity, since $p(u)$, $p(w)$ and $p(t)$ are in general position and $\alpha_t \neq 0$, so $p(v)$ is not on the line through $p(u)p(w)$ (see Lemma 1.4.1). •

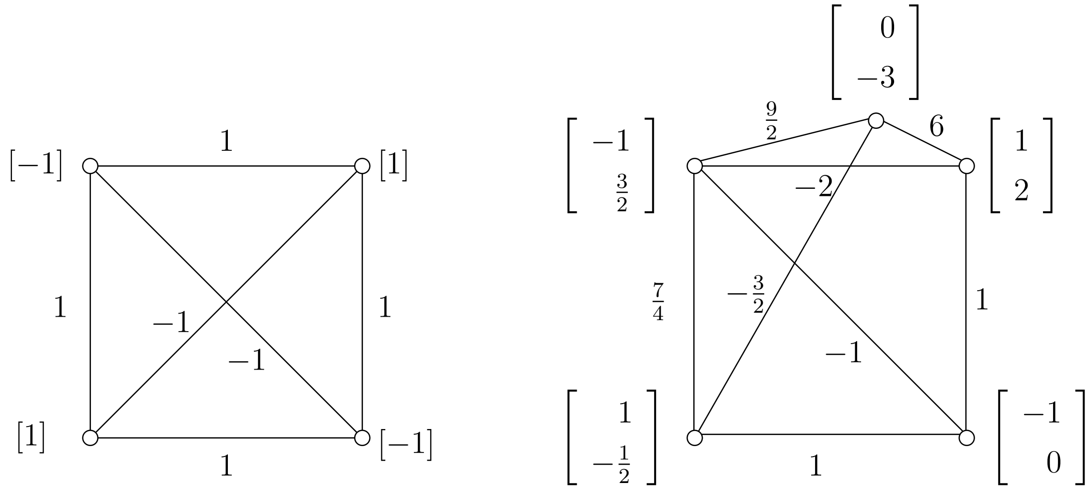

Figure 3.2: The first framework is a Gale framework on K_4 with $p(v_1) = (0,0)$, $p(v_2) = (1,0)$, $p(v_3) = (1,1)$ and $p(v_4) = (0,1)$. The second framework is the Gale framework obtained from the first one by a 1-extension on edge v_1v_3 and vertex v_4 with parameters $\frac{2}{3}$, $-\frac{1}{6}$, $\frac{1}{2}$, 3. The position of the new vertex is $(\frac{2}{3}, \frac{7}{6})$. The values of the (nowhere-zero) stress are shown on the corresponding edges and the column vectors of the Gale transform are indicated at the vertices. Both frameworks are multidirectional and infinitesimally rigid.

3.3 Globally rigid realizations

In this section we describe our algorithm which creates a globally rigid realization of a globally rigid graph G. The algorithm builds a multidirectional Gale framework on G inductively, following the local operations edge addition and 1-extension that we can use to construct G from K_4.

Given a graph $G = (V, E)$ we say that a 1-extension on the edge uw and vertex t is a *triangle-split* if $\{ut, wt\} \subseteq E$ (that is, if u, w, t induce a triangle of G). A graph will be called *triangle-reducible* if it can be obtained from K_4 by a sequence of triangle-splits. We note that triangle-reducible graphs are 3-connected redundantly rigid planar graphs with $2|V| - 2$ edges.

Theorem 3.3.1. *Let $G = (V, E)$ be a globally rigid graph on at least four vertices. Then one can construct, in polynomial time, a globally rigid realization (G, p), where $p(V)$ spans $\mathbb{R}^2$. Furthermore, if G is triangle-reducible, the constructed realization can be chosen to be infinitesimally rigid, too.*

Proof: Let $K_4 = H_1, H_2, ..., H_m = G$ be an inductive construction of G from K_4

using edge-additions and 1-extensions. Such a sequence exists by Theorem 1.5.5. Furthermore, if G is triangle-reducible, we may assume that H_{i+1} is obtained from H_i by a triangle-split, $1 \leq i \leq m-1$. These inductive constructions can be obtained in polynomial time, see [3] and Lemma 3.3.2 below.

Let $(H_1, p_1, \omega_1, A_1)$ be a multidirectional Gale framework on $H_1 = K_4$. If G is triangle-reducible, we choose one with a nowhere-zero stress and for which (H_1, p_1) is infinitesimally rigid and is in general position. The example in Subsection 3.1.1 satisfies all these conditions.

To compute a globally rigid framework on G we follow the inductive construction and perform edge additions and 1-extensions as described in Cases 1-3, to create multidirectional Gale-frameworks $(H_i, p_i, \omega_i, A_i)$ for $1 \leq i \leq m$. By Lemmas 3.2.1, 3.2.2, and Theorem 3.1.4, the framework (H_m, p_m) will be a globally rigid realization of G.

If, in addition, G is triangle-reducible, we only perform 1-extensions, as described in Case 3, with the additional property that the points in each framework (H_i, p_i), $1 \leq i \leq m$, are in general position. In this case Lemma 3.2.3 implies that (H_m, p_m) will also be infinitesimally rigid.

Observe that the algorithm does not need to compute the Gale transforms A_i but updates the stress and the realization. Without giving an explicit upper bound, we note that the numbers (the values of the stress and the coordinates of the vertices) occurring in the algorithm can always be chosen to be of polynomial size. •

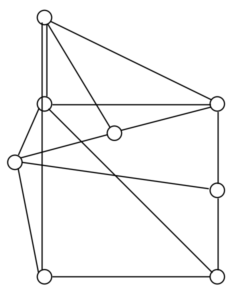

Figure 3.3: A globally rigid realization of a globally rigid graph produced by the algorithm.

We remark that even though the realization given by the algorithm will affinely span $\mathbb{R}^2$, it may be rather degenerate: the positions of several vertices may coincide and certain edges may have length zero. (For example, if a 1-extension is performed on an edge whose stress is zero, the position of the new vertex will be on one of the endpoints of the edge.) This can be overcome in the case of triangle-reducible graphs, for which our algorithm outputs an infinitesimally rigid realization. We believe that, possibly by using a different sufficient condition for global rigidity, it will be possible to obtain such 'non-degenerate' and 'stable' realizations for all globally rigid graphs.

3.3.1 Testing triangle-reducibility

In this subsection we show that testing triangle-reducibility (and finding an inductive construction for triangle-reducible graphs) can be done efficiently in a greedy fashion. Let $G = (V, E)$ be a graph. If G is triangle-reducible and $|V| > 4$ then there must be a vertex v with neighbors x, y, z spanning exactly two edges in G. The following lemma says that we can eliminate any such vertex and get another triangle-reducible graph. Thus triangle-reducibility can be tested with a simple greedy algorithm which also provides a sequence of triangle-splits which generates G.

Lemma 3.3.2. *Let $G = (V, E)$ be a triangle-reducible graph with $|V| \geq 5$ and let $v \in V$ with $N(v) = \{x, y, z\}$. Suppose that $xz, yz \in E$ and $xy \notin E$. Then $G' = G - v + xy$ is triangle-reducible.*

Proof: Let $K_4 = G_0, G_1, \ldots, G_n = G$ be a sequence of graphs, where G_{i+1} is obtained from G_i by a triangle-split, $0 \leq i \leq n-1$. Consider the first graph G_k in the sequence which contains v. It is easy to see that, by modifying G_0 and G_1, if necessary, we may assume that $k \geq 1$. Thus v is created by a triangle split operation on G_{k-1}. Since a triangle split does not decrease the degree of any vertex, and does not add new edges connecting existing vertices, it follows that v has degree three and $N_l(v)$ induces exactly two edges in G_l for all $k \leq l \leq n$, where $N_i(v)$ denotes the set of neighbors of v in some G_i.

Let $N_k(v) = \{u, w, t\}$ and suppose that $ut, wt \in E(G_k)$ and $uw \notin E(G_k)$. Next observe that as long as t remains a neighbor of v, the other two neighbors of v must be non-adjacent. In fact, t must remain a neighbor of v in the rest of the sequence.

Claim 3.3.3. *$vt \in E(G_l)$ for all $k \leq l \leq n$.*

Proof: Let $i \geq k$ be the largest index for which $vt \in E(G_i)$. For a contradiction suppose that $i \leq n-1$. Let $N_i(v) = \{u_i, w_i, t\}$. It follows from the previous observation that we must have $u_i w_i \notin E(G_i)$. Since $vt \notin E(G_{i+1})$, it follows that G_{i+1} is obtained from G_i by 'splitting' the edge vt by a new vertex t' of degree three. Hence $N_{i+1}(v) = \{u_i, w_i, t'\}$ induces at most one edge in G_{i+1}. This contradicts the fact that the neighbors of v induce exactly two edges in G_l for all $k \leq l \leq n$. •

It follows from Claim 3.3.3 that $N_i(v) = \{u_i, w_i, t\}$ and $u_i w_i \notin E(G_i)$ for all $k \leq i \leq n$. Thus $z = t$ holds. Let $G'_i = G_i - v + u_i w_i$, $k \leq i \leq n$. Next we show, by induction on i, that G'_i is triangle-reducible. Since $G'_k = G_{k-1}$, it is true for $i = k$. Suppose that $N_{i+1}(v) = N_i(v)$, i.e. the triangle-split, applied to G_i, leaves the neighbor set of v unchanged. Then G'_{i+1} can be obtained from G'_i by the same triangle split, and hence, by induction, G'_{i+1} is also triangle-reducible. Otherwise G_{i+1} is obtained from G_i by 'splitting' the edge vu_i (or vw_i). Then, without loss of generality, we have $G_{i+1} = G_i + u_{i+1} - vu_i + \{u_{i+1}v, u_{i+1}u_i, u_{i+1}t\}$. Then $w_{i+1} = w_i$ and $G'_{i+1} = G'_i + u_{i+1} - u_i w_i + \{u_{i+1}w_i, u_{i+1}u_i, u_{i+1}t\}$. So G'_{i+1} can be obtained from G'_i by a triangle-split. By induction, this gives that G'_{i+1} is triangle-reducible. Thus $G' = G'_n$ is triangle-reducible, which completes the proof. •

3.3.2 Cauchy and Grünbaum graphs

Another sufficient condition for global rigidity, due to Connelly, is based on stresses as well as convexity. Here we formulate a 2-dimensional version of his result for bar-and-joint frameworks, which can be deduced from Corollary 1 and Theorem 5 of [8].

