# CLOSED FORMS AND TROTTER'S PRODUCT FORMULA

by

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# Introduction

The theory of Hilbert space operators witnessed a major progress with the work of J. von Neumann when he started to use systematically the notions of adjoints, graphs, and functions of operators. One reason behind his success was the observation that bounded sesquilinear forms and bounded linear operators on a complex Hilbert space are, in fact, the same. He applied these techniques with great success to the theory of unbounded self-adjoint operators, as well. Later, however, forms have come back into favour in certain situations. Namely, in the unbounded case, closed, positive forms provide a convenient way to define positive self-adjoint operators via the form representation theorem. The basic applications of this idea have been manifested in the Friedrichs extension of densely defined positive symmetric operators and the form sum construction of two appropriate positive self-adjoint operators. The form sum of two positive self-adjoint operators was later connected to the convergence of Trotter's product formula by a result of Kato. This dissertation presents a collection of my results from this circle of ideas.

The present dissertation is based on the author's papers [14] , [15] , [21], [22]. Papers [14] and [15] consist of results of a joint research with Bálint Farkas at the Department of Applied Analysis, ELTE. Paper [22] is the result of a joint research with Roman Shvidkoy originating at the Internet Seminar Workshop, Blaubeuren, 2001. Paper [21] contains results of the author accomplished during his stay at the University of Ulm with the Marie Curie Host Fellowship. Many other related results are also included. References are given to the best of the author's knowledge.

In Chapter 1 we describe a factorization theorem for positive self-adjoint operators establishing a connection between form methods and operator methods. This construction is due to Z. Sebestyén. It has been applied successfully to many problems both in bounded and semibounded case. Some recent applications, related to subsequent results of the dissertation, are also included.

In Chapter 2 we apply the basic construction of Chapter 1 to the addition problem of positive, symmetric operators. We arrive at a generalized notion of the form sum construction. A commutation property of this sum with bounded operators is proved. We also describe some pathological phenomena concerning the addition of positive self-adjoint operators.

In Chapter 3 we consider closed, positive forms on reflexive Banach spaces. We examine which of the Hilbert space results can be carried over to this general case.

In Chapter 4 we describe the result of Kato which gives a connection between the form sum of two operators and Trotter's product formula. We apply this result to the special case when one of the semigroups is replaced by a bounded orthogonal projection (which can be regarded as a degenerate semigroup). The convergence of Trotter's formula for projections is then further investigated. Some positive results and counterexamples are given.

Chapter 5 contains a similarity result which will be needed subsequently in the characterization of the convergence of Trotter's product formula for projectons. This general similarity result is of independent interest.

Finally, Chapter 6 contains the characterization of the convergence of Trotter's formula for projections in terms of properties of the generator. The result proves, in a sense, the converse of Kato's result.

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# Chapter 1

# Factorization of positive operators

This chapter is of introductory character. It describes the basic construction, due to Z. Sebestyén, which will be indispensable in the course of Chapters 2 and 3. Some applications of the construction, which are closely related to results of Chapters 2 and 3, are also included. In most cases only the outline of the proof is presented, while references are made as to where the detailed proof can be found.

## 1.1 Factorization over an auxillary Hilbert space

Let $H$ denote, here and throughout this dissertation, a complex Hilbert space. The space of bounded linear operators on $H$ will be denoted by $\mathcal{B}(H)$. Let $A$ be a positive, self-adjoint operator (bounded or unbounded), i.e. $A = A^*$ and $(Ax, x) \geq 0$ holds true for all $x \in \operatorname{dom} A$, the domain of the operator $A$.

We construct an auxillary Hilbert space in order to factorize the operator $A$. Define a new scalar product $[\ ,\ ]$ on the range of $A$ by $[Ax, Ay] := (Ax, y)$. It is well defined because if $x_1, x_2, y_1, y_2 \in \operatorname{dom} A$ and $Ax_1 = Ax_2,\ Ay_1 = Ay_2$ then we have $(Ax_1, y_1) = (Ax_2, y_1) = (x_2, Ay_1) = (x_2, Ay_2) = (Ax_2, y_2)$. Also, it is positive definite because $(Ax, x) = 0$ implies $A^{\frac{1}{2}}x = 0$ and therefore $Ax = 0$. Hence $\operatorname{ran} A$, the range of the operator $A$, equipped with the scalar product $[\ ,\ ]$ is a pre-Hilbert space. The completion space of this pre-Hilbert space will be denoted by $H_A$.

There is a natural (identification) mapping $J$ of $\operatorname{ran} A$ (as a subspace of $H_A$) into the original Hilbert space $H$ defined by $Jx = x$ ($x \in \operatorname{ran} A$). As the operator $J : H_A \to H$ is densely defined, the adjoint $J^* : H \to H_A$ exists. For $x \in \operatorname{dom} A$ we have

$$|(J(Ay), x)| = |(Ay, x)| \leq (Ay, y)^{\frac{1}{2}}(Ax, x)^{\frac{1}{2}} = [Ay, Ay]^{\frac{1}{2}}(Ax, x)^{\frac{1}{2}}$$

which means that $x \in \operatorname{dom} J^*$. Hence $J^*$ is also densely defined, and therefore $J^{**}$ exists. Furthermore, for $x \in \operatorname{dom} A$, $(J(Ay), x) = (y, Ax)$, hence $J^*x = Ax$. The operator $J^{**}J^* : H \to H$ is positive, self-adjoint by von Neumann's theorem. Furthermore, for all $x \in \operatorname{dom} A$, $J^{**}J^*(x) = J^{**}(Ax) = J(Ax) = Ax$, that is, the operator $J^{**}J^*$ is a positive self-adjoint extension of $A$. This means that $J^{**}J^* = A$ since $A$ is self-adjoint itself.

We remark that it is not necessary to consider the operator $J^{**}$ at this point. The operator $JJ^*$ is a positive symmetric extension of $A$, therefore $JJ^* = A$ holds also. In Section 1.2, however, we will need the operator $J^{**}$ instead of $J$. For the sake of unified treatment the operator $J^{**}$ is introduced already at this point.

The factorization $J^{**}J^* = A$ implies, by general theory, that $\operatorname{dom} J^* = \operatorname{dom} A^{\frac{1}{2}}$, where $A^{\frac{1}{2}}$ is the unique positive self-adjoint square root of the operator $A$. Moreover, for all $y \in \operatorname{dom} J^*$ we have

$$\|A^{\frac{1}{2}}y\|^2 = \|J^*y\|^2 = \sup\,\{|(Ax, y)|^2 : x \in \operatorname{dom} A, (Ax, x) \leq 1\} \tag{1.1}$$

Therefore we can identify the closed quadratic form corresponding to $A$ in terms of the auxillary operator $J^*$. This fact highlights one major advantage of this construction: it establishes a connection between the 'form approach' and the 'operator approach'.

This factorization argument, with appropriate modifications, has led to various results concerning positive operators. Some of the applications of this argument are included here, and some other will appear in Chapters 2 and 3.

Assume first that $A$ is bounded. The following theorem is taken from [25]. It illustrates the advantages of the definition of the auxillary Hilbert space $H_A$, and, at the same time, the factorization of $A$ over $H_A$.

**Theorem 1.1.1** *Let $A \in \mathcal{B}(H)$ be a positive, self-adjoint operator on the Hilbert space $H$. Assume that $B \in \mathcal{B}(H)$ has no negative real numbers in its spectrum, and that the product $AB$ is self-adjoint. Then $AB$ is automatically positive.*

**Proof.** We only include the sketch of the proof, see [25] for details.

Define an operator $\hat{B} : H_A \to H_A$ by $\hat{B}(Ax) := A(Bx)$, $(\operatorname{dom} \hat{B} = \operatorname{ran} A \subset H_A)$. It is not hard to show that $\hat{B}$ is well-defined, symmetric, and bounded on $\operatorname{ran} A \subset H_A$ (cf. Lemma 2.2.1 and 2.2.2). The continuous extension to $H_A$ is also denoted by $\hat{B}$. It is easy to prove that the inclusion of spectrums $\operatorname{Sp} \hat{B} \subset \operatorname{Sp} B$ holds (cf. Theorem 2.2.3).

Furthermore, the factorization $A = J^{**}J^*$, shows that $AB = J^{**}\hat{B}J^*$ holds: $J^{**}\hat{B}J^*x = J^{**}\hat{B}(Ax) = J^{**}A(Bx) = ABx$ for all $x \in H$.

By assumption, the spectrum of $B$ does not contain negative reals. Therefore we see from the inclusion of the spectrums that $\hat{B}$ is positive, self-adjoint. Hence, the factorization $AB = J^{**}\hat{B}J^*$ gives the desired result. □

**Remark** In [25] the result above is stated for bounded positive operators $A$ only. However, the proof applies to the case of unbounded, positive, self-adjoint operators $A$, as well. Indeed, $(B^*A)^* = AB$, therefore $(B^*A)^{**} = \overline{B^*A} = AB$, by the assumption that $AB$ is self-adjoint. This means that $B^*A$ is essentially self-adjoint, and is a core of $AB$. Hence, it is enough to prove that $B^*A$ is positive. This, however, follows from the fact that $B^*A \subset J^{**}\hat{B}J^*$.

## 1.2 Operator extensions

Next, we turn to the application of the factorization construction in the theory of positive, self-adjoint extensions of positive symmetric operators. The statements of the following theorem appeared in [3] and [26].

**Theorem 1.2.1** *Let $a : H \to H$ be a positive linear operator defined on a (not necessarily dense) subspace $D := \operatorname{dom} a$. The following are equivalent:*

*(i) $a$ can be extended to a positive, self-adjoint operator $A$ in $H$.*

*(ii) The set* $\mathrm{D}_*(a) := \{y \in H : \sup\{|(ax, y)|^2 : x \in D, (ax, x) \leq 1\} < \infty\}$ *is dense in* $H$.

*The operator $a$ has a bounded positive extension $A$ on $H$ if and only if* $\mathrm{D}_*(a) = H$, *which occurs if and only if there exists a constant $m \geq 0$ such that $\|ax\|^2 \leq m(ax, x)$ for all $x \in D$. In this case there exists a bounded positive extension of $a$ whose norm is* $\inf\{m : \|ax\|^2 \leq m(ax, x),\ x \in D\}$.

**Proof.** The proof relies on slight modifications of the basic factorization argument presented at the beginning of the chapter. We only include the main points of the argument here, see [26] for full details.

The implication (i) $\rightarrow$ (ii) follows from the inclusions $\mathrm{dom}\, A \subset \mathrm{D}_*(A) \subset \mathrm{D}_*(a)$ which clearly holds for any positive, self-adjoint extension $A$ of the given operator $a$.

For the proof of (ii) $\rightarrow$ (i) the auxillary space $H_a$ is defined analogously as at the beginning of the chapter.

The scalar product $[ax, ay] := (ax, y)$ is well defined on $\mathrm{ran}\, a$ because $a$ is symmetric. The positive definity of $[\ ,\ ]$ follows from the positivity of $a$ and the assumption that $\mathrm{D}_*(a)$ is dense: indeed, if $(ax, x) = 0$ for some $x \in D$, then for all $y \in \mathrm{D}_*(a)$ we have $(ax, y) = 0$, therefore $ax = 0$. Define $J : H_a \rightarrow H$ as before: $\mathrm{dom} J := \mathrm{ran} a$, and $Jx = x$. It is clear from the definition of adjoint operators that $\mathrm{dom}\, J^* = \mathrm{D}_*(a) \subset H$. It is also clear that $D \subset \mathrm{dom}\, J^*$ and $J^*x = ax$ for all $x \in D$. Now, $\mathrm{dom}\, J^* = \mathrm{D}_*(a)$ is assumed to be dense, therefore $J^{**}$ exists. Finally, it is easy to check that the positive, self-adjoint operator $a_K := J^{**}J^*$ is an extension of $a$. Indeed, $J^{**}J^*x = J^{**}(ax) = J(ax) = ax$ for all $x \in D$.

Also, we see from the factorization that

$$\mathrm{dom}\, a_K{}^{\frac{1}{2}} = \mathrm{dom}\, J^* = \mathrm{D}_*(a), \tag{1.2}$$

$$\|a_K{}^{\frac{1}{2}}y\|^2 = \|J^*y\|^2 = \sup\ \{|(ax, y)|^2 : x \in \mathrm{dom}\, a, (ax, x) \leq 1\} \tag{1.3}$$

holds. Furthermore, the operator $J$ is bounded if and only if there exists a constant $m \geq 0$ such that $\|ax\|^2 \leq m(ax, x)$ for all $x \in D$

The statements concerning bounded positive extensions of $a$, as described in [26] in detail, are fairly straightforward from the construction above. □

We introduce the classical partial ordering of positive, self-adjoint operators as follows:

**Definition 1.2.2** Let $A$ and $B$ be positive, self-adjoint operators on $H$. We say that $A \leq B$ if and only if $\mathrm{dom}\, B^{\frac{1}{2}} \subset \mathrm{dom}\, A^{\frac{1}{2}}$ and $(A^{\frac{1}{2}}x, A^{\frac{1}{2}}x) \leq (B^{\frac{1}{2}}x, B^{\frac{1}{2}}x)$ for all $x \in \mathrm{dom}\, B^{\frac{1}{2}}$.

