Summary of the Ph.D. dissertation

Parity problems of combinatorial polymatroids

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1 Introduction

One of the most important fields of combinatorial optimization is organized around submodular functions. Another area examines the role of parity. Important examples from these two fields are the matroid intersection problem and the matching problem of graphs. As a common generalization, Lawler introduced the (poly)matroid matching problem in the early seventies. Let $f : 2^S \to \mathbb{Z}_+$ be a polymatroid function and let

$$\mathcal{P}(f) = \{x \in \mathbb{R}^S_+ : x(U) \le f(U) \text{ for every } U \subseteq S\},$$

be the associated polymatroid where $x(U) = \sum_{s \in U} x(s)$. We usually restrict ourselves to finite ground-set, the only infinite cases considered are linear polymatroids. The integer vectors of $\mathcal{P}(f)$ with everywhere even coordinates are called the *matchings*, a matching m maximizing $m(S)$ is called a *maximum matching.* For $x \in \mathbb{Z}^S_+$, let $r_f(x) = \max\{y(S) : y \le x : y \in \mathcal{P}(f)\}$ be the rank of x. The even vector c having $r_f(c) = f(S)$ is called a *cover.* Let

$$\begin{aligned}\nu(f) &= \max\{m(S)/2 : m \text{ is a matching of } f\},\\ \varrho(f) &= \min\{c(S)/2 : c \text{ is a cover of } f\}.\end{aligned}$$

The problem is to determine $\nu(f)$ (equivalently $\varrho(f)$) in the sense of good characterization, as well as in algorithmic sense.

If $\mathcal{M}_i$, $i \in \{1, 2\}$ are matroids with rank functions r_i resp., then the matchings of the polymatroid function $r_1 + r_2$ correspond to the common independent sets of $\mathcal{M}_1$ and $\mathcal{M}_2$. If $G = (V, E)$ is an undirected graph, then let $q : 2^E \to \mathbb{Z}$, $q(F) = |\bigcup F|$ for every $F \subseteq E$, i.e., the number of end-vertices of F. Again, the matchings of the graph G correspond to the polymatroid matchings of q.

The task of computing $\nu(f)$ is not tractable in general (Jensen and Korte [7], Lovász [9]), however, if the polymatroid is linear then $\nu(f)$ has a good characterization (Lovász [9]). Lovász has shown also an algorithm for linearly represented polymatroids. It is easy to see, that $\nu(f) \le \left\lfloor \frac{f(S)}{2} \right\rfloor$, $\nu(f|_{2^A}) \le \sum_{j=1}^{t} \nu(f|_{2^{A_j}})$ where $A_1, \dots, A_t$ is a partition of $A \subseteq S$, and $\nu(f) \le r_f(z) + \nu(f/z)$ for $z \in \mathbb{Z}^S_+$. Hence:

(1)

$$\nu(f|_{2^A}) \leq \min\left\{\sum_{j=1}^{t}\left\lfloor\frac{f(A_j)}{2}\right\rfloor : A_1,\dots,A_t \text{ is a partition of } A\right\},$$

(2)

$$\nu(f|_{2^A}) \leq \min\left\{r_f(z) + \sum_{j=1}^{t}\left\lfloor\frac{(f/z)(A_j)}{2}\right\rfloor : z \in \mathbb{Z}_+^S, A_1,\dots,A_t \text{ is a partition of } A\right\}.$$

Polymatroids having equality in (2) has a wider literature, this property is a peculiarity of polymatroids related to the double circuit property. The simpler inequality (1) is also interesting, as we proved in [15] that polymatroids having no non-trivial compatible double circuits have equality here. The dissertation presents new classes of combinatorial polymatroids having equality either in (1) or in (2). The results imply a range of graph theoretical applications.

2 Preliminaries

This part discusses the preliminary notions and statements which are necessary for the comprehension. The quantity $x(S) - r_f(x)$ is called the deficiency of $x \in \mathbb{Z}_+^S$. The 1-deficient vectors are called *flowers*, and a 1-deficient vector $c \in \mathbb{Z}_+^S$ is called a *circuit* if $c - \chi_s \in \mathcal{P}(f)$ for every $s \in \operatorname{supp}(c)$. The 2-deficient vectors are called *double flowers*, and a 2-deficient vector $y \in \mathbb{Z}_+^S$ is said to be a *double circuit* if $y - \chi_s$ is 1-deficient for every $s \in \operatorname{supp}(y)$. Hence, a (double) flower x has a unique (double) circuit $c \leq x$. We speak about a *compatible double circuit (CDC)* if for every $s \in \operatorname{supp}(y)$ the support of the unique circuit of $y - \chi_s$ does not contain s. If y is a CDC, then $\operatorname{supp}(y)$ has a unique partition $U_1, U_2, \dots, U_d$ called *principal partition*, s.t. $y_i = y|_{\operatorname{supp}(y)-U_i}$, $i = 1, 2, \dots, d$ are exactly the circuits of y. If $d \geq 3$ then the CDC is said to be non-trivial (NTCDC for short). Double flowers are called compatible or non-trivial according to the properties of their double circuits. The characterizations to $\nu(f)$ are based on the following decomposition theorem:

Theorem 2.1 (Lovász [9]). *If $f : 2^S \to \mathbb{Z}_+$ is a polymatroid function then at least one of the following cases holds:*

(3i) $f(S) = 2\nu(f) + 1$.

(3ii) *There exists a partition* $S = S_1 \dot\cup S_2$, $S_i \neq \emptyset$ *s.t.* $\nu(f) = \nu(f|_{2^{S_1}}) + \nu(f|_{2^{S_2}})$.

(3iii) *There exists* $s \in \bigcap\{\mathrm{sp}_f(m) : m$ *is a maximum matching*$\}$ *s.t.* $f(s) > 0$.

(3iv) f *has a non-trivial compatible double flower* x *with even coordinates s.t.* $x(S) = 2\nu + 2$.

In the first three cases we can apply natural operations to reduce the problem into smaller ones. In the last case, however, a reduction is possible only for special classes of polymatroids. A possible requirement is the *double circuit property (DCP)* of Dress and Lovász [2]. The polymatroid function f has the DCP if

$$(f/z)\left(\bigcap_{i=1}^{d} \mathrm{sp}_{f/z}(y_i)\right) > 0 \tag{4}$$

holds for each NTCDC y of each contraction f/z of f.

Theorem 2.2 (Lovász [9; 10; 11], Dress and Lovász [2])**.** *For polymatroids having the DCP equality holds in (2) for every* $A \subseteq S$. *Let* $\mathcal{M}$ *be a matroid with ground set* E, $A \subseteq 2^E$, *and let* $f : 2^{E \cup A} \to \mathbb{Z}_+$, $f(F \cup B) = r_{\mathcal{M}}(F \cup \bigcup B)$ *for* $F \subseteq E$ *and* $B \subseteq A$. *If* $\mathcal{M}$ *has the DCP, then we have*

$$\nu(f|_{2^A}) = \min\left(f(Z) + \sum_{j=1}^{t} \left\lfloor \frac{(f/Z)(A_j)}{2} \right\rfloor\right), \tag{5}$$

where the minimum is taken for all $Z \subseteq E$ *and for all partitions* $A_1, \dots, A_t$ *of* A.

If the ground set of a linear polymatroid contains the full linear space, then it has the DCP (Lovász). Though the polymatroids defined by combinatorial problems and those that we meet in daily life are unlikely to be non-linear, the min-max relation of obtained by applying Theorem 2.2 to polymatroids containing the full linear space does not reflect the combinatorial nature of the problem. The obstacle is that Z can be any linear subspace. Moreover, it may happen that we are not aware of a deterministic algorithm to compute a linear representation over a field in which the elementary operations can be carried out efficiently.

Except in part 6 where we are dealing with NTCDC-free polymatroids, our general aim is to show some new polymatroid constructions suggested by combinatorial applications for which an embedding into a DCP polymatroid is possible, which polymatroid is combinatorial and does not contain the full linear space.

3 (k, l)-matroids

Important classes of DCP matroids are the graphic matroids of complete graphs and the transversal matroids satisfying a certain density condition ([2; 9]). These have natural generalizations, the class of (k, l)-matroids (a.k.a. count matroids) as follows. The ground-set of $\mathcal{M}_{k,l}(H)$ is the edge-set of the hypergraph $H = (V, E)$, and $F \subseteq E$ is independent if and only if $|F'| \le k|\bigcup F'| - l$ for every $F' \subseteq F$.