Theorem 3.3.4. *[8] Let (G,p) be a framework whose edges form a convex polygon P in $\mathbb{R}^2$ with some chords. Suppose that there is a non-zero stress ω for (G,p) for which $\omega_{ij} \geq 0$ if $ij \in E$ is an edge on the boundary of P and $\omega_{ij} \leq 0$ if $ij \in E$ is an edge which is a chord of P. Then (G,p) is globally rigid.*

The *Cauchy-graphs* C_n and *Grünbaum graphs* G_n are both defined on vertex set $\{1,\ldots,n\}$ and both contain the edges $\{i, i+1\}$, $i = 1,2,\ldots,n$ (modulo n). In addition, the Cauchy graph contains the chords $\{i, i+2\}$, $i = 1, \ldots, n-2$, and the Grünbaum graph has the edges $1, 3$ and $2, i$ for $i = 4, \ldots, n$.

A *Cauchy-polygon* (*Grünbaum polygon*) is a framework (C_n, p) $((G_n, p))$, where the positions $p_1, \ldots, p_n$ of the vertices are in general position and, in this order, they form the set of vertices of a convex polygon in the plane. See Figure 3.4.

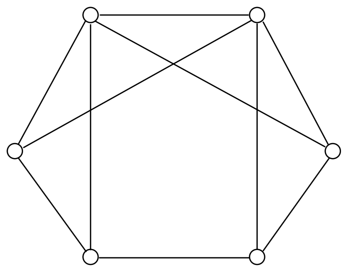

Figure 3.4: A Cauchy-polygon on 6 vertices.

It is easy to check that Cauchy-graphs as well as Grünbaum-graphs are triangle-reducible. One can also show, by induction on n, that any given Cauchy-polygon (C_n, p) (or Grünbaum-polygon (G_n, p)) can be obtained as the output of our algorithm. This gives a different proof of the first part of the next theorem.

Theorem 3.3.5. *(i) [8, Lemma 4, Theorem 5] Every Cauchy-polygon (C_n, p) is globally rigid.*
(ii) Every Grünbaum-polygon (G_n, p) is globally rigid.

Note that our algorithm may also generate non-convex globally rigid realizations of Cauchy-graphs, see Figure 3.5. Thus, in this sense, it gives an extension of Theorem 3.3.5(i).

3.4 Globally rigid tensegrity frameworks

The algorithm described in the previous section can be used to construct (universally) globally rigid tensegrity frameworks as well.

Let $T = (V; B, C, S)$ be a tensegrity graph and let (T, p) be a d-dimensional tensegrity framework. We say that a tensegrity framework (T, q) is *compatible* with

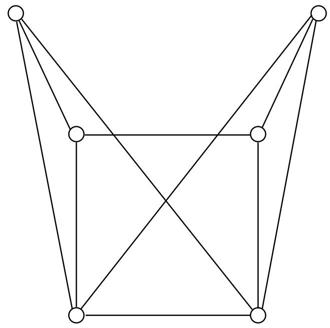

Figure 3.5: A non-convex globally and infinitesimally rigid realization of the Cauchy-graph C_6.

(T, p), if

$$||p(u) - p(v)|| = ||q(u) - q(v)|| \quad \text{for all } uv \in B,$$
$$||p(u) - p(v)|| >= ||q(u) - q(v)|| \quad \text{for all } uv \in C,$$
$$||p(u) - p(v)|| <= ||q(u) - q(v)|| \quad \text{for all } uv \in S.$$

Note that unlike equivalence of frameworks, this is not a symmetric relation. We say that a tensegrity framework (T, p) is *globally rigid* if every tensegrity framework (T, q) which is compatible with (T, p) is congruent to (T, p). A d-dimensional tensegrity framework is *universally globally rigid*, if it is globally rigid in $\mathbb{R}^D$ for all $D \geq d$. As an example, consider the four tensegrity frameworks on Figure 1.7. The first is not globally rigid, since we can 'fold' it along a cable until the other cable has zero length, and the fourth is not globally rigid, because we can move the central vertex to a point very far from the cable-triangle. The other two tensegrity frameworks are universally globally rigid. This shows that unlike infinitesimal rigidity, the global rigidity of tensegrity frameworks is not preserved when we interchange the cables and struts.

First we will state the generalization of Theorem 3.1.1 for tensegrity frameworks.

Theorem 3.4.1. *[8, 12] Let (T, p) be a 2-dimensional multidirectional tensegrity framework on n vertices and let $\omega : E \to \mathbb{R}$ be a function such that $-\omega$ is a stress for (T, p). Let Ω be the stress matrix of ω and suppose that Ω is positive semi-definite and has rank $n - 3$. Then (T, p) is globally rigid.*

Proof: The proof is analogous to the proof of Theorem 3.1.1. We define the same $H(q) = q\widehat{\Omega}q^\top$ quadratic form, and notice that if (T, p') is compatible with (T, p),

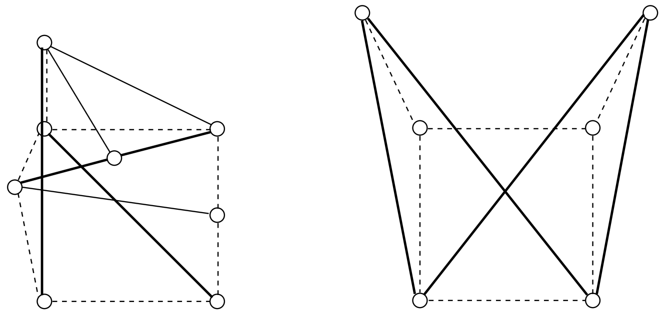

Figure 3.6: Globally rigid tensegrity frameworks obtained by ω-labeling the frameworks on Figures 3.3 and 3.5.

then $H(p') \leq H(p)$. Since $H(p) = 0$ and H is non-negative, we have that $H(p') = 0$ and that (T, p') is equivalent to (T, p) (that is, all the edge lengths are the same). From this point on, the proof is identical. •

We note that if a tensegrity framework satisfies the conditions of Theorem 3.4.1, then it is in fact universally globally rigid. This is called the Fundamental Theorem of Tensegrity Frameworks by Connelly [12]. The condition that $-\omega$ is a stress for (T, p) means that $\omega_{ij} \geq 0$ if $ij \in C$ and $\omega_{ij} \leq 0$ if $ij \in S$, which is Connelly's definition of a stress. In this work, we will stay with the original definition of Roth and Whiteley.

Now suppose that (G, p) is a globally rigid realization of the graph $G = (V, E)$ output by the construction algorithm of the previous section. The algorithm also produces a stress ω, for which the associated stress matrix is positive semi-definite and has rank $n - 3$. The *ω-labeling* of G is a tensegrity graph $T = (V; B, C, S)$, where $B = \{uv \in E \,|\, \omega_{uv} = 0\}$, $C = \{uv \in E \,|\, \omega_{uv} > 0\}$ and $S = \{uv \in E \,|\, \omega_{uv} < 0\}$. Clearly, $-\omega$ is proper stress for (T, p) and by Theorem 3.4.1 (T, p) is (universally) globally rigid. Furthermore, if G is triangle-reducible, then (T, p) will be a cable-strut tensegrity graph whose underlying bar framework is infinitesimally rigid, thus (T, p) is infinitesimally rigid by Theorem 1.6.1. See Figure 3.6 for examples of globally rigid tensegrity frameworks constructed this way.

We close this section by showing that in certain cases we can easily construct universally globally rigid tensegrity frameworks without the use of stresses.

Let $T = (V; B, C, S)$ be a tensegrity graph. The *cable-split* operation deletes a cable $uw \in C$ and adds a new vertex v and three new cables vu, vw and vt. Note that cable-splitting is a 1-extension on the cable-subgraph of T. The following Lemma on cable-splitting of tensegrity frameworks is due to Connelly [7].

Lemma 3.4.2. *Let (T, p) be a 2-dimensional universally globally rigid tensegrity framework. Let T' be a tensegrity graph that is obtained from T by a cable-split operation on the cable uw. Let p' be an extension of p where the placement of the new vertex v is $p'(v) = \frac{1}{2}(p(u) + p(w))$. Then the tensegrity framework (T', p') is universally globally rigid.*

Proof: Suppose that (T', q') is a d-dimensional tensegrity framework that is compatible with (T', p'). Let q be the restriction of q' on the vertex set of T. First we will show that (T, q) is compatible with (T, p). With the exception of the cable uw, all members of T are members of T', so it is sufficient to show that the member constraint is satisfied for uw, which follows from the triangle-inequality on the $q(u)q'(v)q(w)$ triangle:

$$\begin{aligned} ||q(w) - q(u)|| \leq & ||q(w) - q'(v)|| + ||q'(v) - q(u)|| \leq \\ & ||p(w) - p'(v)|| + ||p'(v) - p(u)|| = ||p(w) - p(u)||. \end{aligned}$$

Since (T, p) is universally globally rigid, we conclude that (T, q) is congruent to (T, p), thus the above inequality must hold with equality. Together with the member constraints on uv and vw, this means that $q'(v) = \frac{1}{2}(q(u) + q(w))$ and thus (T', q') is congruent to (T', p'). •

Let $G = (V, E)$ be a 3-connected M-circuit that contains a triangle $v_1v_2v_3$. The following algorithm, which is due to Connelly [7, 11], constructs a universally globally rigid tensegrity framework whith the underlying graph G. By a result of Berg and Jordán [4, Theorem 5.9], G can be constructed from $G_0 = K_4$ with a sequence of 1-extensions, such that the triangle $v_1v_2v_3$ of the starting K_4 is never used in the 1-extensions. Let $G_0, G_1, \ldots, G_r$ be the graph sequence that produces $G = G_r$. Each G_{i+1} is obtained from G_i by a 1-extension on an edge $uw \notin \{v_1v_2, v_1v_3, v_2v_3\}$.