The construction in the proof of the theorem above distinguishes itself by being the smallest positive self-adjoint extension of $a$ (see [26]):

**Corollary 1.2.3** *Let $a : H \to H$ be a positive linear operator defined on a (not necessarily dense) subspace $D := \mathrm{dom}\, a$. Assume that $a$ posesses at least one positive, self-adjoint extension. Then the set of all positive, self-adjoint extensions of $a$ contains a smallest element. The smallest extension is provided by the construction of Theorem 1.2.1, i.e. $a_K = J^{**}J^*$.*

**Proof.** Let $A_1$ be any positive self-adjoint extension of $a$. Then

$$\sup\{|(ax, y)|^2 : x \in \mathrm{dom}\, a, (ax, x) \leq 1\} \leq \sup\{|(A_1x, y)|^2 : x \in \mathrm{dom}\, A_1, (A_1x, x) \leq 1\}$$

and this implies the statement, because the left hand side is the form of $a_K$ and the right hand side is the form of $A_1$. □

This extension, in the case when $a$ has positive lower bound, was first constructed by von Neumann. Later, it was studied in detail by Krein [20]. Hence the following

**Definition 1.2.4** Let $a : H \to H$ be a positive linear operator defined on a (not necessarily dense) subspace $D := \mathrm{dom}\, a$. Assume that $a$ posesses at least one positive, self-adjoint extension. Then $a_K = J^{**}J^*$, the smallest positive extension of $a$, is called the *Krein-von Neumann extension* of $a$.

Next we turn to the case when $\mathrm{dom}\, a$ is dense in $H$. We show that a slight modification of the factorization argument leads to the Friedrichs extension of $a$. The detailed proof of the following result can be found in [24] and [23].

**Theorem 1.2.5** *Let $a : H \to H$ be a positive linear operator defined on a dense subspace $D := \operatorname{dom} a$. Then the set of positive, self-adjoint extensions of $a$ contains a largest element, the Friedrichs extension of $a$.*

**Proof.** This theorem is well known. The customary construction of the Friedrichs extension is via the form representeation theorem. The densely defined, positive form $t[x, y] := (ax, y)$ is shown to be closable, and the Friedrichs extension of $a$ is defined as the positive self-adjoint operator associated with the closure of $t$. The maximality of the Friedrichs extension is an easy consequence (see [13] pp. 89, 90). It is also a consequence of this construction that the Friedrichs extension is the only positive selfadjoint extension of $a$ such that the domain of its square root is the same as the domain of the closure of the form $t$.

Here we give a factorization of the Friedrichs extension in the spirit of Theorem 1.2.1. First we note that $\operatorname{dom} a \subset \mathrm{D}_*(a)$ always holds, hence $\mathrm{D}_*(a)$ is dense in $H$, therefore it is possible to construct the auxillary Hilbert space $H_a$. We define an operator $Q : H \to H_a$ by $\operatorname{dom} Q = D$ and $Qx = ax$ for all $x \in D$. It is not hard to show that $J \subset Q^*$, and that $a_F := Q^* Q^{**}$ is a positive self-adjoint extension of $a$. We have to show that $a_F$ is, in fact, the Friedrichs extension of $a$. It is enough to show that $\operatorname{dom} a_F^{\frac{1}{2}} = \operatorname{dom} \bar{t}$. This follows from $\operatorname{dom} a_F^{\frac{1}{2}} = \operatorname{dom} Q^{**} = \operatorname{dom} \overline{Q} = \{y \in H : \exists (x_n) \subset D, \|x_n - y\| \to 0, (a(x_n - x_m), x_n - x_m) \to 0\} = \operatorname{dom} \bar{t}$. □

We can deduce from the constructions above, that the closed form corresponding to the Friedrichs extension $a_F$ is a restriction of the closed form of the Krein-von Neumann extension $a_K$. Indeed, $Q^{**} \subset J^*$, therefore $(a_F^{\frac{1}{2}} x, a_F^{\frac{1}{2}} y) = [Q^{**}x, Q^{**}y] = [J^*x, J^*y] = (a_K^{\frac{1}{2}} x, a_k^{\frac{1}{2}} y)$ for all $x, y \in \operatorname{dom} a_F^{\frac{1}{2}}$. This simple observation relies on the fact that $Q \subset J^*$.

In a similar manner, it is a natural idea to examine all restrictions $R_{\mathcal{L}} := J^* \restriction \mathcal{L}$ of the operator $J^*$ to each subspace $\operatorname{dom} a \subset \mathcal{L} \subset \operatorname{dom} J^*$, and define the operators $A_{\mathcal{L}} := R_{\mathcal{L}}^* R_{\mathcal{L}}^{**}$. It is easy to see (cf. [7] Proposition 4.1.) that each $A_{\mathcal{L}}$ is a positive self-adjoint extension of $a$, and the closed form corresponding to $A_{\mathcal{L}}$ is a restriction of the form of $a_K$. Also, if $\operatorname{dom} a \subset \mathcal{L} \subset \mathcal{M} \subset \operatorname{dom} J^*$ then $A_{\mathcal{L}} \geq A_{\mathcal{M}}$ holds.

The next theorem gives a characterization of the set of positive self-adjoint extensions $A_{\mathcal{L}}$ (see [7] Theorem 4.4.):

**Theorem 1.2.6** *Let $a$ be a densely defined, positive operator in $H$, and let $A$ denote a positive self-adjoint extension of $a$. The following are equivalent:*

*(i)* $A = R_{\mathcal{L}}^* R_{\mathcal{L}}^{**}$ *for some* $\operatorname{dom} a \subset \mathcal{L} \subset \operatorname{dom} J^*$

*(ii) $A$ is an extremal extension of $a$ in the sense that* $\inf\{(A(x-y), x-y) : \ y \in \operatorname{dom} a\} = 0$ *for all* $x \in \operatorname{dom} A$.

*(iii) The form associated to $A$ is a restriction of the form associated to $a_K$.*

**Proof.** For the proof we refer to [7] Theorem 4.4. □

For further (function-theoretic) investigations of the class of extremal extensions of $a$, and some applications we refer to [7].

Now, we turn to an interesting commutation property of the Krein-von Neumann and the Friedrichs extensions of $a$. These results give the basis behind Theorem 2.2.4 in Chapter 2. We remark that the same property is not known to hold for all extremal extensions of $a$.

**Theorem 1.2.7** *Let $a : H \to H$ be a positive linear operator defined on a subspace* $D := \operatorname{dom} a$, *and assume* $\mathrm{D}_*(a)$ *is dense in $H$. Let $B$ and $C$ be bounded linear operators on $H$ leaving $D$ invariant and such that*

$$aBx = C^* ax, \quad aCx = B^* ax$$

*for all $x \in D$. Then*

$$a_K Bx = C^* a_K x, \quad a_K Cx = B^* a_K x$$

*holds for all* $x \in \operatorname{dom} a_K$.

*If $D$ is dense (ensuring the existence of $a_F$), then*

$$a_F Bx = C^* a_F x, \quad a_F Cx = B^* a_F x$$

*holds for all* $x \in \operatorname{dom} a_F$.

**Proof.** These results are non-trivial. The statement concerning the Krein-von Neumann extension appeared in [26]. It also follows from Theorem 2.2.4 in Chapter 2 on setting $a = b$.

The statement concerning the Friedrichs extension appeared in [24]. □

**Problem 1.2.8** *Is the same commutation property enjoyed by all extremal extensions $A_{\mathcal{L}}$ of $a$?*

The author remarks that the proof employed in the cases of $a_K$ and $a_F$ does not apply, and it is a conjecture of the author that counterexamples exist.

Finally we demonstrate the different extensions introduced above by a particular example (see the discussion in [2] and [7]).

**Example** Let $H := L^2[0,1]$ and $a := -\frac{d^2}{dx^2}$ with dom $a := W_0^{2,2}[0,1]$. Then $a$ is closed, positive, with lower bound $\pi^2$, and defect index dim(ker $a^*$)=2. In fact,
ker $a^*$=span $\{1, x\}$. The Friedrichs extension of $a$ is simply the Dirichlet-Laplacian: dom $a_F = W_0^{1,2}[0,1] \cap W^{2,2}[0,1]$ and the corresponding closed form is given by

$$(a_F^{\frac{1}{2}}u, a_F^{\frac{1}{2}}v) = \int_0^1 u'(x)\overline{v'(x)}dx$$

for all $u, v \in$ dom $a_F^{\frac{1}{2}} = W_0^{1,2}[0,1]$. The Krein-von Neumann extension of $a$ corresponds to the closed form

$$(a_K^{\frac{1}{2}}u, a_K^{\frac{1}{2}}v) = \int_0^1 u'(x)\overline{v'(x)}dx - (u(1) - u(0))(\overline{v(1)} - \overline{v(0)})$$

for all $u, v \in$ dom $a_K^{\frac{1}{2}} = W^{1,2}[0,1]$. The domain of $a_K$ is characterized by the folowing boundary conditions:

$$\text{dom } a_K = \{u \in W^{2,2}[0,1] : u'(0) = u'(1) = u(1) - u(0)\}$$

From the form of $a_K$ we see that $a_K$ is not the same as the Neumann-Laplacian $a_N$ and, furthermore, that the Neumann-Laplacian is not an extremal extension of $a$.

In fact, all extremal extensions of $a$ (except for $a_F$ and $a_K$) are characterized by the following boundary conditions (see [7]):

Let $c := (c_1, c_2) \in \mathbb{C}^2$ be a vector of norm 1, and define $a_c := -\frac{d^2}{dx^2}$ with

$$\text{dom} a_c := \{u \in W^{2,2}[0,1] : c_2 u(0) = c_1 u(1),\ c_1(u'(0) - u(1) + u(0)) = \overline{c}_2(u'(1) - u(1) + u(0))\}.$$

# Chapter 2

# Form sum constructions

This chapter deals with the addition problem of two positive (not necessarily self-adjoint) operators. The results of this chapter are taken from [14].

The addition problem of unbounded self-adjoint operators is highly non-trivial and has been investigated with several approaches (see e.g. [10] and [13]). In the case when both operators are positive (or semi-bounded) self-adjoint, the form sum construction is distinguished by Kato's result [18] on the convergence of Trotter's product formula. This chapter is devoted to the construction and investigation of a generalized form sum of two positive, symmetric operators. The construction is based on the method described in Chapter 1, and it reveals an 'operator approach' to the form sum construction.

Given two positive, selfadjoint operators $A$ and $B$ in the Hilbert space $H$, we may form the operator sum $A + B$ on $\operatorname{dom} A \cap \operatorname{dom} B$. However, the intersection of the domains may be zero-dimensional, and in general nothing can assure us that the sum will be a selfadjoint operator. The so-called form sum construction handles this problem if $\operatorname{dom} A^{\frac{1}{2}} \cap \operatorname{dom} B^{\frac{1}{2}}$ is dense in $H$. Define $q_A(x) = (A^{\frac{1}{2}}x, A^{\frac{1}{2}}x)$ and $q_B(x) = (B^{\frac{1}{2}}x, B^{\frac{1}{2}}x)$ two closed forms; their sum $q_A + q_B$ is a closed form on $\operatorname{dom} A^{\frac{1}{2}} \cap \operatorname{dom} B^{\frac{1}{2}}$, therefore the representation theorem provides a selfadjoint operator $C$, such that $C$ and $A + B$ coincide on $\operatorname{dom} A \cap \operatorname{dom} B$ [13]. The usual notation for the form sum of $A$ and $B$ is $A \dot{+} B$. In Section 2.1, we give a new construction of a generalized form sum of positive, symmetric operators. Section 2.2 deals with commutation properties of this construction. In the last section we give some examples concerning the form sum, and describe the relation between other extensions of operator sums.

We use the notations of Chapter 1. Throughout this chapter, unless otherwise stated, $a, b$ will denote positive, symmetric operators in the Hilbert space $H$, with not neccesarily dense domains. The characterizing properties 1.2, 1.3 of the KREIN-VON NEUMANN extension will be used frequently in this chapter.

## 2.1 The form sum

In the following we propose a new construction for the additon of two positive, symmetric operators. We show that in case of selfadjont operators this construction supplies the form sum of the operators.