Pseudomodularity [1; 6] implies that $\mathcal{M}_{k,l}(H)$ has the DCP if $k = l = 1$ and H has an edge between each pair of vertices, or if $k = 1$, $l = 0$ and each subset of V is a hyperedge with large enough multiplicity. By a construction of Iwata on the homomorphic map, a formula can be given for the case $k = l \ge 2$. For other (k, l)'s we do not know a solution different from the following. We have proved in [14] that if H is dense enough, then $\mathcal{M}_{k,l}(H)$ has the double circuit property:

Theorem 3.1 ([14]). *$\mathcal{M}_{k,l}(H)$ has the DCP if*

(6) $\qquad r_{\mathcal{M}_{k,l}(H)}(E[X]) = k|X| - l$ *holds for each set* $X \subseteq V,\ k|X| - l \ge 0.$

Theorem 3.2 ([14]). *Let $l = ck + d$ where c, d are integers with $0 \le d < k$. Then, (6) holds if E contains*

(7i) all the subsets of V of size $c + 1$ with multiplicity at least $k - d$, and

(7ii) all the subsets of V of size $c + 2$ with multiplicity at least $cd + d - ck$.

The Berge-Tutte formula and the formula for transversal matroid matching are special cases. Hypergraphic matroid parity, Lovász' theorem on triangle cacti is also a consequence. Let us mention new applications of Theorem 3.1 and 3.2.

If $k = 2$, $l = 3$, then we get the 2-dimensional generic rigidity matroid. Let $G = (V, E)$ be a graph, $A_1 \subseteq \binom{E}{2}$, $A_2 \subseteq \binom{\binom{V}{2}}{2}$. If G is generically rigid, then $\nu(r_{\mathcal{M}}|_{2^{A_1}})$ is the largest number of edge-pairs of A_1 which are in a minimally generically rigid subgraph of G. If G is not generically rigid but $(V, E \cup \bigcup A_2)$ is, then $\varrho(r_{\mathcal{M}/E}|_{2^{A_2}})$ gives the minimum number of pairs of A_2 which makes G generically rigid.

Next, let $G = (V, E)$ be an undirected graph, and let $A \subseteq \binom{V}{2}$ be a set of extra edges. What is the minimum size of a set $B \subseteq A$ s.t. the graph obtained from G by contracting the members of B has k edge-disjoint spanning trees? The question is trivial for $k = 1$, NP-hard if k is part of the input. The interesting is that for $k = 2$ the problem reduces to the parity problem of a $(2, 2)$-matroid.

The study of the parity of (k,l)-matroids is due to the 3-dimensional generic rigidity matroid. We are not aware of a good characterization of the rank of the latter matroid, nor an algorithm computing its rank. Jackson and Jordán proved that if $G=(V,E)$ is an undirected graph and each set $X\subseteq V$, $|X|\geq 2$ spans at most $\frac{5|X|-7}{2}$ edges, then E is independent in the 3-dimensional generic rigid matroid. Hence they gave a relatively large subset of the family of independent sets. The maximum size of such an edge set is a parity problem in a $(5,7)$-matroid.

4 Solid polymatroids

The Dilworth truncation of intersecting submodular functions define polymatroids in a natural way. By extracting the most important properties of (k,l)-matroids, we can construct an abstract class of polymatroids having the DCP, with interesting applications. The results of this part are presented in [13].

(8i) Let S be a finite ground-set, let $\emptyset\in\mathcal{L}\subseteq 2^S$ be a family which is closed under taking intersections, and $\bigcup\mathcal{L}=S$. Let $b:\mathcal{L}\to\mathbb{Z}_+$, $b(\emptyset)=0$ be a function having the following *intersecting submodular* property. Let us suppose that, if $U_1,U_2\in\mathcal{L}$ with $U_1\cap U_2\neq\emptyset$, then there exists a member of $\mathcal{L}$ denoted by $U_1\vee U_2$ s.t. $U_1\cup U_2\subseteq U_1\vee U_2$, and

$$b(U_1)+b(U_2)\geq b(U_1\cap U_2)+b(U_1\vee U_2).$$

Then, $\widehat{b}:2^S\to\mathbb{Z}_+$,

$$\widehat{b}(U)=\min_{\mathcal{F}\subseteq\mathcal{L}-\{\emptyset\},\ U\subseteq\bigcup\mathcal{F}}\sum_{U_i\in\mathcal{F}}b(U_i)\tag{9}$$

is a polymatroid function. Let $\mathcal{F}_U$ be a family among those giving equality in (9) which is “maximal in a sense”. The requirements that imply the DCP of $\widehat{b}$ are:

(8ii) Let us suppose that if $U\in\mathcal{L}-\{\emptyset\}$, then $|\mathcal{F}_U|=1$.