Let (T_0, p_0) be a tensegrity framework like the third drawing on Figure 1.7, where $v_1v_2v_3$ is the outer strut-triangle. It is easy to see that the tensegrity framework (T_0, p_0) is universally globally rigid, e.g. by Theorem 3.4.1 choosing an appropriate stress. Following the construction steps of G, we define the sequence $T_0, T_1, \ldots, T_r$, where each T_{i+1} is a cable-split of T_i. Similarly we can define the configuration sequence $p_0, p_1, \ldots, p_r$ as in Lemma 3.4.2. The resulting tensegrity framework (T_r, p_r) is universally globally rigid by Lemma 3.4.2, and consequently the underlying bar framework (G, p_r) is also universally globally rigid. This construction algorithm gives a simplified proof of Theorem 3.3.1 in the special case where G is a 3-connected M-circuit that contains a triangle. See Figure 3.7 for an example output.

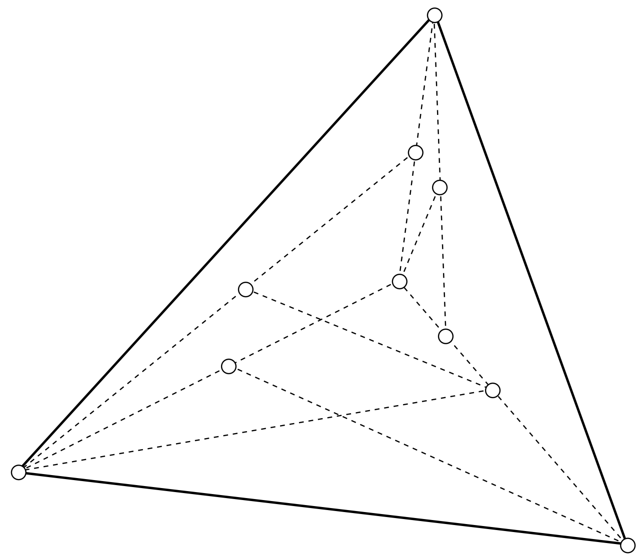

Figure 3.7: Universally globally rigid tensegrity framework obtained by the cable-splitting algorithm.

Chapter 4

Rigidity of Tensegrity Graphs

In this chapter we consider two combinatorial problems related to tensegrity graphs:

(a) Given a graph $G = (V, E)$, how to find a cable-strut labeling $E = C \cup S$ of the edges for which the resulting tensegrity graph $T = (V; C, S)$ is rigid in $\mathbb{R}^d$. (Note that G has such a rigid cable-strut labeling if and only if G is redundantly rigid in $\mathbb{R}^d$.)

(b) Given a cable-strut tensegrity graph $T = (V; C, S)$, decide whether T is rigid in $\mathbb{R}^d$.

Our main result for the first problem is an efficient combinatorial algorithm for finding a rigid cable-strut labeling, if it exists, in the case when $d = 2$. The algorithm is based on a new inductive construction of redundant graphs, i.e. graphs which have a realization as a bar framework in which each bar can be deleted without increasing the degree of freedom. The labeling is constructed recursively, following the steps of the construction, by using labeled versions of 1-extension and 2-sum operations. We note that our algorithm does not yield an infinitesimally rigid realization of the labeled graph but is designed to only find the labels.

The characterization of rigid tensegrity graphs is not known for $d \geq 2$. (The solution for $d = 1$ can be found in [33].) In the second part of this chapter we give a characterization for rigid tensegrity graphs in the case when $d = 2$ and the underlying graph is either the complete graph or the wheel.

In the rest of the chapter we will assume that $d = 2$, and we will call a tensegrity graph rigid, if it is rigid in $\mathbb{R}^2$.

4.1 Operations on tensegrity graphs

In this section we introduce the 'labeled generalizations' of the 1-extension and 2-sum operations and show that they preserve rigidity when applied to tensegrity graphs. These operations will be used in the next sections to define rigid cable-strut labelings of graphs and to prove the characterization of rigid cable-strut tensegrity graphs on the complete graph and the wheel.

Let $T = (V; B, C, S)$ be a tensegrity graph, let $uw \in C \cup S$ be a cable or strut of T and let $t \in V - \{u, w\}$ be a vertex. The *labeled* 1-*extension* operation deletes the member uw, adds a new vertex v and new members vu, vw, vt, satisfying the condition that if uw is a cable then at least one of vu, vw is not a strut, and if uw is a strut then at least one of vu, vw is not a cable. The new member vt may be arbitrary. For example, if we consider cable-strut tensegrity graphs, this definition leads to six possible labeled 1-extensions on a strut uw, as illustrated in Figure 4.1.

Lemma 4.1.1. *Let T be a rigid tensegrity graph and let T' be a tensegrity graph obtained from T by a labeled 1-extension. Then T' is also rigid.*

Proof: Since infinitesimal rigidity (and the labeled 1-extension operation) is preserved by interchanging cables and struts, we may assume that the 1-extension is made on a strut uw of T and vertex $t \in V - \{u, w\}$. Let (T, p) be an infinitesimally rigid realization of T in $\mathbb{R}^2$. By Theorem 1.6.1 there is a proper stress ω of (T, p) and (G, p) is an infinitesimally rigid bar framework, where $G = \overline{T}$ is the underlying graph of T. By Theorem 1.6.2 we may assume that $p(u), p(w), p(t)$ are not collinear. In the rest of the proof we will also assume that the new members vu, vw, vt are all struts. The proof is similar for each of the six possible labeled 1-extensions.

Let us extend the configuration p by putting $p(v) = \alpha p(u) + (1-\alpha)p(w)$ for some $0 < \alpha < 1$. Let G' be the underlying graph of T', which can be obtained from G by a 1-extension. We can also extend the stress ω of (T, p) to (T', p) by defining $\omega_{vu} = \omega_{uw}/(1-\alpha)$, $\omega_{vw} = \omega_{uw}/\alpha$ and $\omega_{vt} = 0$.

Since $p(u), p(w), p(t)$ are not collinear, the bar framework (G', p) is infinitesimally rigid, see Theorem 1.4.4. Furthermore, the extended stress is nearly proper on (T', p): the only member with a zero stress is vt. This implies that (T'', p) is infinitesimally rigid, where T'' is obtained from T' by replacing the strut vt by a bar.

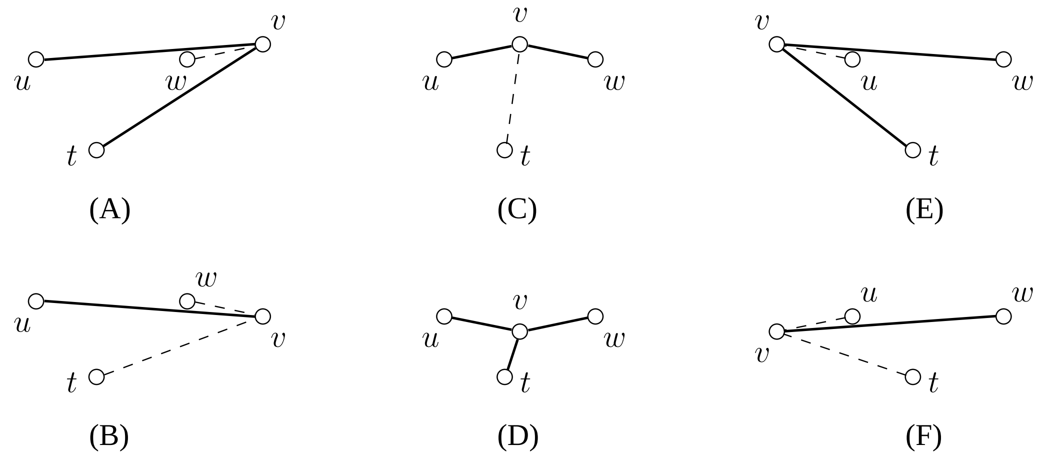

Figure 4.1: The six possible labeled 1-extensions on the strut uw and the feasible positions of v.

To obtain a proper stress we need to modify the realization a bit by replacing $p(v)$ by a point in the interior of the triangle $p(u)p(w)p(t)$. By Theorem 1.6.2 this can be done without destroying the infinitesimal rigidity of (T'', p). Consider a proper stress ω' of this modified realization of T''. Since we have three members incident with v, and vu, vw are struts, we must have a positive stress on vt. Thus we may replace vt by a strut and obtain the required infinitesimally rigid realization of T'.

The other labeled 1-extensions can be treated in a similar manner by appropriately defining $\alpha \in \mathbb{R} - \{0, 1\}$ and moving $p(v)$ out of the line of $p(u)p(w)$ in such a way that the signs of the stresses on the members incident to v are as required. See Figure 4.1. •

We will also need an operation that glues together two tensegrity graphs along a pair of members. Let $T_1 = (V_1; B_1, C_1, S_1)$ and $T_2 = (V_2; B_2, C_2, S_2)$ be two tensegrity graphs with $V_1 \cap V_2 = \emptyset$ and let $u_1v_1 \in S_1$ and $u_2v_2 \in C_2$ be two designated members, a strut in T_1 and a cable in T_2. The 2-*sum* of T_1 and T_2 (along the strut-cable pair u_1v_1 and u_2v_2) is the tensegrity graph obtained from $T_1 - u_1v_1$ and $T_2 - u_2v_2$ by identifying u_1 with u_2 and v_1 with v_2. See Figure 4.2. We denote a 2-sum of T_1 and T_2 by $T_1 \oplus_2 T_2$. Since we will apply the 2-sum operation to non-rigid tensegrity graphs as well, we first prove the following lemma.