Let $a$ and $b$ be two positive, symmetric operators, and suppose that $\mathrm{D}_*(a) \cap \mathrm{D}_*(b)$ is dense in $H$. This implies, a fortiori, that $\mathrm{D}_*(a)$ and $\mathrm{D}_*(b)$ are dense, so that the auxillary Hilbert spaces $H_a$, $H_b$ are possible to construct, and the corresponding KREIN-VON NEUMANN extensions $a_K$ and $b_K$ exist (cf. Theorem 1.2.1). Consider the space $H_a \oplus H_b$, and the operator

$$J : H_a \oplus H_b \to H, \text{ with } \operatorname{dom} J = \operatorname{ran} a \oplus \operatorname{ran} b, \quad J(ax \oplus by) = ax + by. \tag{2.1}$$

It is easy to prove that $J^*$ is densely defined, in fact, $\mathrm{D}_*(a) \cap \mathrm{D}_*(b) = \operatorname{dom} J^*$. To see this, let $x \in \operatorname{dom} a, y \in \operatorname{dom} b$ and $u \in \mathrm{D}_*(a) \cap \mathrm{D}_*(b)$, then

$$|(J(ax \oplus by), u)|^2 = |(ax, u) + (by, u)|^2 \leq 2|(ax, u)|^2 + 2|(by, u)|^2 \leq \\ 2m_u(ax, x) + 2n_u(by, y) \leq m[ax \oplus by, ax \oplus by],$$

with $m = 2\max(m_u, n_u)$. This shows that $u \in \operatorname{dom} J^*$, hence $\mathrm{D}_*(a) \cap \mathrm{D}_*(b) \subseteq \operatorname{dom} J^*$. For the reverse, let $u \in \operatorname{dom} J^*$ and $x \in \operatorname{dom} a$, then

$$|(ax, u)|^2 = |(J(ax \oplus \mathbf{0}), u)|^2 \leq m[ax \oplus \mathbf{0}, ax \oplus \mathbf{0}] = m[ax, ax] = m(ax, x),$$

with a suitable $m \geq 0$, therefore $u \in \mathrm{D}_*(a)$. Similarly, we obtain that $u \in \mathrm{D}_*(b)$. Thus we have shown that $\mathrm{D}_*(a) \cap \mathrm{D}_*(b) \supseteq \operatorname{dom} J^*$.

We see that $J^{**}$ exists. Now, we calculate $J^*$ on $\mathrm{dom}\, a \cap \mathrm{dom}\, b$. Let $u \in \mathrm{dom}\, a \cap \mathrm{dom}\, b$ and $x \in \mathrm{dom}\, a, y \in \mathrm{dom}\, b$, then

$$(J(ax \oplus by), u) = (ax, u) + (by, u) = [ax, au] + [by, bu] = [ax \oplus by, au \oplus bu],$$

consequently $J^* u = au \oplus bu$.

According to the VON NEUMANN theorem $J^{**}J^*$ is positive and selfadjoint. We claim that $J^{**}J^*$ is an extension of $a + b$. Indeed, let $u \in \mathrm{dom}\, a \cap \mathrm{dom}\, b$, then

$$J^{**}J^* u = J^{**}(au \oplus bu) = J(au \oplus bu) = au + bu = (a + b)u.$$

In order to prove that our construction is a generalization of the form sum of selfadjoint operators, we need the following lemma on the KREIN-VON NEUMANN extension.

**Lemma 2.1.1** *If $a, b$ are positive, symmetric operators, and $\mathrm{D}_*(a)$ and $\mathrm{D}_*(b)$ are dense in $H$, then $\mathrm{D}_*(a \oplus b)$ is dense in $H \oplus H$ and*

$$a_K \oplus b_K = (a \oplus b)_K.$$

**Proof.** First we show that $(a \oplus b)_K$ exists. It is enough to prove that $\mathrm{D}_*(a \oplus b) = \mathrm{dom}\,(a_K \oplus b_K)^{\frac{1}{2}}$ since the latter is dense in $H \oplus H$.

We observe first that $(a_K \oplus b_K)^{\frac{1}{2}} = (a_K^{\frac{1}{2}} \oplus b_K^{\frac{1}{2}})$, indeed both are positive and selfadjoint with the same square $a_K \oplus b_K$.

Now, using the definition, we can write:

$$\mathrm{D}_*(a \oplus b) = \tag{2.2}$$

$$\{x \oplus y : \exists m_{x,y} \ \ |((a \oplus b)(u \oplus v), x \oplus y)|^2 \le m_{x,y}((a \oplus b)(u \oplus v), u \oplus v), \forall u \oplus v \in \mathrm{dom}\, a \oplus b\} =$$

$$\{x \oplus y : \exists m_{x,y} \ \ |(au, x) + (bv, y)|^2 \le m_{x,y}((au, u) + (bv, v)), \forall u \oplus v \in \mathrm{dom}\, a \oplus \mathrm{dom}\, b\}.$$

Also, we know that

$$\mathrm{dom}\,(a_K \oplus b_K)^{\frac{1}{2}} = \mathrm{dom}\,(a_K^{\frac{1}{2}} \oplus b_K^{\frac{1}{2}}) = \tag{2.3}$$

$$\mathrm{dom}\, a_K^{\frac{1}{2}} \oplus \mathrm{dom}\, b_K^{\frac{1}{2}} = \mathrm{D}_*(a) \oplus \mathrm{D}_*(b) =$$

$$\{x : \exists m_x \ \ |(au, x)|^2 \le m_x(au, u), \forall u \in \mathrm{dom} a\} \oplus \{y : \exists m_y \ \ |(bv, y)|^2 \le m_y(bv, v), \forall v \in \mathrm{dom} b\}.$$

Putting $u = 0$ and respectively $v = 0$ in 2.2, we see that

$$\mathrm{D}_*\,(a \oplus b) \subseteq \mathrm{dom}\,(a_K \oplus b_K)^{\frac{1}{2}}.$$

To show

$$\mathrm{D}_*\,(a \oplus b) \supseteq \mathrm{dom}\,(a_K \oplus b_K)^{\frac{1}{2}},$$

we let $m_{x,y} = 2\max(m_x, m_y)$, and use 2.2, 2.3 and the convexity of the function $\alpha \mapsto \alpha^2$ on $\mathbb{R}_+$. We have seen consequently that $\mathrm{D}_*(a\oplus b) = \mathrm{dom}(a_K \oplus b_K)^{\frac{1}{2}}$. So the KREIN-VON NEUMANN extension of $a \oplus b$ exists, and we know that $\mathrm{D}_*\,(a \oplus b) = \mathrm{dom}\,(a \oplus b)_K^{\frac{1}{2}}$.

To see that $(a \oplus b)_K = a_K \oplus b_K$, we have to check that

$$\mathrm{dom}\,(a \oplus b)_K^{\frac{1}{2}} = \mathrm{dom}\,(a_K \oplus b_K)^{\frac{1}{2}}$$

and furthermore that

$$\|(a_K \oplus b_K)^{\frac{1}{2}} z\|^2 = \|(a \oplus b)_K^{\frac{1}{2}} z\|^2$$

holds for all $z \in \mathrm{dom}\,(a \oplus b)_K^{\frac{1}{2}}$.

The equality of the domains follows from the above argument.

Now, we prove the required identity. Let $x \oplus y \in \mathrm{dom}\,(a \oplus b)_K^{\frac{1}{2}}$. Then

$$\|(a_K \oplus b_K)^{\frac{1}{2}}(x \oplus y)\|^2 = \|(a_K^{\frac{1}{2}} \oplus b_K^{\frac{1}{2}})(x \oplus y)\|^2 = \|a_K^{\frac{1}{2}} x \oplus b_K^{\frac{1}{2}} y\|^2 = \|a_K^{\frac{1}{2}} x\|^2 + \|b_K^{\frac{1}{2}} y\|^2 \quad (2.4)$$

Now we calculate $\|(a \oplus b)_K^{\frac{1}{2}}(x \oplus y)\|^2$. The inequality

$$\|(a \oplus b)_K^{\frac{1}{2}}(x \oplus y)\|^2 \leq \|a_K^{\frac{1}{2}} x\|^2 + \|b_K^{\frac{1}{2}} y\|^2 \quad (2.5)$$

follows immediately from the minimality of the KREIN-VON NEUMANN extension and the fact that $a_K \oplus b_K$ is a positive, selfadjoint extension of $a \oplus b$.

To see the reverse inequality, we consider the following. We can assume that $\|a_K^{\frac{1}{2}} x\|^2 + \|b_K^{\frac{1}{2}} y\|^2 > 0$, therefore we let

$$t = \frac{\|a_K^{\frac{1}{2}} x\|^2}{\|a_K^{\frac{1}{2}} x\|^2 + \|b_K^{\frac{1}{2}} y\|^2}, \quad \text{thus} \quad 1 - t = \frac{\|b_K^{\frac{1}{2}} y\|^2}{\|a_K^{\frac{1}{2}} x\|^2 + \|b_K^{\frac{1}{2}} y\|^2}.$$

Then

$$\sup\{|((a_K^{\frac{1}{2}} \oplus b_K^{\frac{1}{2}})(u \oplus v), (a_k^{\frac{1}{2}} \oplus b_K^{\frac{1}{2}})(x \oplus y))|^2 : u \in \mathrm{dom} a, v \in \mathrm{dom} b, (a_K u, u) + (b_K v, v) \leq 1\} \geq$$

$$\sup\{|(a_K^{\frac{1}{2}}u, a_K^{\frac{1}{2}}x) + (b_K^{\frac{1}{2}}v, b_K^{\frac{1}{2}}y)|^2 : u \in \operatorname{dom} a, v \in \operatorname{dom} b, (a_K u, u) \le t,\ (b_K v, v) \le 1-t\}$$

Now multiplying $u$ and $v$ by a suitable $\alpha_u, \alpha_v \in \mathbb{C}$ of absolute value 1, we continue:

$$\sup\{|(a_K^{\frac{1}{2}}u, a_K^{\frac{1}{2}}x) + (b_K^{\frac{1}{2}}v, b_K^{\frac{1}{2}}y)|^2 : u \in \operatorname{dom} a,\ v \in \operatorname{dom} b, (a_K u, u) \le t, (b_K v, v) \le 1-t\} =$$

$$\sup\{(|(a_K^{\frac{1}{2}}u, a_K^{\frac{1}{2}}x)| + |(b_K^{\frac{1}{2}}v, b_K^{\frac{1}{2}}y)|)^2 : u \in \operatorname{dom} a, v \in \operatorname{dom} b, (a_K u, u) \le t, (b_K v, v) \le 1-t\} =$$

$$(\sup\{|(a_K^{\frac{1}{2}}u, a_K^{\frac{1}{2}}x)| : u \in \operatorname{dom} a, (a_K u, u) \le t\} + \sup\{|(b_K^{\frac{1}{2}}v, b_K^{\frac{1}{2}}y)| : v \in \operatorname{dom} b, (b_K v, v) \le 1-t\})^2 =$$

$$t\|a_K^{\frac{1}{2}}x\|^2 + 2\sqrt{t(1-t)}\|a_K^{\frac{1}{2}}x\|\|b_K^{\frac{1}{2}}y\| + (1-t)\|b_K^{\frac{1}{2}}y\|^2 = \|a_K^{\frac{1}{2}}x\|^2 + \|b_K^{\frac{1}{2}}y\|^2. \tag{2.6}$$

We have used that

$$\sup\{|(a_K^{\frac{1}{2}}u, a_K^{\frac{1}{2}}x)|^2 : u \in \operatorname{dom} a, (a_K u, u) \le t\} = \sup\{|(a_K u, x)|^2 : u \in \operatorname{dom} a, (a_K u, u) \le t\} =$$

$$t\|a_K^{\frac{1}{2}}x\|^2 = \sup\{|(a_K^{\frac{1}{2}}u, a_K^{\frac{1}{2}}x)|^2 : u \in \operatorname{dom} a_K^{\frac{1}{2}}, (a_K^{\frac{1}{2}}u, a_k^{\frac{1}{2}}u) \le t\}$$

and the same for $b_K$. Putting together 2.4, 2.5 and 2.6 we obtain:

$$\|(a_K \oplus b_K)^{\frac{1}{2}}(x \oplus y)\|^2 = \|(a \oplus b)_K^{\frac{1}{2}}(x \oplus y)\|^2$$

completing the proof. □

With the help of Lemma 2.1.1 we are able to prove that the constructed operator $J^{**}J^*$ is indeed a generalization of the notion of the form sum of two positive self-adjoint operators.

**Theorem 2.1.2** *Let $a$ and $b$ be positive, symmetric operators such that $\mathrm{D}_*(a) \cap \mathrm{D}_*(b)$ is dense in $H$, and let $J$ be as in 2.1, then the form sum of $a_K$ and $b_K$ is $J^{**}J^*$, i.e.*

$$a_K \dotplus b_K = J^{**}J^*.$$

**Proof.** Again we prove that $\operatorname{dom}(a_K \dotplus b_K)^{\frac{1}{2}} = \operatorname{dom}(J^{**}J^*)^{\frac{1}{2}}$, and $(a_K \dotplus b_K)^{\frac{1}{2}}x = (J^{**}J^*)^{\frac{1}{2}}x$ for each $x \in \operatorname{dom}(a_K \dotplus b_K)$.