(8iii) Let $U_1,U_2,U_3\in\mathcal{L}$ be s.t. $b(U_{i,j})>0$ for every $U_{i,j}\in\mathcal{L}$ with $U_i\cap U_j\subseteq U_{i,j}$, $1\leq i<j\leq 3$. Then, we suppose the existence of a member of $\mathcal{L}$ denoted by $\sqcup(U_1,U_2,U_3)$ s.t. $U_1\cup U_2\cup U_3\subseteq\sqcup(U_1,U_2,U_3)$, and

$$\sum_{1\leq i<j\leq 3}b(U_i\cap U_j)+b(\sqcup(U_1,U_2,U_3))\leq\sum_{i=1}^{3}b(U_i)+b(U_1\cap U_2\cap U_3).\tag{10}$$

The quintuplet $(S, \mathfrak{L}, b, \vee, \sqcup)$ and also $\widehat{b}$ is said to be *solid* if (8i-8iii) is satisfied.

Theorem 4.1. *The class of solid polymatroids is closed under taking contractions. Prematroids of solid polymatroids are solid. Solid polymatroids have the DCP.*

(k, l)-matroids satisfying (6) are solid. Let us consider other applications to matchings of solid polymatroids. Lovász has formalized Mader's vertex-disjoint $\mathcal{A}$-paths problem [12] as a parity problem, and derived Mader's formula by characterizing the NTCDC's and following a non-constructive approach. Schrijver represented the arising polymatroid which yielded therefore an algorithm to Mader's problem. However, the algorithm does not have a combinatorial manner, it does not give the combinatorial dual solution. We have shown in [13] that the polymatroid in point has a solid embedding, and a possibility of deriving Mader's result. If there would be a parity algorithm for DCP or solid matroids, it would give a new combinatorial algorithm for Mader's problem.

5 Parity constrained connectivity orientations

Parity constrained orientation problems have a central role in the thesis. The goal is to characterize the hypergraphs admitting an orientation which meets some connectivity requirement and has in addition prescribed parity of out-degree for each vertex. Let $H = (V, E)$ be a hypergraph (not having the empty set as a hyperedge). While dealing with orientations, by a hyperedge $e \in E$ we always mean a multiset, i.e. the vertex v is contained in e with multiplicity $e(v)$. By an *orientation* $\vec{H}$ of H we mean that for each hyperedge $h \in E$ a vertex $v \in h$ is chosen which is called the *head* of h and the vertices of the multiset $h - \chi_v$ are the *tails*. A hyperedge h *enters* X if X contains the head of h and does not contain all of its tails, and h *leaves* X if it enters $V - X$. The set of hyperedges entering and leaving X is denoted resp. by $\delta^{in}_{\vec{H}}(X)$ and $\delta^{out}_{\vec{H}}(X)$.

The connectivity requirement is prescribed by the function $p : 2^V \to \mathbb{Z}_+$, s.t. $p(\emptyset) = p(V) = 0$. An orientation $\vec{H}$ of H is said to *cover* p if $\delta^{in}_{\vec{H}}(X) \geq p(X)$ for every $X \subseteq V$. We say that the function $p : 2^V \to \mathbb{Z}_+$ is *intersecting supermodular* if

$$p(X) + p(Y) \geq p(X \cap Y) + p(X \cup Y) \tag{11}$$

for every $X, Y \subseteq V$ with $X \cap Y \neq \emptyset$. Similarly, $p : 2^V \to \mathbb{Z}_+$ is *co-intersecting supermodular* if (11) holds for $X, Y \subseteq V$ with $X \cup Y \neq V$, and p is *crossing supermodular* if (11) holds for every $X, Y \subseteq V$ with $X \cap Y \neq \emptyset$ and $X \cup Y \neq V$.

Let $g : 2^V \to \mathbb{Z}$, $g(X) = \sum_{e \in E} e(X) - |E[X]| - p(X)$. It is clear, that if $x \in \mathbb{Z}^V$ is the out-degree vector of an orientation covering p, then

$$x \geq 0, \tag{12}$$

$$x(X) \leq g(X), \quad \text{for every } \emptyset \neq X \subseteq V, \tag{13}$$

$$x(V) = g(V), \tag{14}$$

moreover, if $x \in \mathbb{Z}_+^V$ satisfies (12-14), then H has an orientation covering p s.t. the out-degree of each vertex $v \in V$ is $x(v)$. If $g \geq 0$, and p is crossing supermodular, then (12-14) either determines an empty polyhedron or a base polyhedron. The existence of an orientation can be derived from the "theory of submodularity":

Theorem 5.1 (Frank, Király, and Király [5]). *Let $H = (V, E)$ be a hypergraph and let $p : 2^V \to \mathbb{Z}_+$, $p(\emptyset) = p(V) = 0$. If p is intersecting supermodular, then H has an orientation covering p if and only if*

$$g(V) \leq \sum_{j=1}^{t} g(X_j) \tag{15}$$

holds for every partition $X_1, X_2, \ldots, X_t$ of V. If p is co-intersecting supermodular, then H has an orientation covering p if and only if

$$(t-1)g(V) \leq \sum_{j=1}^{t} g(V - X_j) \tag{16}$$

holds for every partition $X_1, X_2, \ldots, X_t$ of V. If p is crossing supermodular, then H has an orientation covering p if and only if (15) and (16) hold for every partition.