Lemma 4.1.2. *Let (T_1, p_1) and (T_2, p_2) be regular realizations of tensegrity graphs*

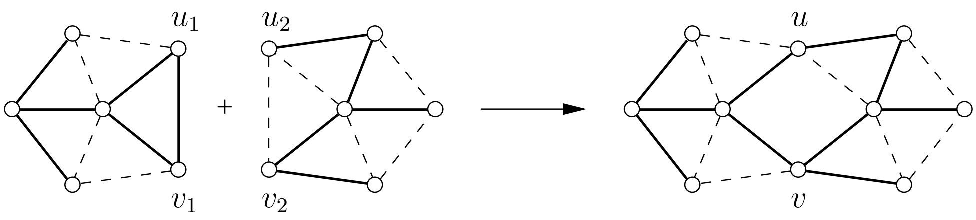

Figure 4.2: The 2-sum of two tensegrity graphs along the pair u_1v_1, u_2v_2.

T_1, T_2 with a proper stress. Then $T = T_1 \oplus_2 T_2$ also has a regular realization with a proper stress.

Proof: By Theorem 1.6.2 we may assume that (T_i, p_i) is strongly regular for $i = 1, 2$. Let ω_i be a proper stress of (T_i, p_i), $i = 1, 2$. By scaling, translating, and rotating the frameworks, if necessary, we may assume that $p_1(u_1) = p_2(u_2)$ and $p_1(v_1) = p_2(v_2)$. These operations will not destroy strong regularity and ω_i remains a proper stress in the realization of T_i for $i = 1, 2$. By scaling the stresses we can also assume that $\omega_1(u_1v_1) = -\omega_2(u_2v_2) = 1$. Since the realizations are strongly regular, it follows from Theorem 1.6.1(ii) that u_1v_1 and u_2v_2 are both redundant.

Let T' be the tensegrity graph obtained from T_1, T_2 by identifying u_1 with u_2, and v_1 with v_2. Consider the realization (T', p) of T' obtained by merging the frameworks (T_i, p_i), $i = 1, 2$, along the points $p_1(u_1), p_1(v_1)$. We can find strongly regular realizations of T' arbitrarily close to (T', p) without changing the positions of $p(u), p(v)$. Now we can use Theorem 1.6.2, applied to each (T_i, p_i), and the fact that u_iv_i is redundant in (T_i, p_i), $i = 1, 2$, to deduce that there is an $\epsilon > 0$ for which any regular realization of T' in the ϵ-neighborhood has a proper stress whose value is equal to 1 on the strut u_1v_1 and -1 on the cable u_2v_2. Since the stresses on u_1v_1 and u_2v_2 cancel each other, we have that (T, p) is a regular realization of $T = T_1 \oplus_2 T_2$ with a proper stress. This proves the lemma. •

Theorem 1.6.1 and the gluing lemma [44, Lemma 3.1.4] gives the following corollary.

Lemma 4.1.3. *Suppose that T_1 and T_2 are rigid tensegrity graphs. Then $T = T_1 \oplus_2 T_2$ is also rigid.*

4.2 Cable-strut labeling of redundant graphs

In this section we will consider the problem of finding a rigid cable-strut labeling for a graph. By Theorem 1.6.3 we may suppose that the input of the cable-strut labeling problem is a redundantly rigid graph G. Our goal is to find an inductive construction of G, i.e. a sequence $G_1, G_2, ..., G_r$ of graphs for which G_{i+1} is obtained from G_i by some graph operation, $1 \leq i \leq r-1$, and $G_r = G$, in such a way that (i) each G_i has a 'good' cable-strut labeling, (ii) the operations are chosen so that, given a 'good' cable-strut labeling of G_i, a 'good' cable-strut labeling of G_{i+1} is easy to find, (iii) the cable-strut labeling of G gives rise to a rigid tensegrity graph. To make this idea work we need to consider redundant graphs, a family of graphs which properly contains redundantly rigid graphs, and call a labeling 'good' if the corresponding tensegrity graph has a regular realization with a proper stress.

We say that G is *redundant* if it has at least one edge and each edge of G is in an M-circuit. It follows that a graph G is redundantly rigid if and only if G is rigid and redundant.

To prove the main theorem of this section, we need the following two results on redundant graphs.

Lemma 4.2.1. *Suppose that G is a 2-connected redundant graph. Let $\{u, v\}$ be a 2-separator of G and let $\tilde{H}_1$ and $\tilde{H}_2$ be the cleavage graphs obtained by cleaving G along $\{u, v\}$. Then at least one of the following holds:*
(i) $\tilde{H}_i$ is redundant for $i = 1, 2$;
(ii) there is a 2-separation (H_1, H_2) of G with $V(H_1) \cap V(H_2) = \{u, v\}$ for which H_i is redundant for $i = 1, 2$.

Proof: First we prove that each edge $f \in E(\tilde{H}_1) - uv$ belongs to an M-circuit in $\tilde{H}_1$. Since G is redundant, there is an M-circuit C in G which contains f. If C is a subgraph of $\tilde{H}_1$ then we are done. If not, then $\{u, v\}$ is a 2-separator of C. In this case it follows from 2.4.3 that the cleavage graphs C_1 and C_2 obtained by cleaving C along $\{u, v\}$ are both M-circuits. Hence C_1 is an M-circuit in $\tilde{H}_1$ which contains f. By symmetry we also have that each edge $f' \in E(\tilde{H}_2) - uv$ belongs to an M-circuit in $\tilde{H}_2$.

Thus, if uv belongs to an M-circuit in both cleavage graphs then (i) holds. Now suppose that, say, uv is in no M-circuit in $\tilde{H}_1$. As above, this implies that if $uv \in$

$E(G)$ then all M-circuits of G containing uv must be in $\tilde{H}_2$ and if $uv \notin E(G)$ then all M-circuits of G containing some edge of $E(\tilde{H}_1) - uv$ must be in $\tilde{H}_1 - uv$.

By moving the edge uv from one side of the 2-separation to the other, if necessary, we may assume that there is a 2-separation (H_1, H_2) of G with $V(H_1) \cap V(H_2) = \{u, v\}$ and $uv \notin E(H_1)$. The arguments above now imply that H_1 and H_2 are both redundant. Thus (ii) holds. •

The next result follows by observing that the proof of [25, Theorem 3.2] goes through under the weaker hypothesis that G is redundant, and by using [25, Lemma 3.1].

Theorem 4.2.2. *[25] Suppose that G is a 3-connected redundant graph. Then G is redundantly rigid.*

Now we are ready to give an algorithmic proof for the existence of a 'good' labeling of a redundant graph G. It will also imply an inductive construction for redundant graphs as well as an efficient combinatorial algorithm for finding a rigid cable-strut labeling in the special case when G is redundantly rigid.

Theorem 4.2.3. *Let $G = (V, E)$ be a redundant graph in $\mathbb{R}^2$. Then the edge set of G has a cable-strut labeling $E = C \cup S$ for which the tensegrity graph $T = (V; C, S)$ has a regular realization with a proper stress.*

Proof: We prove the theorem by induction on $|V|$. Since G is redundant and the smallest M-circuit is K_4, we must have $|V| \geq 4$ with equality only if $G = K_4$. The statement is straightforward for K_4 (see Figure 1.7), so we may assume that $|V| \geq 5$ and that the theorem holds for all redundant graphs containing less vertices than G.

First suppose that G has at least two blocks (i.e. maximal 2-connected subgraphs), denoted by $H_1, H_2, ..., H_t$. Since M-circuits are 2-connected, each block is redundant. Thus, by induction, we can find a cable-strut labeling of each block H_i such that the corresponding tensegrity graph T_i has a regular realization with a proper stress. Let T be the tensegrity graph on G whose cable-strut labeling is induced by the T_i's. Since a proper stress remains a proper stress after translating a framework, and since the blocks of G are edge-disjoint, we may obtain a realization (T, p) of T with a proper stress by simply translating and merging the realizations of the T_i's at the cut-vertices of G. Since the regular realizations of T form a dense open set, we can use Theorem 1.6.2, applied to each of the realizations of the T_i's, to make the realization regular. This shows that G has the required labeling.

Hence we may assume that G is 2-connected. If G is 3-connected then G can be obtained from K_4 by 1-extensions and edge additions by Theorems 4.2.2, 1.5.5. Thus we can obtain a rigid cable-strut tensegrity graph T with underlying graph G by starting with a rigid cable-strut labeling of K_4 and using labeled 1-extensions as well as cable or strut additions, following the inductive construction of G. Lemma 4.1.1 implies that the labeled graph is indeed rigid. (The addition of new members clearly preserves rigidity.) Since an infinitesimally rigid realization of T is regular and has a proper stress by Theorem 1.6.1, the existence of the required cable-strut labeling of G follows.

It remains to consider the case when G is 2-connected and has a 2-separation $\{u, v\}$. Let $\tilde{H}_1, \tilde{H}_2$ be the cleavage graphs obtained by cleaving G along $\{u, v\}$. By Lemma 4.2.1 either G can be obtained as the edge-disjoint union of two redundant graphs with two vertices in common or both cleavage graphs are redundant. In the former case we can proceed as in the case of 1-separations: by induction, we can find cable-strut labelings of the smaller graphs for which the required realizations exist. These labelings induce a cable-strut labeling T of G. Furthermore, by first rotating, translating, and scaling the frameworks, if necessary, we can merge the realizations to obtain a realization of T with a proper stress. By perturbing this realization, and using Theorem 1.6.2, we can make the realization regular, too. This shows that G has the required labeling.