We know that $\operatorname{dom}(a_K \dotplus b_K)^{\frac{1}{2}} = \operatorname{dom} a_K^{\frac{1}{2}} \cap \operatorname{dom} b_K^{\frac{1}{2}}$, and $\operatorname{dom}(J^{**}J^*)^{\frac{1}{2}} = \operatorname{dom} J^* = \mathrm{D}_*(a) \cap \mathrm{D}_*(b)$, as we have seen in the argument following 2.1. Moreover $\operatorname{dom} a_K^{\frac{1}{2}} = \mathrm{D}_*(a)$ and $\operatorname{dom} b_K^{\frac{1}{2}} = \mathrm{D}_*(b)$, which implies the desired equality of the domains.

Using Lemma 2.1.1, we have that

$$\|(J^{**}J^*)^{\frac{1}{2}}x\|^2 = [J^*x, J^*x] =$$
$$\sup\{|[au \oplus bv, J^*x]|^2 : u \in \operatorname{dom} a, v \in \operatorname{dom} b, [au \oplus bv, au \oplus bv] \leq 1\} =$$
$$\sup\{|(au + bv, x)|^2 : u \in \operatorname{dom} a, v \in \operatorname{dom} b, (au, u) + (bv, v) \leq 1\} =$$
$$\sup\{|((a \oplus b)(u \oplus v), x \oplus x)|^2 : u \oplus v \in \operatorname{dom} a \oplus \operatorname{dom} b, ((a \oplus b)(u \oplus v), (u \oplus v)) \leq 1\} =$$
$$\|(a \oplus b)_K^{\frac{1}{2}}(x \oplus x)\|^2 = \|(a_K^{\frac{1}{2}} \oplus b_K^{\frac{1}{2}})(x \oplus x)\|^2 = \|a_K^{\frac{1}{2}}x\|^2 + \|b_K^{\frac{1}{2}}x\|^2.$$

Therefore

$$\|(J^{**}J^*)^{\frac{1}{2}}x\|^2 = \|a_K^{\frac{1}{2}}x\|^2 + \|b_K^{\frac{1}{2}}x\|^2,$$

which is, by definition, equal to $\|(a_K \dotplus b_K)^{\frac{1}{2}}x\|^2$. The theorem is proved. □

The following theorem is an immediate consequence of Theorem 2.1.2, because for any positive, selfadjoint operator $a$, the Krein-von Neumann extension $a_K$ and $a$ coincide.

**Theorem 2.1.3** *If $a$ and $b$ are positive, selfadjoint operators with $\operatorname{dom} a^{\frac{1}{2}} \cap \operatorname{dom} b^{\frac{1}{2}}$ dense in $H$, then the corresponding operator $J^{**}J^*$ is just the form sum of $a$ and $b$.*

The previous theorem shows that the following notation is consistent with the notation for the form sum construction. From now on we will use $a \dotplus b$ for the above constructed operator $J^{**}J^*$, even if $a, b$ are positive, symmetric operators. We reformulate Theorem 2.1.2 as follows.

**Theorem 2.1.4** *If $a$ and $b$ are positive, symmetric operators with $\mathrm{D}_*(a) \cap \mathrm{D}_*(b)$ dense in $H$, then $a \dotplus b = a_K \dotplus b_K$.*

**Remark** Considering the extensions of direct sum of operators, an analogous statement can be proved for the Friedrichs extension, as for the Krein-von Neumann extension in Lemma 2.1.1. Namely, if $a, b$ are densely defined, positive, symmetric operators, then

$$a_F \oplus b_F = (a \oplus b)_F.$$

For the proof we only have to check the equality of the domains of the square root operators.

$$\operatorname{dom}(a \oplus b)_F^{\frac{1}{2}} = \{x \oplus y \in H \oplus H : \exists x_n \oplus y_n \in \operatorname{dom} a \oplus b, x_n \oplus y_n \to x \oplus y,$$

$$((a \oplus b)(x_n \oplus y_n - x_m \oplus y_m), x_n \oplus y_n - x_m \oplus y_m) \to 0\} =$$

$$\{x \oplus y \in H \oplus H : \exists x_n \in \operatorname{dom} a, y_n \in \operatorname{dom} b, x_n \to x, y_n \to y,$$

$$(a(x_n - x_m), x_n - x_m) + (b(y_n - y_m), y_n - y_m) \to 0\} =$$

$$\{x \in H : \exists x_n \in \operatorname{dom} a, x_n \to x, (a(x_n - x_m), x_n - x_m) \to 0\} \oplus$$

$$\oplus \{y \in H : \exists y_n \in \operatorname{dom} b, y_n \to y, (b(y_n - y_m), y_n - y_m) \to 0\} = \operatorname{dom} a_F^{\frac{1}{2}} \oplus \operatorname{dom} b_F^{\frac{1}{2}}$$

## 2.2 Commutation properties

In this section we prove certain commutation properties of the generalized form sum. Theorem 2.2.4 is the analogue of Theorem 1.2.7. The ideas used in this section are essentially taken from [26], where the commutation property is proved for the Krein-von Neumann extension. In turn, the first part of Theorem 1.2.7 follows from Theorem 2.2.4 on setting $a = b$. The situation is as follows: given $E, F \in \mathcal{B}(H)$ and two positive, symmetric operators $a$ and $b$, with $\mathrm{D}_*(a)$ and $\mathrm{D}_*(b)$ dense in $H$, such that both $E$ and $F$ leave $\operatorname{dom} a$ and $\operatorname{dom} b$ invariant. Suppose furthermore that the following equations hold for all $x \in \operatorname{dom} a$ and $y \in \operatorname{dom} b$:

$$E^* ax = aFx, \quad F^* ax = aEx, \quad E^* by = bFy, \quad F^* by = bEy.$$

We remark that throughout this section it is illuminating to think of the less general case of $E = F$ (cf. Theorem 1.1.1).

Now, we define $\hat{E}$ and $\hat{F}$ on $H_a \oplus H_b$ as follows.

$$\operatorname{dom} \hat{E} = \operatorname{ran} a \oplus \operatorname{ran} b, \quad \hat{E}(ax \oplus by) = aEx \oplus bEy,$$

and

$$\operatorname{dom} \hat{F} = \operatorname{ran} a \oplus \operatorname{ran} b, \quad \hat{F}(ax \oplus by) = aFx \oplus bFy.$$

It is obvious that $\hat{E}$ and $\hat{F}$ leave $\operatorname{ran} a \oplus \operatorname{ran} b$ invariant. The following lemma shows that both $\hat{E}$ and $\hat{F}$ are well-defined and continuous on a dense subspace of $H_a \oplus H_b$.

**Lemma 2.2.1** *With the notations above, $\hat{E}$ and $\hat{F}$ are well defined, and $\hat{E}, \hat{F} \in \mathcal{B}(H_a \oplus H_b)$.*

**Proof.** The proof of this lemma could be considerably shortened by referring to the result [Theorem 2 in [26]]. However, for the sake of completeness we include the detailed proof.

$$[\hat{F}(ax \oplus by), \hat{F}(ax \oplus by)] = [aFx \oplus bFy, aFx \oplus bFy] = [aFx, aFx] + [bFy, bFy] =$$

$$(aFx, Fx) + (bFy, Fy) = (E^*ax, Fx) + (E^*by, Fy) = (ax, EFx) + (by, EFy) =$$

$$[ax, aEFx] + [by, bEFy] = [ax \oplus by, aEFx \oplus bEFy] \leq$$

$$[ax \oplus by, ax \oplus by]^{\frac{1}{2}} [aEFx \oplus bEFy, aEFx \oplus bEFy]^{\frac{1}{2}} =$$

$$[ax \oplus by, ax \oplus by]^{\frac{1}{2}} [\hat{E}\hat{F}(ax \oplus by), \hat{E}\hat{F}(ax \oplus by)]^{\frac{1}{2}} \tag{2.7}$$

Substituting $\hat{E}\hat{F}$ for $\hat{F}$, and repeating the argument in 2.7, we obtain

$$[\hat{E}\hat{F}(ax \oplus by), \hat{E}\hat{F}(ax \oplus by)] \leq [ax \oplus by, ax \oplus by]^{\frac{1}{2}} [(\hat{E}\hat{F})^2(ax \oplus by), (\hat{E}\hat{F})^2(ax \oplus by)]^{\frac{1}{2}}$$

From this, by induction:

$$[\hat{F}(ax \oplus by), \hat{F}(ax \oplus by)] \leq [ax \oplus by, ax \oplus by]^{\frac{1}{2} + \cdots \frac{1}{2^n}} [(\hat{E}\hat{F})^{\frac{2^n}{2}}(ax \oplus by), (\hat{E}\hat{F})^{\frac{2^n}{2}}(ax \oplus by)]^{\frac{1}{2^n}} =$$

$$[ax \oplus by, ax \oplus by]^{1-\frac{1}{2^n}} [a(EF)^{\frac{2^n}{2}} x \oplus b(EF)^{\frac{2^n}{2}} y, a(EF)^{\frac{2^n}{2}} x \oplus b(EF)^{\frac{2^n}{2}} y]^{\frac{1}{2^n}} =$$

$$[ax \oplus by, ax \oplus by]^{1-\frac{1}{2^n}} [(F^*E^*)^{\frac{2^n}{2}} ax \oplus (F^*E^*)^{\frac{2^n}{2}} by, a(EF)^{\frac{2^n}{2}} x \oplus b(EF)^{\frac{2^n}{2}} y]^{\frac{1}{2^n}} =$$

$$[ax \oplus by, ax \oplus by]^{1-\frac{1}{2^n}} (ax \oplus by, (EF)^{2^n} x \oplus (EF)^{2^n} y)^{\frac{1}{2^n}} \leq$$

$$[ax \oplus by, ax \oplus by]^{1-\frac{1}{2^n}} \|ax \oplus by\|^{\frac{1}{2^n}} \|(EF)^{2^n} x \oplus (EF)^{2^n} y\|^{\frac{1}{2^n}} =$$

$$[ax \oplus by, ax \oplus by]^{1-\frac{1}{2^n}} \|ax \oplus by\|^{\frac{1}{2^n}} \|((EF)^{2^n} \oplus (EF)^{2^n})(x \oplus y)\|^{\frac{1}{2^n}} \leq$$

$$[ax \oplus by, ax \oplus by]^{1-\frac{1}{2^n}} \|ax \oplus by\|^{\frac{1}{2^n}} \|(EF)^{2^n} \oplus (EF)^{2^n}\|^{\frac{1}{2^n}} \|x \oplus y\|^{\frac{1}{2^n}} =$$

$$[ax \oplus by, ax \oplus by]^{1-\frac{1}{2^n}} \|ax \oplus by\|^{\frac{1}{2^n}} \|(EF \oplus EF)^{2^n}\|^{\frac{1}{2^n}} \|x \oplus y\|^{\frac{1}{2^n}}$$

If we take the limit $n \to \infty$, we obtain:

$$[\hat{F}(ax \oplus by), \hat{F}(ax \oplus by)] \leq r(EF \oplus EF)[ax \oplus by, ax \oplus by],$$

where $r(EF \oplus EF)$ stands for the spectral radius of $EF \oplus EF$. This proves both statements for $\hat{F}$. The proposition for $\hat{E}$ can be proved analogously. (To be very precise, we have shown that $\hat{E}$ and $\hat{F}$ are continuously defined on a dense subspace of $H_a \oplus H_b$, but they are automatically extended to the whole space.) □

Now, we compute the adjoints of $\hat{E}$ and $\hat{F}$ in $\mathcal{B}(H_a \oplus H_b)$:

**Lemma 2.2.2** $\hat{E}^* = \hat{F}$ *and* $\hat{F}^* = \hat{E}$.

**Proof.** It is enough to prove $\hat{F}^* = \hat{E}$, as $\hat{E}, \hat{F} \in \mathcal{B}(H_a \oplus H_b)$. We check that $\hat{F}^* x = \hat{E}x$ on the dense subspace $\operatorname{ran} a \oplus \operatorname{ran} b$. Let $ax \oplus by \in \operatorname{ran} a \oplus \operatorname{ran} b$, then for all $au \oplus bv \in \operatorname{ran} a \oplus \operatorname{ran} b$

$$[au \oplus bv, \hat{F}^*(ax \oplus by)] = [\hat{F}(au \oplus bv), ax \oplus by] = [aFu \oplus bFv, ax \oplus by] =$$

$$[aFu, ax] + [bFv, by] = (aFu, x) + (bFv, y) = (E^*au, x) + (E^*bv, y) = (au, Ex) + (bv, Ey) =$$

$$[au, aEx] + [bv, bEy] = [au \oplus bv, aEx \oplus bEy] = [au \oplus bv, \hat{E}(ax \oplus by)],$$

and that was to be proved. □

Before proving the commutation preserving property of the generalized form sum we make a short observation in the case when $a = b$ and $E = F$. Note, that in the special case when $E = F$ Lemma 2.2.2 means that $\hat{E}$ is a bounded selfadjoint operator on $H_a \oplus H_b$. Furthermore, in the case when $a = b$ and $E = F$ holds, we can replace the auxillary space $H_a \oplus H_b$ by simply $H_a$, and define $\hat{E}(ax) := a(Ex)$ with $\operatorname{dom} \hat{E} = \operatorname{ran} a$. In this case, the arguments above show that $\hat{E}$ is a bounded self-adjoint operator on $H_a$. We can relate the spectrum of $\hat{E}$ to the spectrum of $E$ (see [25] for the bounded case, and [15] for a more general case).