Now we turn to the parity constrained case. Frank, Sebő, and Tardos have shown that if p can be positive only on singletons and complements of singletons, then the problem reduces to the matching problem of graphs.

Frank, Jordán, and Szigeti [4] examined the existence of a rooted k-edge-connected orientations of graphs. We have shown that this case, and even its hypergraph version reduces to the parity problem of a solid polymatroid. This methods allows us to put also lower and upper bounds on the out-degrees. Király and Szabó gave a surprising generalization [8]. They proved the sufficiency of the natural partition formula when p is an intersecting supermodular function.

Theorem 5.2 (Király and Szabó [8]). *Let $H = (V, E)$ be a hypergraph, let $p : 2^V \to \mathbb{Z}_+$ be an intersecting supermodular function with $p(\emptyset) = p(V) = 0$, and finally $T \subseteq V$. Then, H has an orientation covering p with odd out-degrees exactly in the vertices of T, if and only if*

$$g(V) \leq \sum_{j=1}^{t} g(X_j) - |\{j : g(X_j) \not\equiv |T \cap X_j|\}| \tag{17}$$

holds for every partition $X_1, X_2, \ldots, X_t$ of V.

From the viewpoint of polymatroid parity, this problem behaves completely different from the previous ones, and is discussed in the next part.

Very little is known about parity constrained orientation problems when p is crossing supermodular. An important open problem is to characterize the graphs admitting a strongly connected orientation with even out-degrees. It is not hard to see that the problem is random polynomial. In the dissertation we give a combinatorial characterization in the special case of planer graphs by showing that the out-degree sequences of good orientations has a solid embedding.

6 Polymatroids without NTCDCs

The results of this part are from [15]. We have already seen that some parity constrained connectivity orientation problems reduce to parity of solid polymatroids. We have mentioned also that Theorem 5.2 is unlikely to do so as the polymatroid of degree sequences of the orientations covering p does not satisfy any algebraic property that would imply the DCP. We need a different approach. We have shown in [15] that if p is intersecting submodular, $g \geq 0$, and g is non-decreasing on non-empty sets, then the polymatroid described by (12-13) has no NTCDCs. Moreover, the partition formula characterizes $\nu(f)$ for polymatroids having no NTCDCs:

Theorem 6.1 (M. and Szabó [15]). *Let $f : 2^S \to \mathbb{Z}_+$ be a polymatroid function without NTCDCs. Then, equality holds in (1) for every $A \subseteq S$.*

Then, Theorem 5.2 is an easy consequence. There are other application of polymatroids without NTCDCs. While the natural polymatroid parity formulations of the ordinary matching problem of graphs admit NTCDCs, there are also formulations without NTCDCs.

Another application is Fekete's [3] following problem. Let $l \in \{2,3\}$, $G = (V, E)$ be an undirected graph, and for $Z \subseteq V$, let K_Z be the graph with vertex-set Z having $4-l$ parallel edges between any two vertices. Let $r(E) < 2|V|-l$ where r is the rank function of $\mathcal{M}_{2,l}$. Then, we can ask for the minimum $Z \subseteq V$ s.t. $G+K_Z$ has rank $2|V|-l$. For $l = 2$, this is equivalent to shrinking a minimum vertex set Z into one vertex s.t. G/Z has two edge-disjoint spanning trees. Loosely speaking, for $l = 3$, this is the problem of pinning down a minimum set Z generically, s.t. the resulting graph is generically rigid. The latter is a generic version of Lovász' pinning down problem [9]. Fekete proved that both problems reduce to the matching problem of graphs. We have shown that the arising polymatroids are in fact NTCDC-free, therefore his formulas are immediate consequences of our theory.

7 Open questions

As DCP polymatroids and polymatroids having no NTCDCs are not sub– nor superclasses of each other, it is an interesting question whether they admit a common generalization with tractable parity problem. It is also open that under what extent can we solve the parity problems of DCP polymatroids algorithmically.

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