In the latter case we can also find, by induction, cable-strut labelings of the cleavage graphs which have the desired realizations. By Theorem 1.6.2 we may assume that these realizations are strongly regular. After interchanging cables and struts in one of the cleavage graphs, if necessary, we can take the 2-sum of the labeled cleavage graphs to obtain a labeling T' of $G' = G - uv$ which has a regular realization (T', p) with a proper stress, by Lemma 4.1.2. If $uv \notin E(G)$ then this provides the required labeling of G. Now suppose that $uv \in E(G)$. By Theorem 1.6.2 we may assume that p is chosen so that $(T' + uv, p)$ is strongly regular. Then uv is redundant in $(T' + uv, p)$, and hence there is a stress ω' of $(T' + uv, p)$ whose value on uv is not zero. By adding ω' to ω with a small coefficient we obtain a proper stress of $(T' + uv, p)$.

Thus adding a new cable (or strut) uv to the labeled graph T' gives the required labeling of G in this case. This completes the proof of the theorem. •

The proof implies that if G is redundant then G be obtained from disjoint K_4's by recursively applying 1-extensions or edge-additions within some connected component, 2-sums to two connected components, and merging components along at most two vertices. This inductive construction can be obtained in polynomial time by using efficient algorithms to find 2-separators [24] and test redundancy [3]. By following the steps of the construction it is then straightforward to find a good labeling of G by applying labeled 1-extensions, cable or strut additions, taking 2-sums (possibly after interchanging cables and struts in one of the summands), and merging.

When G is redundantly rigid, Theorem 1.6.1, Theorem 4.2.3, and the above argument imply:

Theorem 4.2.4. *Let $G = (V, E)$ be a redundantly rigid graph in $\mathbb{R}^2$. Then the edge set of G has a cable-strut labeling $E = C \cup S$ for which the tensegrity graph $T = (V; C, S)$ is rigid. Furthermore, such a cable-strut labeling of E can be found in polynomial time.*

We note that it is fairly easy to extend the above results to the case when the input graph may contain multiple edges and/or a designated set of edges labeled as bars and the goal is to find a cable-strut labeling of the remaining edges so that the union of the bars, cables, and struts gives rise to a rigid tensegrity graph.

We also remark that the proof of Theorem 4.2.3 can be used to verify the existence of rigid cable-strut labelings with various structural properties. For example, consider a 3-connected M-circuit G on at least five vertices. Then G has a rigid cable-strut labeling in which the cables as well as the struts induce a spanning tree of G. This follows from the facts that (i) the wheel graph W_5 on five vertices (which is obtained from K_4 by a 1-extension) has a rigid cable-strut labeling in which the cables as well as the struts induce a spanning tree, (ii) G can be obtained from W_5 by 1-extensions by Theorem 1.5.5, (iii) when labeling the edges of G using this inductive construction it is possible to choose labeled 1-extensions so that the spanning trees are 'preserved'. One can similarly verify that if G contains a triangle then G has a rigid cable-strut labeling in which the cables induce a single triangle and all other members are struts.

This follows by using [4, Theorem 5.9], which implies that the inductive construction can be chosen so that the edges of a designated triangle of the starting K_4 are never involved in the 1-extensions.

4.3 Rigid cable-strut tensegrity graphs

As we noted earlier, the characterization of the rigid (cable-strut) tensegrity graphs is still open, even in two dimensions. In one dimension it turns out that a cable-strut tensegrity graph T is rigid if and only if its underlying graph is rigid (i.e. connected) and each of its M-connected components (i.e. blocks) contains at least one cable and at least one strut, see [33]. (Note that by using this result the solution of the labeling problem in one dimension is straightforward.)

The same conditions, however, are not sufficient to guarantee the rigidity of a cable-strut tensegrity graph T in two dimensions. This follows by observing that a cable-strut tensegrity graph with just one cable (or strut) can never be rigid, see Lemma 4.3.3. In addition, there is no lower bound on the number of cables and struts which would imply the rigidity of a tensegrity graph, even if its underlying graph is M-connected. This follows by observing that the 2-sum of a rigid cable-strut tensegrity graph with an M-connected underlying graph and a tensegrity graph on K_4 which contains only struts, is not rigid. However, the following might be true.

Conjecture 4.3.1. *There exists a (smallest) integer k such that every tensegrity graph T containing at least k cables and at least k struts, and with a 3-connected and redundantly rigid underlying graph, is rigid in $\mathbb{R}^2$.*

In the rest of this chapter we will give a characterization of rigid tensegrity graphs whose underlying graph is a complete graph or a wheel. The wheels show that if k exists, it must be at least five. See drawing (J) of Figure 4.4. We will also show that every rigid tensegrity graph on the wheel can be constructed from a rigid tensegrity graph on K_4 with a sequence of labeled 1-extensions. We conjecture that this is true for all 3-connected M-circuits.

Conjecture 4.3.2. *Let T be a rigid cable-strut tensegrity graph with an underlying graph which is a 3-connected M-circuit. Then T can be constructed from a rigid cable-strut tensegrity graph on K_4 with a sequence of labeled 1-extensions.*

4.3.1 Necessary conditions

In this subsection we give some necessary conditions for the rigidity of cable-strut tensegrity graphs. Let $T = (V; C, S)$ be a cable-strut tensegrity graph, (T, p) an infinitesimally rigid tensegrity framework and ω a proper stress for (T, p). The stress equation for a vertex $v \in V$ can be written as:

$$\sum_{uv \in S} \omega_{uv}(p(u) - p(v)) = \sum_{uv \in C} (-\omega_{uv})(p(u) - p(v)), \tag{4.1}$$

where the empty sum is defined as the zero vector. For the framework (T, p) we define the *strut cone* at v as

$$S_p(v) = \{\sum_{uv \in S} \lambda_{uv}(p(u) - p(v)) \,|\, \lambda_{uv} > 0\}.$$

Similarly we can define the *cable cone* at v, $C_p(v)$. With this notation, equation (4.1) can be written as

$$S_p(v) \cap C_p(v) \neq \emptyset. \tag{4.2}$$

Let V_S and V_C be the set of vertices that are incident to a strut (cable, respectively). It is easy to see that if $v \in V_S - V_C$, that is, all the members incident to v are struts, then $S_p(v) = \mathbb{R}^2$ must hold. This has the important corollary that $p(v)$ is in the convex hull of $p(V_C)$, for all $v \in V$, and that the convex hull of $p(V)$, $p(V_C)$ and $p(V_S)$ are the same. To see this, suppose indirectly that there is $v \in V$, such that $p(v)$ is outside the convex hull of $p(V_C)$. We can choose v such that the distance between $p(v)$ and $\operatorname{conv} p(V_C)$ is maximal. Since $v \notin V_C$, all the members incident to v are struts and hence $S_p(v) = \mathbb{R}^2$, which means that there is another $w \in V_S$, which is further from $p(V_C)$ than $p(v)$. With a similar argument, we can see that if $v \in V_S - V_C$, then $p(v)$ is in the interior of the convex hull of $p(V_C)$.

Using the above observations, we are ready to prove the following necessary condition about the number of cables and strut.

Lemma 4.3.3. *Let $T = (V; C, S)$ be a cable-strut tensegrity graph that is rigid in $\mathbb{R}^2$. Then $|C| \geq 2$ and $|S| \geq 2$. Moreover, if $|C| = 2$ ($|S| = 2$), then the two cables (struts, respectively) are independent.*

Proof: Let (T, p) be an infinitesimally rigid tensegrity framework. Because interchanging the cables and struts preserves infinitesimal rigidity, we can assume that

$|S| \geq |C|$. If $C = \emptyset$, then a simple scaling on the framework shows that (T, p) is not infinitesimally rigid.

Now let us suppose that $C = \{uw\}$. In this case, the convex hull of $p(V_C)$ is the line segment $[p(u), p(w)]$, and hence, from the above argument we get that $p(v)$ is on the line $p(u)p(w)$, for all $v \in V$. Since a rigid tensegrity graph has at least four vertices, this contradicts the infinitesimal rigidity of the bar framework $(\overline{T}, p)$.

Finally, suppose that $C = \{ut, tw\}$. Then for all $v \in V - \{u, t, w\}$, $p(v)$ is in the interior of the $p(u)p(t)p(w)$ triangle, which means that $S_p(u) \cap C_p(u) = \emptyset$, contradicting equation (4.2). •

Lemma 4.3.4. *Let $T = (V; C, S)$ be a cable-strut tensegrity graph and let $v \in V$ be a vertex of degree 3 with neighbors $u, w, t \in V$. If $vu, vw, vt, ut, wt \in S$, then T is rigid in $\mathbb{R}^2$ if and only if $T' = (V - \{v\}; C, S + \{uw\})$ is rigid in $\mathbb{R}^2$.*

Proof: The if part follows from applying a labeled 1-extension on T' which yields T. To prove the only if part, consider an infinitesimally rigid and generic tensegrity framework (T, p) together with a proper stress ω, such that $\omega_{vt} = 1$. Let $W = \{v, u, w, t\}$ and let H be the tensegrity graph on K_W where vu, vw, vt are cables and uw, ut, wt are struts. Let ν be a proper stress of $(H, p|_W)$ such that $\nu_{vt} = -1$. Now $\omega + \nu$ is a stress on $T + uw$, where uw is a strut, with the property that it is zero on vu, vw, vt and non-zero everywhere else. Therefore the restriction of $\omega + \nu$ on T' is a proper stress of $(T', p|_{V-\{v\}})$. By Lemma 1.6.1 $(\overline{T} - \{vt\}, p)$ is infinitesimally rigid, and by applying an inverse 0-extension on v we get that $(\overline{T'}, p|_{V-\{v\}})$ is also infinitesimally rigid. By Lemma 1.6.1 we get that $(T', p|_{V-\{v\}})$ is an infinitesimally rigid tensegrity framework and thus T' is a rigid tensegrity graph in $\mathbb{R}^2$. •

4.3.2 Complete cable-strut tensegrity graphs

In this section we show that the necessary condition in Lemma 4.3.3 is also sufficient for complete cable-strut tensegrity graphs.