**Theorem 2.2.3** *The spectrum* $\operatorname{Sp}(\hat{E})$ *is contained in* $\operatorname{Sp}(E) \cap \mathbb{R}$.

**Proof.** Since $\hat{E}$ is self-adjoint it is clear that $\mathrm{Sp}\,(\hat{E}) \subseteq \mathbb{R}$. On the other hand, take any real $\lambda$ from the resolvent set of $E$ and $x \in \mathrm{dom}\, A$, then

$$A(E-\lambda I)x = (E-\lambda I)^* Ax, \text{ hence } A(E-\lambda I)^{-1}x = \left[(E-\lambda I)^{-1}\right]^* Ax$$

which means that we can define the operator $[(E-\lambda I)^{-1}]\hat{}$. A short computation gives that

$$\left[(E-\lambda I)^{-1}\right]\hat{} = \left(\hat{E}-\lambda\hat{I}\right)^{-1},$$

indeed for $x \in \mathrm{dom}\, A$

$$\left[(E-\lambda I)^{-1}\right]\hat{}\left(\hat{E}-\lambda\hat{I}\right)Ax = A\,(E-\lambda I)^{-1}\,(E-\lambda I)\,x = Ax, \text{ and}$$

$$\left(\hat{E}-\lambda\hat{I}\right)\left[(E-\lambda I)^{-1}\right]\hat{}Ax = A\,(E-\lambda I)\,(E-\lambda I)^{-1}\,x = Ax.$$

This proves the statement. □

Now we prove that commutation is preserved when taking the generalized form sum of operators.

**Theorem 2.2.4** *Let $a, b$ be positive, symmetric operators with $\mathrm{D}_*\,(a) \cap \mathrm{D}_*\,(b)$ dense in $H$, and suppose that $E, F \in \mathcal{B}(H)$, such that both $E$ and $F$ leave* dom $a$ *and* dom $b$ *invariant, and for all $x \in \mathrm{dom}\, a$ and $y \in \mathrm{dom}\, b$*

$$E^*ax = aFx, \quad F^*ax = aEx, \quad E^*by = bFy, \quad F^*by = bEy.$$

*Then*

$$E^*(a \dotplus b) \subseteq (a \dotplus b)F \quad \text{and} \quad F^*(a \dotplus b) \subseteq (a \dotplus b)E.$$

**Proof.** First we show the following:

$$E^*J \subseteq J\hat{F}, \quad F^*J \subseteq J\hat{E}, \quad \hat{E}J^* \subseteq J^*E, \quad \hat{F}J^* \subseteq J^*F.$$

Indeed, let $ax \oplus by \in \mathrm{ran}\, a \oplus \mathrm{ran}\, b$, then

$$J\hat{F}(ax \oplus by) = J(aFx \oplus bFy) = aFx + bFy = E^*ax + E^*by = E^*(ax+by) = E^*J(ax \oplus by).$$

Observing the domains, we have consequently $E^*J \subseteq J\hat{F}$. An analogous proof can be given for $F^*J \subseteq J\hat{E}$. For the remaining inclusions, we write:

$$\hat{E}J^* = \hat{F}^*J^* \subseteq (J\hat{F})^* \subseteq (E^*J)^* = J^*E,$$

as $E$ is bounded, hence $\hat{E}J^* \subseteq J^*E$, and with the same reasoning $\hat{F}J^* \subseteq J^*F$.

Finally we turn to the proof of the theorem. Using the previously proved statement, we have

$$E^*J^{**} \subseteq (J^*E)^* \subseteq (\hat{E}J^*)^* = J^{**}\hat{E}^* = J^{**}\hat{F}.$$

Note that we have used that $\hat{E}$ is continuous according to Lemma 2.2.1. We complete the proof by writing

$$E^*(a \dot{+} b) = E^*J^{**}J^* \subseteq J^{**}\hat{F}J^* \subseteq J^{**}J^*F = (a \dot{+} b)F,$$

that is $E^*(a \dot{+} b) \subseteq (a \dot{+} b)F$, and with the same argument $F^*(a \dot{+} b) \subseteq (a \dot{+} b)E$. □

The following result, which is just a special case of Theorem 2.2.4 with $E = F = S = S^*$, shows the reason why we talk about "commutation properties" above.

**Theorem 2.2.5** *Let $S$ be a bounded, selfadjoint operator over the Hilbert space $H$, such that $S$ leaves both* $\operatorname{dom} a$ *and* $\operatorname{dom} b$ *invariant, and furthermore*

$$Sax = aSx, \quad Sby = bSy$$

*hold for all $x \in \operatorname{dom} a$ and $y \in \operatorname{dom} b$. Also, assume that $\mathrm{D}_*(a) \cap \mathrm{D}_*(b)$ is dense in $H$. Then*

$$S(a \dot{+} b) \subseteq (a \dot{+} b)S.$$

In Theorem 2.2.4, we require that the bounded operators $E, F$ leave $\operatorname{dom} a$ and $\operatorname{dom} b$ invariant. It is interesting to see what other subspace $D$ can replace $\operatorname{dom} a$ and $\operatorname{dom} b$. It is clear that a sufficient condition on $D$ is that $a \dot{+} b = (a\restriction_D) \dot{+} (b\restriction_D)$. The following theorem characterizes such subspaces $D$.

**Theorem 2.2.6** *Let $a$ and $b$ be positive, symmetric operators with $\mathrm{D}_*(a) \cap \mathrm{D}_*(b)$ dense in $H$, and suppose that $D \subseteq \mathrm{dom}\, a \cap \mathrm{dom}\, b$ is a linear subspace. Then $a\restriction_D \dot{+} b\restriction_D = a \dot{+} b$ if and only if for all $x \in H$*

$$\sup\{|(au,x)|^2 : u \in \mathrm{dom}\, a, (au,u) \leq 1\} + \sup\{|(bv,x)|^2 : v \in \mathrm{dom}\, b, (bv,v) \leq 1\} =$$
$$\sup\{|(au,x)|^2 : u \in D, (au,u) \leq 1\} + \sup\{|(bv,x)|^2 : v \in D, (bv,v) \leq 1\} \tag{2.8}$$

**Proof.** Before all, observe that $\mathrm{D}_*(a) \subseteq \mathrm{D}_*(a\restriction_D)$ and $\mathrm{D}_*(b) \subseteq \mathrm{D}_*(b\restriction_D)$. Indeed:

$$\mathrm{D}_*(a) = \{y \in H : \exists m_y |(ax,y)|^2 \leq m_y(ax,x), \forall x \in \mathrm{dom}\, a\} \subseteq \tag{2.9}$$
$$\{y \in H : \exists m_y |(ax,y)|^2 \leq m_y(ax,x), \forall x \in D\} = \mathrm{D}_*(a\restriction_D),$$

and the same for $\mathrm{D}_*(b)$ and $\mathrm{D}_*(b\restriction_D)$.

Suppose now that condition 2.8 is satisfied. Then for the reverse inclusion $\mathrm{D}_*(a) \cap \mathrm{D}_*(b) \supseteq \mathrm{D}_*(a\restriction_D) \cap \mathrm{D}_*(b\restriction_D)$ we let $x \in \mathrm{D}_*(a\restriction_D) \cap \mathrm{D}_*(b\restriction_D)$, which is the same as saying that the right hand side of 2.8 is finite for this $x$. But then, from assumption 2.8 it follows that the left hand side of 2.8 is also finite, implying $x \in \mathrm{D}_*(a) \cap \mathrm{D}_*(b)$. By our construction for the form sum

$$\mathrm{dom}\,((a \dot{+} b)^{\frac{1}{2}}) = \mathrm{D}_*(a) \cap \mathrm{D}_*(b), \quad \text{and} \quad \mathrm{dom}\,(a\restriction_D \dot{+} b\restriction_D)^{\frac{1}{2}} = \mathrm{D}_*(a\restriction_D) \cap \mathrm{D}_*(b\restriction_D),$$

hence $\mathrm{dom}\,(a \dot{+} b)^{\frac{1}{2}} = \mathrm{dom}\,(a\restriction_D \dot{+} b\restriction_D)^{\frac{1}{2}}$. Let $x \in \mathrm{dom}\,(a \dot{+} b)^{\frac{1}{2}}$, then by the proof of Theorem 2.1.2 and 1.2 and 1.3

$$\|(a \dot{+} b)^{\frac{1}{2}} x\|^2 = \|a_K^{\frac{1}{2}} x\|^2 + \|b_K^{\frac{1}{2}} x\|^2 = \tag{2.10}$$
$$\sup\{|(au,x)|^2 : u \in \mathrm{dom}\, a, (au,u) \leq 1\} + \sup\{|(bv,x)|^2 : v \in \mathrm{dom}\, b, (bv,v) \leq 1\} =$$
$$\sup\{|(au,x)|^2 : u \in D, (au,u) \leq 1\} + \sup\{|(bv,x)|^2 : v \in D, (bv,v) \leq 1\} =$$
$$\|(a\restriction_D)_K^{\frac{1}{2}} x\|^2 + \|(b\restriction_D)_K^{\frac{1}{2}} x\|^2 = \|(a\restriction_D \dot{+} b\restriction_D)^{\frac{1}{2}} x\|^2.$$

Consequently we have $a\restriction_D \dot{+} b\restriction_D = a \dot{+} b$.

For the reverse direction, we suppose that $a\restriction_D \dot{+} b\restriction_D = a \dot{+} b$. Then for all $x \in \mathrm{dom}\,(a \dot{+} b)^{\frac{1}{2}}$ we have $\|(a \dot{+} b)^{\frac{1}{2}} x\|^2 = \|(a\restriction_D \dot{+} b\restriction_D)^{\frac{1}{2}} x\|^2$, and the same argument as in 2.10 shows that 2.8 is satisfied. □

## 2.3 Remarks on operator sums

Our construction for the form sum is based on the idea used when constructing the KREIN-VON NEUMANN extension $a_K$ of a positive, symmetric operator $a$. Analogously we consider the construction corresponding to the Friedrichs extension $a_F$ of $a$ (cf Theorem 1.2.5). We suppose that dom $a$ and dom $b$ are dense. Again we have the Hilbert space $H_a \oplus H_b$, and we define analogously as in [23], [24]

$$Q : H \to H_a \oplus H_b, \text{ with } \operatorname{dom} Q = \operatorname{dom} a \cap \operatorname{dom} b, \ \ Qx = ax \oplus bx.$$

Obviously $Q$ is a restriction of $J^*$. The question is, what can be said about $Q^*Q^{**}$.

**Theorem 2.3.1** *Suppose that $a$ and $b$ are positive, symmetric operators, and* $\operatorname{dom} a \cap \operatorname{dom} b$ *is dense in $H$. Then $Q^*Q^{**} = (a+b)_F$.*

**Proof.** First we show that under these circumstances $Q^*Q^{**}$ exists and is a positive, selfadjoint operator. From the VON NEUMANN theorem, it is clear that if $Q^*Q^{**}$ exists then it is selfadjoint, and obviously positive. $Q^*$ exists, since dom $Q$ is dense. We compute dom $Q^*$, and as it will be dense, we conclude that $Q^{**}$ exists. First we compute $Q^*$ on $\operatorname{ran} a \oplus \operatorname{ran} b$. Let $ax \oplus by \in \operatorname{ran} a \oplus \operatorname{ran} b$ and $z \in \operatorname{dom} a \cap \operatorname{dom} b$

$$[Qz, ax \oplus by] = [az \oplus bz, ax \oplus by] = [az, ax] + [bz, by] = (az, x) + (bz, y) =$$

$$(z, ax) + (z, by) = (z, ax + by),$$

which shows that $\operatorname{ran} a \oplus \operatorname{ran} b \subseteq \operatorname{dom} Q^*$ and $Q^*(ax \oplus by) = ax + by$. Therefore $Q^*$ is densely defined. We see that $Q^*Q^{**}$ is an extension of $a + b$:

$$Q^*Q^{**}z = Q^*Qz = Q^*(az \oplus bz) = az + bz.$$

Because of the extremality of the FRIEDRICHS extension, we only have to prove that

$$\operatorname{dom} (a+b)_F^{\frac{1}{2}} = \operatorname{dom} (Q^*Q^{**})^{\frac{1}{2}}.$$

We can write

$$\operatorname{dom}(Q^*Q^{**})^{\frac{1}{2}} = \operatorname{dom}Q^{**} = \operatorname{dom}\bar{Q} = \{y \in H : \exists y_n \in \operatorname{dom}Q, y_n \to y, Qy_n \text{ convergent}\} =$$