Lemma 4.3.5. *Let $T = (V; C, S)$ be a cable-strut tensegrity graph and $u_1, u_2, u_3 \in V$. If T is rigid, then $T' = T + v + \{vu_1, vu_2, vu_3\}$ is also rigid for all cable-strut labeling* *of the new edges.*

Proof: If there are $1 \leq i < j \leq 3$ such that $u_iu_j \notin C \cup S$, then we add a cable u_iu_j to T if vu_i is a cable otherwise we add a strut u_iu_j. Since adding a member preserves rigidity, $T + u_iu_j$ is rigid. Performing a labeled 1-extension on u_iu_j generates the new tensegrity graph. So we may assume that $\{u_1u_2, u_1u_3, u_2u_3\} \subset C \cup S$. If there are $1 \leq i < j \leq 3$ such that vu_i and vu_j have different labels, then performing a labeled 1-extension on u_iu_j and then putting back the member u_iu_j generates the new tensegrity graph, so we can assume that all the new edges have the same label, say they are struts. If there is $u_iu_j \in S$, then we can apply the 1-extension and put back the deleted member to get the new tensegrity graph. The remaining case is when we add three struts vu_1, vu_2, vu_3 to the cable triangle u_1u_2, u_1u_3 and u_2u_3. If (T, p) is an infinitesimally rigid realization of T in $\mathbb{R}^2$, we can extend it to an infinitesimally rigid realization of T' by putting $p(v)$ anywhere in the convex hull of $p(u)$, $p(w)$ and $p(t)$ such that the bar framework $(\overline{T}', p)$ is rigid. •

Corollary 4.3.6. *Let $T = (V; C, S)$ be a cable-strut tensegrity graph on K_n for some $n \geq 5$. If there is a subgraph $H \leq T$ that is rigid in $\mathbb{R}^2$, then T is rigid in $\mathbb{R}^2$.*

Lemma 4.3.7. *Let $T = (V; C, S)$ be a cable-strut tensegrity graph on K_5, where $|S| \geq |C| \geq 3$. Then T is rigid in $\mathbb{R}^2$.*

Proof: Since the number of vertices is odd, there must be a vertex with an even number of cables. We will consider the following three cases: there is a vertex with no cables, there is a vertex with two cables or there is a vertex with four cables. In each case, by Corollary 4.3.6, we need to consider only the non-rigid tensegrity graphs on the other four vertices. The tensegrity graphs that need to be proven rigid are shown on Figure 4.3. In each case we will show the rigidity by either finding a rigid subgraph or by deleting two edges and applying an inverse 1-extension to get a rigid tensegrity graph. In the first case we delete vw and uz and apply an inverse 1-extension on w by deleting w and adding a cable uz. In the seventh case we delete ut and wz, then delete w and add a strut ut. In the ninth case we delete ut and vz, then delete u and add a cable vz. In all the other cases $T - u$ is rigid. •

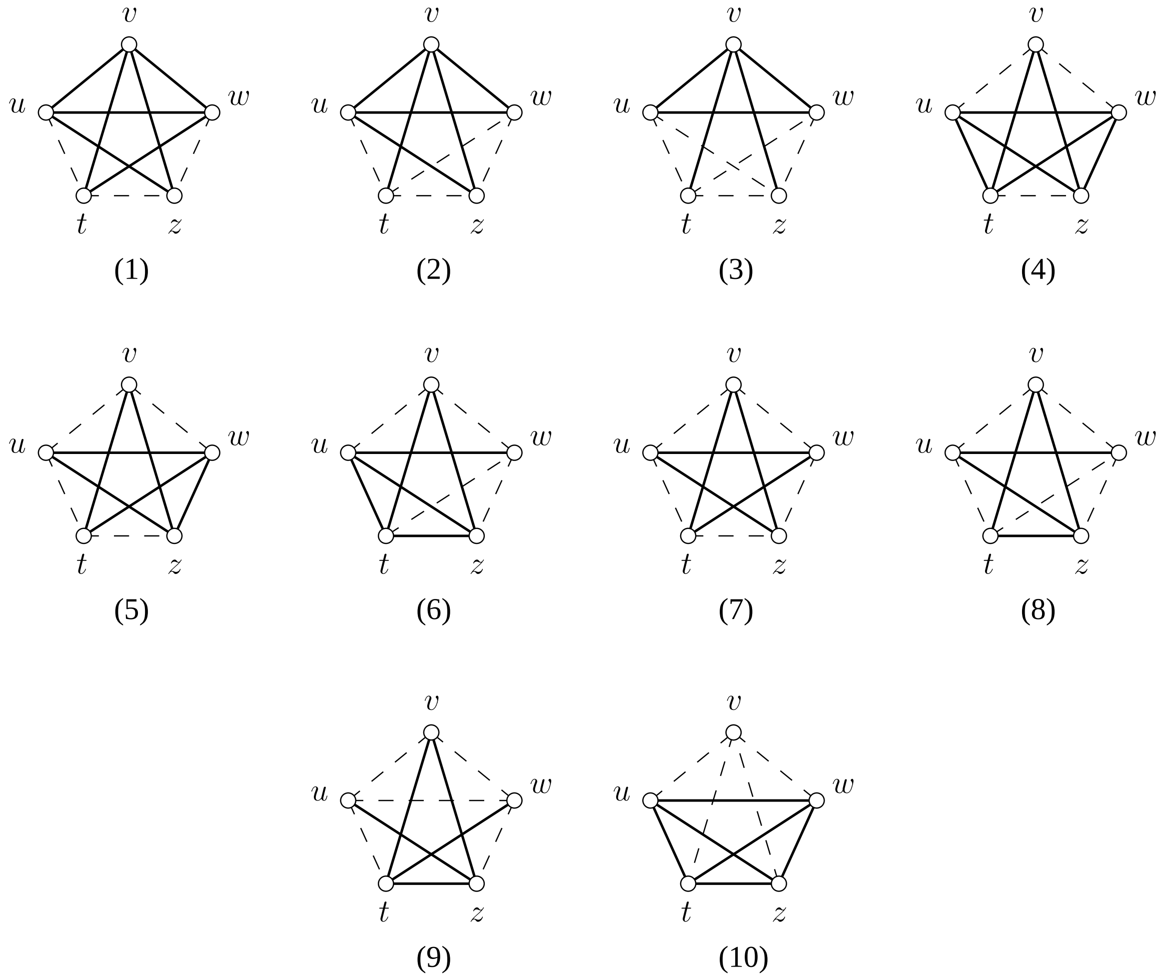

Figure 4.3: The tensegrity graphs on K_5, where $3 \leq |C| \leq 5$ and $T - v$ is not rigid.

Theorem 4.3.8. *Let $T = (V; C, S)$ be a cable-strut tensegrity graph on K_n for some $n \geq 5$. T is rigid in $\mathbb{R}^2$ if and only if $|C| \geq 3$ and $|S| \geq 3$ or there are four distinct vertices $u, v, w, t \in V$ such that $C = \{uv, wt\}$ or $S = \{uv, wt\}$.*

Proof: The only if part follows from Lemma 4.3.3. Since the role of cables and struts are interchangeable, we may assume that $|S| \geq |C|$. If $C = \{uv, wt\}$, then the tensegrity graph spanned by u, v, w and t is rigid, and therefore T is rigid. To prove that T is rigid whenever $|S| \geq |C| \geq 3$, we use induction on the number of vertices. The $n = 5$ case follows from Lemma 4.3.7, so we may assume that $n \geq 6$. Let $v \in V$ be an arbitrary vertex. Since $|S| \geq |C|$, the tensegrity graph $T - v$ has at least three struts. If $T - v$ has two independent or at least three cables, then $T - v$

is rigid by the induction hypothesis and therefore T is rigid, by Corollary 4.3.6. If $T - v$ has two adjacent cables uw and wt, then let vz be a third cable of T, and let $x \in V - \{v, u, w, t, z\}$ be another vertex. Now $T - x$ has at least three cables and at least three struts, so it is rigid by the induction hypothesis. If $T - v$ has only one cable uw, then let vt and vz be two more cables of T, and then $T - x$ will be rigid for all $x \in V - \{v, u, w, t, z\}$. If $T - v$ has no cables, then let vu, vw, vt be three cables of T, and the tensegrity graph spanned by v, u, w and t is rigid. In each case we have a rigid tensegrity subgraph of T, and therefore T is rigid by Corollary 4.3.6. •

4.3.3 Tensegrity graphs on the wheel

Now we will turn our attention to an other family of cable-strut tensegrity graphs, those with the underlying graph $W_n = C_{n-1} + v_0 + \{v_0 v \mid v \in V(C_{n-1})\}$, the wheel on n vertices. It consists of a cycle C_{n-1} plus one *central vertex* v_0 and $n-1$ *central edges* $v_0 v$ for each $v \in V(C_{n-1})$. The vertices and edges of C_{n-1} are called *side vertices* and *side edges*, respectively.

As a first step we will prove the non-rigidity of certain tensegrity graphs on W_5.

Lemma 4.3.9. *The tensegrity graphs on Figure 4.4 are not rigid in* $\mathbb{R}^2$.

Proof: It follows from Lemma 4.3.3 that the tensegrity graphs (A) - (F) on Figure 4.4 are not rigid in $\mathbb{R}^2$.

Let T be an arbitrary tensegrity graph from Figure 4.4 and let us suppose indirectly that T is rigid in $\mathbb{R}^2$. We will denote the central vertex of T by v_0 and the side vertices with t, u, v and w starting from the upper left vertex and going clockwise. Consider a generic configuration p, such that the bar framework $(\overline{T}, p)$ is infinitesimally rigid, and a proper stress ω of (T, p). Let T' be the extension of T with a bar uw and let ω' be the extension of ω with $\omega'_{uw} = 0$. Let H_1 be the subgraph of $\overline{T'}$ spanned by u, v, w and v_0 – a complete graph on these four vertices – and let ν be a stress on (H_1, p). Since H_1 is an M-circuit and p is generic, ν is nowhere zero. Let ν' be the extension of ν to $\overline{T'}$ with zeros on ut, $v_0 t$ and wt. Clearly ν' is a stress on $(\overline{T'}, p)$. We can choose ν in such a way that $\nu'_{v_0 v} = -\omega'_{v_0 v}$. Since the vectors $p(u) - p(v)$ and $p(w) - p(v)$ are linearly independent and both $\{\nu'_{uv}, \nu'_{wv}, \nu'_{v_0 v}\}$

and $\{\omega'_{uv}, \omega'_{wv}, \omega'_{v_0v}\}$ give a linear dependence among $p(u) - p(v)$, $p(w) - p(v)$ and $p(v_0) - p(v)$, we can deduce that $\nu'_{uv} = -\omega'_{uv}$ and $\nu'_{wv} = -\omega'_{wv}$. Let H_2 be $\overline{T'} - v$ and let μ be the restriction of $\omega' + \nu'$ on H_2. Now μ is a stress on (H_2, p) and since H_2 is an M-circuit, μ is nowhere zero. By labeling the edges of H_2 with cables where $\mu < 0$ and struts where $\mu > 0$ we get a rigid tensegrity graph U with the underlying graph H_2. We notice that the sign of μ is the same as the sign of ω on tu, tv_0 and tw and it is the same as the sign of ν on uw. To arrive to a contradiction in case of each tensegrity graph (G) - (J) on Figure 4.4, let us consider the possible signs of μ on uw, uv and vw.

In case of (H) and (I), we notice that $p(v)$ must be inside the convex hull of its neighbors and therefore the only possible sign distribution of ν, given that $\nu_{v_0v} = -\omega_{v_0v} < 0$, is that ν is negative on uv, wv and v_0v and is positive on uw, uv_0 and v_0w. Therefore uw and v_0w are both struts in U, while uv_0 can be either a cable or a strut. This all means that if we take the tensegrity graphs (H) and (I), then delete v, add a strut uw and possibly change the cable uv_0 into a strut, we get a rigid tensegrity graph in each case. It can be easily seen that this leads to a contradiction in both cases, since all the possible resulting tensegrity graphs on K_4 are not rigid in $\mathbb{R}^2$.

In the case of (G) and (J), there is two possible sign-pattern on ν based on whether $p(u)$ is in the convex hull of $p(v)$, $p(w)$ and $p(v_0)$, or outside it. In the first case ν is positive on uv, uv_0 and uw and negative on v_0v, vw and v_0w, while in the second case it is positive on uv and v_0w, and negative on vw, v_0v, uv_0 and uw. In case of (G) it means that in the derived tensegrity graph U either uw and uv_0 are both struts and v_0w can be either a cable or a strut, or uw is a cable, v_0w is a strut and uv_0 can be either a cable or a strut. In all of these four cases the resulting tensegrity graph U is not rigid, which is a contradiction. In case of (J) in the derived tensegrity graph U either uw is a strut, v_0w is a cable and uv_0 can be either a cable or a strut, or uw and uv_0 are both cables and v_0w can be either a cable or a strut. In all of these four cases the resulting tensegrity graph U is not rigid, which is a contradiction. •

We note that the converse of Lemma 4.3.9 is also true in the sense that a tensegrity graph $T = (V; C, S)$ (where $|S| \geq |C| \geq 1$) with the underlying graph W_5 is rigid if

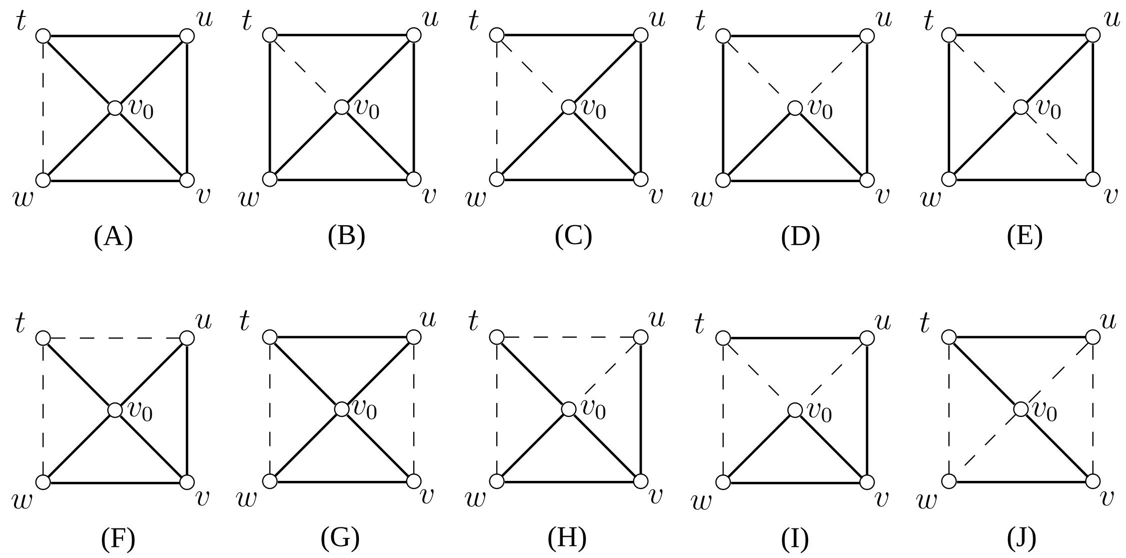

Figure 4.4: The non-rigid tensegrity graphs on W_5 where $|S| \geq |C| \geq 1$.

it is not isomorphic to one of the tensegrity graphs on Figure 4.4. This can be seen by choosing an appropriate side vertex in each case and applying an inverse labeled 1-extension that leads to a rigid tensegrity graph on K_4.

Lemma 4.3.10. *Let $T = (V; C, S)$ be a cable-strut tensegrity graph on W_n for some $n \geq 4$. If $C = \{v_0v_1, v_2v_3\}$ where v_1, v_2, v_3 are distinct side vertices, then T is rigid in $\mathbb{R}^2$.*

Proof: We use induction on n. The $n = 4$ case follows from the characterization of rigid tensegrity graphs on $W_4 = K_4$. If $n \geq 5$, then we can find consecutive side vertices u, v, w such that uv, vw, uv_0, vv_0, wv_0 are all struts. By removing v and adding the strut uw we get a cable-strut tensegrity graph T' on W_{n-1} which is rigid by the induction hypothesis. By Lemma 4.3.4 it follows that T is rigid in $\mathbb{R}^2$. •

Lemma 4.3.11. *Let $T = (V; C, S)$ be a cable-strut tensegrity graph on W_n for some $n \geq 5$. If $|C| = \{v_1v_2, v_3v_4\}$ where v_1, v_2, v_3, v_4 are distinct side vertices, then T is not rigid in $\mathbb{R}^2$.*

Proof: We use induction on n. The $n = 5$ case follows from Lemma 4.3.9. If $n \geq 6$, then we can find consecutive side vertices u, v, w such that uv, vw, uv_0, vv_0, wv_0 are all

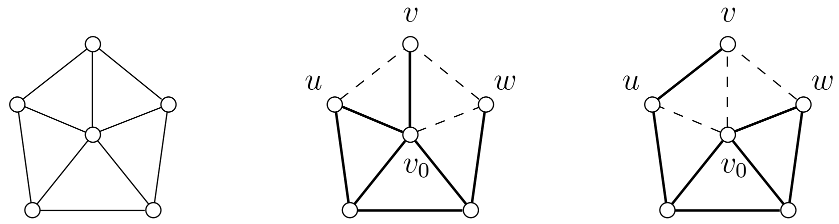

Figure 4.5: The graph W_6 and the two types of forbidden 3-paths on it.

struts. By removing v and adding the strut uw we get a cable-strut tensegrity graph T' on W_{n-1} with two independent side cables. Using the induction hypothesis we get that T' is not rigid in $\mathbb{R}^2$ and by Lemma 4.3.4 it follows that T is not rigid in $\mathbb{R}^2$. •

Given three adjacent side vertices u, v, w of W_n, a *forbidden* 3*-path* of W_n is defined as either $uvwv_0$ or uv_0vw (see Figure 4.5).

Lemma 4.3.12. *Let $T = (V; C, S)$ be a cable-strut tensegrity graph on W_n for some $n \geq 5$. If $|C| = 3$ then T is rigid in $\mathbb{R}^2$ if and only if the cables do not form a forbidden* 3*-path.*

Proof: To prove necessity we use induction on n. The $n = 5$ case follows from Lemma 4.3.9. If $n \geq 6$, then we can find consecutive side vertices u, v, w such that uv, vw, uv_0, vv_0, wv_0 are all struts. By removing v and adding the strut uw we get a cable-strut tensegrity graph T' on W_{n-1} with a forbidden cable 3-path. Using the induction hypothesis we get that T' is not rigid in $\mathbb{R}^2$ and by Lemma 4.3.4 it follows that T is not rigid in $\mathbb{R}^2$.