$\{y \in H : \exists y_n \in \operatorname{dom} Q, y_n \to y, [ay_n \oplus by_n - ay_m \oplus by_m, ay_n \oplus by_n - ay_m \oplus by_m] \to 0\} =$

$\{y \in H : \exists y_n \in \operatorname{dom} a \cap \operatorname{dom} b, y_n \to y, (a(y_n - y_m), y_n - y_m) + (b(y_n - y_m), y_n - y_m) \to 0\} =$

$\{y \in H : \exists y_n \in \operatorname{dom}(a+b), y_n \to y, ((a+b)(y_n - y_m), y_n - y_m) \to 0\} = \operatorname{dom}(a+b)_F^{\frac{1}{2}},$

which remained to complete the proof. □

Finally, we examine the connection between different extensions of the operator sum. Supose that $A$ and $B$ are positive, selfadjoint operators, and let $A+B$ denote the operator sum on $D = \operatorname{dom} A \cap \operatorname{dom} B$. Suppose that $D$ is dense in $H$, so that the FRIEDRICHS extension $(A+B)_F$ of $A+B$ exists. KATO [17] shows an example when $A \dot{+} B \neq (A+B)_F$. In view of Theorem 2.3.1 one could expect that, analogously, $J^{**}J^* = (A+B)_K$ holds. By Theorem 2.1.3 this would mean that $A \dot{+} B = (A+B)_K$ holds. However, we will prove that in general $A \dot{+} B \neq (A+B)_K$. Note that if we assume only that $\operatorname{dom} A^{\frac{1}{2}} \cap \operatorname{dom} B^{\frac{1}{2}}$ is dense in $H$ – assuring the existence of $A \dot{+} B$ – the KREIN-VON NEUMANN extension will still exist. Indeed, it is easy to see that

$$\mathrm{D}_*(A+B) = \{y \in H : \exists m_y \, |((a+b)x, y)|^2 \leq m_y((a+b)x, x), \ \forall x \in D\} \supseteq$$

$$\mathrm{D}_*(A) \cap \mathrm{D}_*(B) = \operatorname{dom} A^{\frac{1}{2}} \cap \operatorname{dom} B^{\frac{1}{2}},$$

so $\mathrm{D}_*(A+B)$ is dense in $H$. However, it may well happen that $\operatorname{dom} A^{\frac{1}{2}} \cap \operatorname{dom} B^{\frac{1}{2}}$ is dense in $H$ while $\operatorname{dom} A \cap \operatorname{dom} B = \{\mathbf{0}\}$. In this case $A \dot{+} B \neq (A+B)_K = \mathbf{0}$, providing a trivial counter-example. For this reason, in the sequel we keep the assumption that $D$ is dense in $H$.

**Example** 1. Let $a$ be a densely defined, closed, symmetric operator with positive lower bound. Suppose moreover that $a$ is not selfadjoint. Then the deficiency index $\dim(\ker a^*)$ of $a$ is greater than zero. Consider $a_K$ and $a_F$, both are positive and selfadjoint, and $D = \operatorname{dom} a_K \cap \operatorname{dom} a_F \supseteq \operatorname{dom} a$, therefore $D$ is dense in $H$. Furthermore, we have that $a_K \dot{+} a_F = 2a_F$, because

$$\operatorname{dom} a_K^{\frac{1}{2}} \cap \operatorname{dom} a_F^{\frac{1}{2}} = \operatorname{dom} a_F^{\frac{1}{2}}, \text{ and } \|a_K^{\frac{1}{2}}x\|^2 = \|a_F^{\frac{1}{2}}x\|^2$$

for all $x \in \operatorname{dom} a_F^{\frac{1}{2}}$. On the other hand, $(a_K + a_F)_K = 2a_K$, because $a_K + a_F$ is a symmetric extension of $2a$, hence $2a_K = (2a)_K \leq (a_K + a_F)_K$. Conversely, $(a_K + a_F)_K \leq 2a_K$, because $a_K + a_F$ is a restriction of $2a_K$. Thus we have that $a_K \dot{+} a_F \neq (a_K + a_F)_K$, as desired. As a simple specific example one can take the extensions $a_F$ and $a_K$ described in the Example at the end of Chapter 1.

**Example** 2. A similar approach can provide an example when $A \dot{+} B \neq (A + B)_F$. The example above fails as $a_K \dot{+} a_F = 2a_F$ and $(a_K + a_F)_F = 2a_F$ as well. However, take any intermediate extension $a_M$ of $a$ instead of $a_F$. Then we have $a_K \dot{+} a_M \leq 2a_M$ because

$$\mathrm{dom}\,(a_K \dot{+} a_M)^{\frac{1}{2}} = \mathrm{dom}\,a_K^{\frac{1}{2}} \cap \mathrm{dom}\,a_M^{\frac{1}{2}} = \mathrm{dom}\,a_M^{\frac{1}{2}}$$

and

$$\|(a_K \dot{+} a_M)^{\frac{1}{2}} x\|^2 = \|a_K^{\frac{1}{2}} x\|^2 + \|a_M^{\frac{1}{2}} x\|^2 \leq \|(2a_M)^{\frac{1}{2}} x\|^2$$

for all $x \in \mathrm{dom}\,a_M^{\frac{1}{2}}$. Furthermore, $(a_K + a_M)_F \geq 2a_M$ because both $a_K + a_M$ and $2a_M$ are extensions of $2a_M\restriction_{\mathrm{dom}\,a_K \cap \mathrm{dom}\,a_M} = a_K + a_M$, here we have used that

$$a_K\restriction_{\mathrm{dom}\,a_K \cap \mathrm{dom}\,a_M} = a^*\restriction_{\mathrm{dom}\,a_K \cap \mathrm{dom}\,a_M} = a_M\restriction_{\mathrm{dom}\,a_K \cap \mathrm{dom}\,a_M},$$

so the inequality follows from the extremality of the Friedrichs extension. Thus we have

$$a_K \dot{+} a_M \leq 2a_M \leq (a_K + a_M)_F. \tag{2.11}$$

How can we assure that equality does not hold at both inequalities in 2.11? It is easy to see from the argument above that a sufficient condition for $a_M$ is that the form $q_{a_M}$ of $a_M$ is not a restriction of the form $q_{a_K}$ of $a_K$. In other words, it is sufficient that $a_M$ is not an extremal extension of $a$ (cf. Theorem 1.2.6). When $\dim(\ker a^*) > 0$, such an $a_M$ is always available (see [2]). Just take any strictly positive, closed form $q_0$ on $\ker a^*$ (e.g. the original inner product) and define a new form $q$ on $\ker a^* + \mathrm{dom}\,a_F^{\frac{1}{2}}$ as follows

$$q(x + y) = q_0(x) + \|a_F^{\frac{1}{2}} y\|^2, \quad x \in \ker a^*, y \in \mathrm{dom}\,a_F^{\frac{1}{2}}.$$

We have used that $\ker a^* \cap \mathrm{dom}\,a_F^{\frac{1}{2}} = \{\mathbf{0}\}$. Using the representation theorem, we get the required $a_M$. (Note that $a_K$ belongs to the choice $q_0 \equiv 0$.) Thus we see that a desired counter-example can be given whenever $\dim \ker a^* > 0$. As a simple specific example one can take $a_K$ as described in the Example at the end of Chapter 1, and $a_M := a_N$ the Neumann-Laplacian, which is not an extremal extension of $a$.

# Chapter 3

# Positive forms on Banach spaces

The representation theorem establishes a correspondance between positive, self-adjoint operators and closed, positive forms on Hilbert spaces. The aim of this chapter is to show that some of the results remain true if the underlying space is a reflexive Banach space. In particular, the construction of the Friedrichs extension and the form sum of positive operators can be carried over to this case.

Let $X$ denote a reflexive complex Banach space, and $X^*$ its conjugate dual space (i.e. the space of all continuous, conjugate linear functionals over $X$). We will use the notation $(v, x) := v(x)$ for $v \in X^*$ , $x \in X$, and $(x, v) := \overline{v(x)}$. Let $A$ be a densely defined linear operator from $X$ to $X^*$. Notice that in this context it makes sense to speak about positivity and self-adjointness of $A$. Indeed, $A$ defines a sesquilinear form on $\operatorname{dom} A \times \operatorname{dom} A$ via $t_A(x, y) = (Ax)(y) = (Ax, y)$ and $A$ is called positive if $t_A$ is positive, i.e. if $(Ax, x) \geq 0$ for all $x \in \operatorname{dom} A$. Also, the adjoint $A^*$ of $A$ is defined (because $A$ is densely defined) and is a mapping from $X^{**}$ to $X^*$, i.e. from $X$ to $X^*$. Thus, $A$ is called self-adjoint if $A = A^*$. Similarly, the operator $A$ is called symmetric if the form $t_A$ is symmetric.

In Section 3.1 we deal with closed, positive forms and associated operators, and we establish a generalized version of the representation theorem. In Section 3.2 we apply the representation theorem in two situations: first we construct the Friedrichs extension of a positive, symmetric operator, then we define the form sum of two positive, self-adjoint operators. We show that the factorization argument of Chapter 1 remains valid in this context, as well. In the last section we give applications of the results in the theory of partial differential equations and in probability theory.

The results of this chapter are taken from [15].

## 3.1 Representation theorem

Let $D \subseteq X$ be a dense subspace, and let $t : D \times D \to \mathbb{C}$ be a sesquilinear form on $D$ (where $t$ is linear in the first variable and conjugate linear in the second). Assume that $t$ is positive with positive lower bound, i.e. $t(x,x) \geq \gamma \|x\|^2,\ \gamma > 0$. Assume also that $t$ is "closed" in the sense that $(D, t(\cdot,\cdot)) =: H$ is a Hilbert space (i.e. it is complete). In this case, the injection $i : H \to X$ is continuous, so $H$ can be regarded as a subspace of $X$. For brevity we will use the notation $[\cdot,\cdot]$ for $t(\cdot,\cdot)$. An operator $A$ from $X$ to $X^*$ can be associated to the form $t$ in a natural way: let $x \in D$ and take the functional $[x,y],\ y \in D$; if this functional is continuous in the norm of $X$ then there is an element $z$ in $X^*$ for which $[x,y] = z(y) =: (z,y)$, in this case, let $Ax := z$.

**Theorem 3.1.1** *With notations as above the operator $A : X \to X^*$ is a positive, self-adjoint operator.*

**Proof.** Let $v \in X^*$ be an arbitrary element. Now, $(v,x)$ $x \in D$ is a continuous, conjugate linear functional on $H$. Indeed,

$$|(v,x)| \leq \|v\|\ \|x\| \leq \frac{1}{\sqrt{\gamma}}[x]\ \|v\| = K[x],$$

where $[x]$ denotes the norm of $H$, i.e. $[x] = [x,x]^{1/2}$. Thus, by the theorem of Riesz we have an element $f \in H$ such that $(v,x) = [f,x]$. Define an operator $B$ from $X^*$ to $X$ by $Bv := f$. Then $B$ is defined everywhere on $X^*$, and $B$ is positive and bounded with $\|B\| \leq \frac{1}{\gamma}$. Indeed, $(z,Bz) = [Bz,Bz] = [Bz]^2 \geq 0$, and

$$\|Bz\|^2 \leq \frac{1}{\gamma}[Bz]^2 = \frac{1}{\gamma}(Bz,z) \leq \frac{1}{\gamma}\|Bz\|\ \|z\|.$$

Hence, $B$ is a bounded, positive, self-adjoint operator. Furthermore, $B$ is injective. To see this, suppose that $Bz = 0$. Then $0 = [Bz, g] = (z, g)$ for every $g \in H$, and $H$ is dense in $X$ therefore $z = 0$. This means that the inverse $B^{-1}$ exists and is a linear mapping from $X$ to $X^*$. We will show that $A = B^{-1}$. Let $x \in \operatorname{dom} A$, then $[x, y] = (t, y)$ for some $t \in X^*$ and $Ax = t$. Also, $(t, y) = [Bt, y]$ so $Bt = x$, and hence $A \subseteq B^{-1}$. Conversely, if $x \in \operatorname{dom} B^{-1}$ then $x = Bz$ for some $z \in X^*$ and $[x, y] = [Bz, y] = (z, y)$ is continuous in $y$ therefore $x \in \operatorname{dom} A$ and $Ax = z = B^{-1}x$, which proves that $B^{-1} \subseteq A$. To complete the proof we have the following lemma, which is well known in Hilbert spaces. □

**Lemma 3.1.2** *If $B : X^* \to X$ is a bounded, injective, self-adjoint operator then $A := B^{-1}$ is also a self-adjoint operator from $X$ to $X^*$.*

**Proof.** First we show that $\operatorname{ran} B$ is dense in $X$. Indeed, if for some $v \in X^*$ we have $(Bz, v) = 0$ for every $z \in X^*$, then $(Bz, v) = (z, Bv) = 0$ so $Bv = 0$ and $v = 0$. Hence $A$ is densely defined. Also, $A$ is symmetric, because if $x \in \operatorname{dom} A$ then $x = Bz$ for some $z \in X^*$ and $(Ax, x) = (z, Bz) \in \mathbb{R}$. Thus $A \subseteq A^*$. To see the reverse inclusion, let $y \in \operatorname{dom} A^*$ and let $x = Bz$ run through the elements of $\operatorname{dom} A$. Then $(Ax, y) = (z, y)$ and also

$$(Ax, y) = (x, A^*y) = (Bz, A^*y) = (z, BA^*y)$$

which means that $y = BA^*y$, so $y \in \operatorname{dom} A$. □

We remark that the previous arguments can be carried out whenever $(X, Y)$ is a dual pair of locally convex, topological linear spaces. In this case, one has to replace the condition on the positivity of the lower bound by the natural assumption that the injection $i$ introduced above is continuous.