We prove sufficiency for all $n \geq 4$ with induction on n. The $n = 4$ case follows from the characterization of rigid tensegrity graphs on $W_4 = K_4$. For $n \geq 5$ we will consider all the possible cases where $|C| = 3$ and the cables do not form a forbidden 3-path and in each case we will find a vertex on which we can apply the inverse labeled 1-extension operation to get a rigid tensegrity graph. Let us denote the number of central cables by c_1 and the number of side cables by c_2.

If $c_1 = 0$ and $c_2 = 3$, then there is a side vertex v which has at most one incident side cable. Removing v and adding a strut if v had no incident cable, and a cable otherwise we get a tensegrity graph which is rigid by the induction hypothesis.

If $c_1 = 3$ and $c_2 = 0$, then there is a side vertex v with three incident struts.

Removing v and adding a strut we get a tensegrity graph which is rigid by the induction hypothesis.

If $c_1 = 1$ and $c_2 = 2$, then let v_0v denote the central cable. If v has no incident side cables, then we find three consecutive side vertices v_1, v_2, v_3 such that v_1v_2 is a strut and v_2v_3 is a cable. Removing v_2 and adding the strut v_1v_3 we get a tensegrity graph that is rigid by Lemma 4.3.10. If v has two incident side cables then there is a side vertex w with three incident struts. Removing w and adding a strut we get a tensegrity graph which is rigid by the induction hypothesis. If v has one incident side cable vw, and u is the other neighbor of w on the side, then wu can not be a cable so removing w and adding the strut vu we get a tensegrity graph that is rigid by Lemma 4.3.10.

If $c_1 = 2$ and $c_2 = 1$, we distinguish two cases. The first case is where the side cable is not incident to any of the central cables. If v_0v is a central cable, then removing v and adding a strut we get a tensegrity graph that has a non-incident pair of central and side cables, and thus it is rigid by Lemma 4.3.10. The second case is where there is a central cable v_0v and an incident side cable vw. Let u be the other neighbor of v. Then v_0u can not be cable, since otherwise uv_0vw would be a forbidden 3-path. So v_0 and uv are struts. Moreover, if t is the neighbor of u other than v, then ut is also strut. Now removing u and the strut vt we get a tensegrity graph that is rigid by the induction hypothesis. •

Lemma 4.3.13. *Let $T = (V; C, S)$ be a cable-strut tensegrity graph on W_n for some $n \geq 6$ and $|S| \geq |C| \geq 4$. Then T is rigid in $\mathbb{R}^2$.*

Proof: We use induction on n. For the $n = 6$ case we use the following claim, which can be verified by considering all cable-strut tensegrity graphs on W_6. We omit the details.

Claim 4.3.14. *All cable-strut tensegrity graphs on W_6 with the property that $|S| \geq |C| \geq 4$ can be constructed from a rigid cable-strut tensegrity graph on K_4 with two steps of labeled 1-extensions.*

Now let us suppose that $n \geq 7$ and $|S| \geq |C| \geq 5$. Let v be a side vertex that is incident to both a cable and a strut. By applying an inverse labeled 1-extension on v, we can get a tensegrity graph $T' = (V - \{v\}; C' \cup S')$ on W_{n-1}, such that $|S'| = |S| - 1$ and $|C'| = |C| - 1$. Using the induction hypothesis we get that T' is rigid in $\mathbb{R}^2$ and hence T is rigid in $\mathbb{R}^2$.

The remaining case to prove is when $n \geq 7$ and $|S| \geq |C| = 4$. Since there are altogether $2n - 2$ edges, this implies that $|S| \geq 8$. In this case we can find a side vertex v, such that all members incident to v are struts or v is incident to one side cable and two struts. In both cases we can apply an inverse labeled 1-extension on v to get a tensegrity graph $T' = (V - \{v\}; C' \cup S')$ on W_{n-1}, such that $|S'| = |S| - 2 \geq 6$ and $|C'| = |C| = 4$. Using the induction hypothesis we get that T' is rigid in $\mathbb{R}^2$ and hence T is rigid in $\mathbb{R}^2$. •

Note that the above Lemma is not true for $n = 5$, since the tensegrity graph (J) on Figure 4.4 has 4 cables and 4 struts, but it is not rigid in $\mathbb{R}^2$.

Putting together Lemmas 4.3.3, 4.3.10, 4.3.11, 4.3.12 and 4.3.13 we get a complete characterization of the rigid tensegrity graphs on the wheel, which can be summarized in the following theorem.

Theorem 4.3.15. *Let $T = (V; C, S)$ be a cable-strut tensegrity graph on W_n for some $n \geq 6$ and $|S| \geq |C|$. T is rigid in $\mathbb{R}^2$ if and only if $|C| \geq 4$, or $|C| = 3$ and the cables do not form a forbidden 3-path, or $C = \{v_0v, uw\}$ where u, v, w are distinct side vertices.*

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Summary

The dissertation focuses on combinatorial problems arising from various rigidity related properties of graphs and frameworks. A d-dimensional framework (G, p) is a graph $G = (V, E)$ together with a map $p : V \to \mathbb{R}^d$. We view (G, p) as a realization of G in $\mathbb{R}^d$.

Two realizations of G are equivalent if the corresponding edges in the two frameworks have the same length. A pair of vertices $\{u, v\}$ is globally linked in G if the distance between the points corresponding to u and v is the same in all pairs of equivalent generic realizations of G. The graph G is globally rigid if all of its pairs of vertices are globally linked. In the first part of the dissertation we extend the characterization of globally rigid graphs in the plane given by Jackson and Jordán by characterizing globally linked pairs in M-connected graphs (i.e. graphs with a connected rigidity matroid), and give a conjecture for the general case. We also prove that in minimally rigid graphs the only globally linked pairs are those connected by edges, which verifies the conjecture for this graph family. We use the characterization of globally linked pairs to solve the unique localizability problem for M-connected graphs, which has applications in sensor network localizations. We also determine the number of distinct realizations of an M-connected graph.

In the second part of the dissertation we consider the algorithmic problem of constructing a 2-dimensional globally rigid realization of a globally rigid graph. If the input graph is also triangle-reducible (it can be constructed from K_4 with special types of 1-extensions), then the output of the described algorithm will be infinitesimally rigid, too. By a result of Cheung and Whiteley, this means that all other frameworks in a neighborhood are globally rigid as well.

Tensegrity frameworks are defined on a set of points in $\mathbb{R}^d$ and consist of bars, cables, and struts, which provide upper and/or lower bounds for the distance between their endpoints. Tensegrity graphs are edge-labeled graphs that encode these restrictions. The third part of the dissertation focuses on two combinatorial problems related to the rigidity of tensegrity graphs in the plane. A tensegrity graph is rigid, if it has a rigid realization. First we give a polynomial algorithm to find a cable-strut labeling of the edges of a (redundantly rigid) graph that gives rise to a rigid tensegrity graph. After that, we characterize the rigid cable-strut tensegrity graphs in the plane where the underlying graph is either a complete graph, or a wheel graph.

Összefoglaló

Doktori értekezésemben gráfok és síkbeli realizációik különféle merevséggel kapcsolatos tulajdonságait vizsgálom. A G gráf egy d-dimenziós realizációján egy (G, p) párt értünk, ahol $p : V \to \mathbb{R}^d$ leképezés.

Egy G gráf két realizációja ekvivalens, ha a megfelelő élek hossza megegyezik. Egy $\{u, v\}$ csúcspár globálisan linkelt G-ben, ha távolságuk megegyezik bármely két ekvivalens generikus realizációban. Egy G gráf globálisan merev, ha bármely két csúcspárja globálisan linkelt. A dolgozat első részében kiterjesztem a síkbeli globálisan merev gráfok Bill Jacksontól és Jordán Tibortól származó karakterizációját az M-összefüggő gráfok (olyan gráfok, amelyek merevségi matroidja összefüggő) globálisan linkelt csúcspárjainak jellemzésével, és adok egy sejtést az általános esetre vonatkozóan. Ezután a sejtés speciális eseteként bebizonyítom, hogy a minimálisan merev gráfokban csakis az éllel összekötött csúcspárok lehetnek globálisan linkeltek. A globálisan linkelt csúcspárok jellemzését felhasználva megoldom az egyértelmű lokalizálhatóság kérdését M-összefüggő gráfokra, melynek fontos alkalmazásai vannak a szenzor hálózatok lokalizációjánál. Végül meghatározom az M-összefüggő gráfok különböző (nem kongruens) realizációinak számát.

A disszertáció második részében leírok egy algoritmust a 2-dimenzióban globálisan merev gráfok egy globálisan merev realizációjának megkonstruálására. Ha a bemeneti gráf háromszög-reducibilis (megkapható K_4-ből speciális típusú 1-kiterjesztésekkel), akkor az algoritmus kimenete választható infinitezimálisan merevnek is. Cheung és Whiteley egy eredménye alapján ez azt jelenti, hogy a kapott realizáció egy környezetében minden más realizáció is globálisan merev.

A gráf realizációk általánosításaként kapjuk a tensegrity szerkezeteket, ahol néhány él esetén megengedjük, hogy a hosszuk nőjön, míg más élek esetén, hogy csökkenjen. Ezeket a típusú éleket nevezzük rugóknak, illetve kábeleknek. Az ezen feltételeket elkódoló élcímkézett gráfokat nevezzük tensegrity gráfoknak. A dolgozat harmadik részében két, a tensegrity gráfok síkbeli merevségével kapcsolatos problémát vizsgálok. Egy tensegrity gráfot akkor nevezünk merevnek, ha realizálható merev tensegrity szerkezetként. Először leírok egy polinomiális algoritmust, amely megtalálja egy (redundánsan merev) gráf éleinek egy olyan kábel-rugó felosztását, amelyre a kapott tensegrity gráf merev. Végül megadom azon merev kábel-rugó tensegrity gráfok jellemzését, amelyek alapgráfja a teljes gráf vagy a kerék gráf.