It is possible to introduce a more general notion of positive, closed forms (in order to include forms with lower bound 0). A positive form $t : D \times D \to \mathbb{C}$ will be called closed if whenever $x_n \subseteq D$ and $x_n \to x$ in $X$ and $t(x_n - x_m, x_n - x_m) \to 0$ then $x \in D$ and $t(x_n - x, x_n - x) \to 0$ (notice that when $t$ has positive lower bound then this definition agrees with the previous one). We will see from Lemma 3.2.2 that it is possible to associate a closed form with every positive self-adjoint operator. Conversely, however, it is an open problem whether the representation theorem remains valid in this context.

**Problem 3.1.3** *Assume $t$ is a positive, closed form on a dense subspace $D \subset X$. Is it true that the operator associated with $t$ is selfadjoint?*

## 3.2 The Friedrichs extension and the form sum

In this section we apply the representation theorem in two situations. First we construct the Friedrichs extension of a densely defined positive operator. We are restricted to the case when $a$ has positive lower bound.

**Theorem 3.2.1** *Let $a : X \to X^*$ be a positive, densely defined operator with positive lower bound, $(ax, x) \geq \gamma \|x\|^2$, $\gamma > 0$ for every $x \in \operatorname{dom} a$. Then $a$ admits a positve self-adjoint extension with the same lower bound.*

**Proof.** The form $t_a(x, y) := (ax, y)$ defines a pre-Hilbert space on $\operatorname{dom} a$. Denote the completion of this space by $H$, and the arising inner product by $[\cdot, \cdot]$. The injection $i : \operatorname{dom} a \to X$ extends by continuity to $H$ and the extension will be denoted by $I_a$. We prove that $I_a$ is injective. Notice first that $[t, y] = (at, I_a y)$ for all $t \in \operatorname{dom} a$, $y \in H$. Indeed, take a sequence $y_n \in \operatorname{dom} a$ , $y_n \to y$ in $H$ (which implies convergence in $X$ as well), then

$$(at, I_a y) = \lim(at, I_a y_n) = \lim[t, y_n] = [t, y].$$

Now assume that $I_a y = 0$. Then

$$[y]^2 = \lim[y_n, y] = \lim(ay_n, I_a y) = 0$$

therefore $y = 0$ which means that $I_a$ is injective. Thus $H$ can be regarded as a subspace of $X$ and Theorem 3.1.1 can be applied. It is clear that the arising self-adjoint operator $A_F$ is an extension of $a$ and we also see from the proof of Theorem 3.1.1 that $(A_F x, x) \geq \gamma \|x\|^2$ for all $x \in \operatorname{dom} A$. This operator will be called the Friedrichs extension of $a$. □

Next, we show that the factorization argument described in Chapter 1 remains valid in this context. For bounded positive self-adjoint operators from $X$ to $X^*$ the following lemma was also proved in [27], and it plays a key role in the characterization of covariance operators of Banach space valued random variables. It is also remarkable that this factorization argument is applicable without the condition of positive lower bound.

**Lemma 3.2.2** *Let $A$ be a positive self-adjoint operator from $X$ to $X^*$ (it is not necessary that $A$ has positive lower bound). Then there exists an auxiliary Hilbert space $H$ and an operator $J : H \to X^*$ such that $A = JJ^*$.*

**Proof.** Define an inner product on ran $A$ by $[Ax, Ay] := (Ax, y)$. It is well defined because if $Ax_1 = Ax_2$ and $Ay_1 = Ay_2$ then

$$(Ax_1, y_1) = (Ax_2, y_1) = (x_2, Ay_1) = (x_2, Ay_2) = (Ax_2, y_2).$$

Furthermore it is positive definite, because if $[Ax, Ax] = (Ax, x) = 0$ then by the Cauchy inequality we have

$$|(Ax, y)|^2 \leq (Ax, x)(Ay, y) = 0$$

for all $y \in \text{dom } A$ which implies that $Ax = 0$. Thus $(\text{ran } A, [\cdot, \cdot])$ is a pre-Hilbert space. Denote the completion of this space by $H_A$. Define the operator $J : H_A \to X^*$ by dom $J$ = ran $A$ and $J(Ax) := Ax$ for all $Ax \in \text{ran } A$. Then, by definition dom $J^* = \{y \in X : |(Ax, y)|^2 \leq M_y(Ax, x) \text{ for all } x \in \text{dom } A\}$, in particular dom $A \subseteq$ dom $J^*$ and $J^*y = Ay$ for all $y \in \text{dom } A$. Thus $JJ^*$ is an extension of $A$ and $JJ^*$ is symmetric. It is also clear that a self-adjoint operator is maximal symmetric just as in the context of Hilbert spaces. This means that $A = JJ^*$. □

One could think that the Krein-von Neumann and Friedrichs extensions of an arbitrary positive, densely defined operator are now possible to construct in a similar manner as in Theorem 1.2.1 and 1.2.5. Notice, however, that one link is missing:

**Problem 3.2.3** *(Generalized von Neumann theorem) Assume that $T$ is a densely defined, closed operator from $X$ to a Hilbert space $H$. Is it true that $T^*T : X \to X^*$ is selfadjoint?*

Notice that in the context of Hilbert spaces dom $A^{\frac{1}{2}}$ = dom $J^*$ and $(A^{\frac{1}{2}}x, A^{\frac{1}{2}}x) = [J^*x, J^*x]$.

It is natural to associate the sesquilinear form $t_A(x, y) := [J^*x, J^*y]$, $x, y \in \text{dom } J^*$ with the operator $A$. This form is closed because the adjoint operator $J^*$ is closed. Also, if two positive self-adjoint operators $A$ and $B$ have the same form then the operators are necessarily equal. Indded, for $x \in \operatorname{dom} A$ and $y \in \operatorname{dom} B$ we have $(Ax, y) = [J_A^* x, J_A^* y] = [J_B^* x, J_B^* y] = (x, By)$ which means that $B \subseteq A^* = A$, hence $A = B$.

It is also possible to obtain the form of $A$ without referring to the operator $J^*$.

**Lemma 3.2.4** *With notations as above we have*

$$\operatorname{dom} J_A^* = \left\{ y \in X : \sup_{x \in \operatorname{dom} A, (Ax,x) \leq 1} |(Ax, y)|^2 < \infty \right\}$$

*and*

$$[J_A^* y, J_A^* y]_A = \sup_{x \in \operatorname{dom} A, (Ax,x) \leq 1} |(Ax, y)|^2$$

**Proof.** The characterization of dom $J^*$ is clear from

$$\operatorname{dom} J^* = \{ y \in X : |(Ax, y)|^2 \leq M_y (Ax, x) \text{ for all } x \in \operatorname{dom} A \}.$$

To see the other equality notice that ran $A$ is dense in $H_A$, therefore we have

$$[J_A^* y, J_A^* y]_A = \sup_{(Ax,x) \leq 1} |[J^* y, Ax]|_A^2 = \sup_{(Ax,x) \leq 1} |(y, Ax)|^2$$

□

Next we turn to the construction of the form sum of two positive self-adjoint operators.

The form sum construction can be carried out if both forms are closed and at least one of them has positive lower bound.

Assume that $A$ is a positive self-adjoint operator with positive lower bound, and $B$ is an operator associated with a positive, closed form $t_B$. Assume also that $H_{A,B} := \operatorname{dom} J_A^* \cap \operatorname{dom} t_B$ is dense in $X$. Then it is easy to see that $(H_{A,B}, t_A + t_B)$ is complete, thus the representation theorem can be applied. The arising positive self-adjoint operator will be called the form sum of $A$ and $B$, and will be denoted by $A \dotplus B$.

## 3.3 Application of the results

### Covariance operators.

Consider a probability measure space $\langle \Omega, \mathcal{A}, \mu \rangle$, and let $\xi : \Omega \to X$ a random variable i.e. a weakly measurable function. Suppose that $\xi$ possesses a weak expectation, in other words

$$\mathrm{E}\,\xi := \int_\Omega \xi \,\mathrm{d}\mu$$

exists as a Pettis integral. Note that if $X$ is reflexive, according to Dunford and Gelfand, this is equivalent to requiring the existence of

$$\int_\Omega f(\xi)\,\mathrm{d}\mu$$

for all $f \in X^*$. Further, we make assumptions on the second moments, and suppose that the set

$$D = \left\{ f : f \in X^*, \int_\Omega |f(\xi)|^2 \,\mathrm{d}\mu < +\infty \right\}$$

is dense in $X^*$. We do not require that $D = X^*$ (cf. [27]).

As an example, take $X = \ell_2$, $\Omega = \{\omega_n : n = 1, 2, \ldots\}$ and $\mu(\{\omega_n\}) = ce^{-(3/2)n}$ with a suitable constant $c$. Setting $\xi(\omega_n)_k = n^k/k!$, it is easy to compute that, in this case, $D \neq X^*$ is dense.

In the sequel we assume that $\mathrm{E}\xi = 0$, since we could take $\xi - \mathrm{E}\xi$ instead of $\xi$. Define the sesquilinear form

$$t(f, g) = \mathrm{E}\,\left(f(\xi)\bar{g}(\xi)\right)$$

for $f, g \in D$.

**Theorem 3.3.1** *$t$ is a positive, closed, sesquilinear form on $D \times D$.*

**Proof.** Positivity is trivial. Suppose that $f_n \in D$ converges to $f \in X^*$ and $\mathrm{E}\,|f_n(\xi) - f_m(\xi)|^2 \to 0$, then $f_n(\xi)$ has a limit $g \in \mathcal{L}_2(\Omega, \mu)$, and moreover $g$ and $f(\xi)$ conincide almost everywhere, hence $\mathrm{E}\,|f(\xi)|^2 < +\infty$, implying $f \in D$ and $\mathrm{E}\,|f_n(\xi) - f(\xi)|^2 \to 0$. □

If $t$ possesses a positive lower bound and $X$ is reflexive, then the application of Theorem 3.1.1 provides a representing, self-adjoint operator $A$ from $X^*$ to $X^{**} = X$, which is called the covariance operator of the random variable $\xi$ (cf. [27]). Note that if $X$ is a Hilbert space then the original version of the representation theorem provides the covariance operator of $\xi$ associated to the closed form $t$ (even if $t$ has lower bound 0).

It is clear from the definitions that the covariance operator of the sum of independent random variables is the form sum of the covariance operators.

**Theorem 3.3.2** *Let $\xi$ and $\eta$ are independent random variables with covariance operators $A$ and $B$ respectively. Then the covariance operator of $\xi + \eta$ is $A \dot{+} B$.*

## Elliptic operators.

This is a classical application of the Friedrichs extension (see [8]). Take $X = L_p(\Omega), 1 \leq p < +\infty$ where $\Omega$ is a bounded domain with smooth boundary in $\mathbb{R}^n$. Define the operator $A$ from $L_p(\Omega)$ to $L_q(\Omega)$ by dom $A = C_0^\infty(\Omega)$ and

$$Af = -\sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}(a_{ik}\frac{\partial f}{\partial x_k}) + bf$$

where $a_{ik} \in C^1(\Omega)$, $b \in L^1_{loc}(\Omega)$, $b \geq 0$ and

$$\sum_{i,k=1}^{n} a_{ik}(x)\beta_i\overline{\beta_k} \geq \gamma \sum_{i}^{n} |\beta_i|^2, \gamma > 0$$

everywhere in $\Omega$ (uniform ellipticity). In this case we have

$$(Af, f) = \int_\Omega \left( -\sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ik}\frac{\partial f}{\partial x_k}\right) + bf \right) \overline{f}\, \mathrm{d}x =$$

$$\int_\Omega \left( \sum_{i,k=1}^{n} a_{ik}\frac{\partial f}{\partial x_i}\overline{\frac{\partial f}{\partial x_k}} + b|f|^2 \right) \mathrm{d}x \geq \gamma \int_\Omega \sum_{i=1}^{n} \left|\frac{\partial f}{\partial x_i}\right|^2 \mathrm{d}x.$$

Now, for $p \leq 2n/(n-2)$ we have

$$\int_\Omega \sum_{i=1}^{n} \left|\frac{\partial f}{\partial x_i}\right|^2 \mathrm{d}x \geq c\|f\|_p^2, \quad c > 0$$

by the Sobolev imbedding theorem (see e.g. [1] pp. 95-99). Thus $A$ has positive lower bound. The Friedrichs extension of $A$ is surjective, and this means that the equation

$$-\sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ik}\frac{\partial f}{\partial x_k}\right) + bf = g$$

has a weak solution for every $g \in L_q(\Omega)$ whenever $q \geq 2n/(n+2)$.

# Chapter 4

# Trotter's formula for projections

The form sum construction of positive, selfadjoint (and, more generally, $m$-sectorial) operators in Hilbert spaces is distinguished by Kato's famous result on the convergence of Trotter's product formula (see [18] Theorem and Addendum; cf. also the subsection 'Closed forms' in Section 4.1 below).

The aim of this chapter is to examine the convergence of Trotter's product formula when one of the $C_0$-semigroups is replaced by a projection (which can always be regarded as a constant degenerate semigroup). The motivaton to study Trotter's formula in this setting arises from the fact that for 'nice' open sets $\Omega \subset \mathbb{R}^n$ the $C_0$-semigroup on $L^2(\Omega)$ generated by the Laplacian with Dirichlet boundary conditions can be obtained as a limit of a formula of this type.

Let $A$ be the generator of a $C_0$-semigroup $(e^{tA})_{t\geq 0}$ on a Banach space $E$, and let $B \in \mathcal{B}(E)$. Then $A + B$ generates a $C_0$ semigroup which is given by Trotter's product formula

$$e^{t(A+B)} = \lim_{n\to\infty} (e^{\frac{t}{n}A} e^{\frac{t}{n}B})^n \tag{4.1}$$

where the limit is taken in the strong operator topology. A possible direction of generalization of this well-known result is discussed in [4] and [6]. Namely, the convergence of Trotter's product formula is examined in the case when the $C_0$-semigroup $e^{tB}$ is replaced by the simplest of degenerate semigroups, i.e. a projection $P \in \mathcal{B}(E)$. For convenience we include the basic notions here:

A family of operators $S(t)_{t>0}$ is called a *semigroup* on $E$ if $S:(0,\infty)\to\mathcal{B}(E)$ is strongly continuous and satisfies the semigroup property $S(t+s) = S(t)S(s)$ for all $s,t>0$. If, in addition, $S(0) := \lim_{t\to 0} S(t)$ exists strongly, then we say that $S(t)_{t>0}$ (or $S(t)_{t\geq 0}$) is a *continuous degenerate semigroup.* In this case $S(0)$ is a bounded projection, its image $E_0 := S(0)E$ is invariant under $S(t)$ $(t\geq 0)$, and the restriction of $S(t)_{t\geq 0}$ to $E_0$ is a $C_0$-semigroup on $E_0$ and $S(t)$ equals 0 on $E_1 := (I - S(0))E$ (see [16], Theorem 10.5.5). A trivial example of a continuous degenerate semigroup is given by $S(t) := P$ $(t>0)$, where $P$ denotes a bounded projection.

Now, in 4.1 we replace the $C_0$-semigroup $e^{tB}$ by the continuous degenerate semigroup $S(t) = P$ $(t>0)$, and we examine the convergence of the formula

$$\lim_{n\to\infty} (e^{\frac{t}{n}A}P)^n \tag{4.2}$$

under various assumptions on $A$ and $P$. (If 4.2 converges, then the limit can be regarded, in a sense, as the 'restriction' of the semigroup $e^{tA}$ to the subspace $PE$. Of course, in the trivial case when $e^{tA}$ and $P$ commute, the formula 4.2 does converge to the restriction of $e^{tA}$ to $PE$.) In Section 2 we describe some interesting conditions under which 4.2 converges strongly. For example, if $A$ is the generator of the Gaussian semigroup on $L^2(\mathbb{R}^n)$ and $Pf = 1_\Omega f$ where $\Omega\subset\mathbb{R}^n$ is a bounded open domain with Lipschitz boundary, we will see that 4.2 converges strongly to the semigroup generated by the Dirichlet Laplacian on $L^2(\Omega)$. In Section 3 we provide some non-trivial examples where 4.2 fails to converge.

This chapter is based on [22].

## 4.1 Convergence results

### Closed forms

In this subsection we describe an important case when Trotter's product formula converges. The results in this subsection are direct consequences of [8, Theorem and Addendum]. We describe the basic notions briefly:

Let $H$ be a Hilbert space and let

$$a : D(a) \times D(a) \to \mathbb{C}$$

be a sesquilinear mapping where $D(a)$, the domain of $a$, is a subspace of $H$. We assume that $a$ is semibounded, i.e. that there exists $\lambda \in \mathbb{R}$ such that

$$\|u\|_a^2 := \text{Re } a(u,u) + \lambda(u,u)_H > 0$$

for all $u \in D(a)$, $u \neq 0$. Moreover, we assume that $a + \lambda$ is sectorial and closed, i.e., that $|\text{Im } a(u,u)| \leq M(\text{Re } a(u,u) + \lambda(u,u)_H)$ and $(D(a), \|\cdot\|_a)$ is complete. In short, we will call $a$ a *closed form.* Let $K = \overline{D(a)}$ be the closure of $D(a)$ in $H$. Denote by $A$ the operator on $K$ associated with $a$, i.e.

$$D(A) = \{u \in D(a) : \exists v \in K \text{ such that } a(u,\phi) = (v,\phi)_H \text{ for all } \phi \in D(a)\}$$

and $Au = v$. Then $-A$ generates a $C_0$-semigroup $e^{-tA}$ on $K$. Denote by $Q$ the orthogonal projection on $K$. Now, define the operator $e^{-ta}$ on $H$ by

$$e^{-ta}x = e^{-tA}Qx, \quad x \in H, \ t \geq 0$$

Then $e^{-ta}$ is a continuous degenerate semigroup on $H$. We call it the *degenerate semigroup generated by $a$ on $H$.*

Now, let $b$ be a second closed form on $H$. Define $a+b$ on $H$ by $D(a+b) = D(a) \cap D(b)$ and $(a+b)(u,v) = a(u,v) + b(u,v)$. Then it is easy to see that $a+b$ is a closed form again. Now the following product formula holds (see [8, Theorem and Addendum]):

**Theorem 4.1.1** *Let $x \in H$. Then*

$$e^{-t(a+b)}x = \lim_{n\to\infty} (e^{-\frac{t}{n}a}e^{-\frac{t}{n}b})^n x$$

*for all $t > 0$.*

We apply this result in a particular situaion. Let $P$ be an orthogonal projection. Define the form $b$ by $D(b) = PH$ and $b(u,v) = 0$ for all $u, v \in PH$. Then $e^{-tb} = P$ for all $t \geq 0$. Therefore, as a corollary of Theorem 4.1.1 we have

**Theorem 4.1.2** *For any orthogonal projection $P$ and closed form $a$, the limit*

$$S(t)x = \lim_{n\to\infty} (e^{-\frac{t}{n}a}P)^n x$$

*exists for all $x \in H$ and $t > 0$, and $S(t)_{t>0}$ is the continuous degenerate semigroup generated by the form $a|_{PH}$.*

There is another possible way to formulate this result. Let $T(z)_{z\in\Sigma_\tau}$ be a holomorphic $C_0$-semigroup on $H$, defined on a sector $\Sigma_\tau := \{z \in \mathbb{C} : z \neq 0, |\arg z| < \tau\}$, $\tau \in (0, \frac{\pi}{2}]$. Assume that $\|(T(z)\| \leq 1$ for all $z \in \Sigma_\tau$. Then the generator $A$ of $T(z)$ is associated with a densely defined, semibounded, closed form $a$ (see [17], Chapters VI. and IX., and also [5], Theorem 1.2), so we have the following corollary (see [6] Theorem 4):

**Corollary 4.1.3** *Let $-A$ be the generator of a holomorphic $C_0$-semigroup $(e^{-zA})_{z\in\Sigma_\tau}$ on a Hilbert space $H$, where $\tau \in (0, \frac{\pi}{2}]\}$, and assume that $\|e^{-zA}\| \leq 1$ for all $z \in \Sigma_\tau$. Let $P$ be an orthogonal projection. Then*

$$S(t)x = \lim_{n\to\infty} (e^{-\frac{t}{n}A}P)^n x$$

*exists for all $x \in H$ and $t > 0$, and $S(t)_{t>0}$ is a continuous degenerate semigroup on $H$.*

## Bounded generators

Just as one would expect, in terms of convergence of 4.2 there is a universally 'nice' situation, namely the case of bounded generators.

**Theorem 4.1.4** *Let $A \in \mathcal{B}(E)$ be the generator of a $C_0$-semigroup $(e^{tA})_{t\geq 0}$ and let $P \in \mathcal{B}(E)$ be a projection. Then*

$$\lim_{n\to\infty} (e^{\frac{t}{n}A}P)^n x = e^{PAPt}Px$$

*for all $x \in E$ and uniformly for $t \in [0, T]$ for each $T \geq 0$.*

**Proof.** Case 1. Assume first that both $e^{tA}$ and $P$ are contractive. Let $V(t) := Pe^{tA}P \in \mathcal{B}(PE)$ and apply Chernoff's product formula (see e.g. [12], Theorem III.5.2) to the family $V(t)$ on the space $PE$. Note that $V(0) = I_{PE}$ , $\|V(t)\| \leq 1$ (for all $t \geq 0$), and

$\lim_{h\to 0} \frac{V(h)x_1 - x_1}{h} = PAx_1 = PAPx_1$ for all $x_1 \in PE$, and $PAP$ is a bounded operator on $PE$. Now, by Chernoff's product formula $\lim_{n\to\infty}[V(\frac{t}{n})]^n x_1 = e^{PAPt}x_1$ for all $x_1 \in PE$ and uniformly for $t \in [0, T]$. Furthermore, for any given $x \in E$ we can decompose $x$ as $x = Px + (I-P)x =: x_1 + x_2$ and we have $(e^{\frac{t}{n}A}P)^n x = (e^{\frac{t}{n}A}P)^n x_1 = e^{\frac{t}{n}A}(Pe^{\frac{t}{n}A}P)^{n-1}x_1$. Now, for large $n$ we have

$$\|e^{PAPt}Px - (Pe^{\frac{t}{n}A}P)^n x_1\| = \|e^{PAPt}x_1 - (Pe^{\frac{t}{n}A}P)^n x_1\| < \varepsilon$$

for $t \in [0, T]$, and also

$$\|e^{\frac{t}{n}A}(Pe^{\frac{t}{n}A}P)^{n-1}x_1 - (Pe^{\frac{t}{n}A}P)^n x_1\| = \|(I-P)e^{\frac{t}{n}A}(Pe^{\frac{t}{n}A}P)^{n-1}x_1\| =$$
$$\|(I-P)(e^{\frac{t}{n}A} - I)(Pe^{\frac{t}{n}A}P)^{n-1}x_1\| \leq \|I-P\| \cdot \|e^{\frac{t}{n}A} - I\| \cdot \|x_1\| < \varepsilon$$

Case 2. In the general case we first introduce an equvivalent norm on $E$ such that $P$ becomes contractive, then we use a rescaling argument to achieve that the semigroup becomes contractive. Indeed, with the new norm $\|x\|_0 := \|Px\| + \|(I-P)x\|$ $E$ is a Banach space, $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent, and $P$ is contractive on $E_{\|\cdot\|_0}$. Now, for $\lambda > \|A\|_0$ the rescaled semigroup $e^{-\lambda t}e^{At}$ is contractive on $E_{\|\cdot\|_0}$, therefore the result of Case 1 can be applied, and the result follows. □

**Remark** 1. By similar arguments one can prove the following statement: if $(e^{tA})_{t\geq 0}$ is a $C_0$-semigroup on $E$ and $P$ is a finite dimensional projection with ran $P \subset D(A)$ then $\lim_{n\to\infty}(e^{\frac{t}{n}A}P)^n x = e^{PAPt}Px$ where $e^{PAPt}$ is meant to be the $C_0$-semigroup on $PE$ generated by the *bounded* operator $PAP$. See also Remark 4 below.

## Positive semigroups

The results in this subsection are taken from [4].

Let $(X, \Sigma, \mu)$ be $\sigma$-finite measure space and let $(e^{tA})_{t\geq 0}$ be a positive $C_0$-semigroup on $E = L^p(X)$ where $1 \leq p < \infty$. Let $\Omega \subset X$ be measureable. Then $Pf := \mathbf{1}_\Omega f$ defines a projection on $E$, where $\mathbf{1}_\Omega$ denotes the characteristic function of $\Omega$. In this subsection we will use the notation $L^p(\Omega)$ both in the usual sense and and in the sense to denote the subspace of functions $f$ in $L^p(X)$ such that $f = 0$ almost everywhere in $\Omega^c$. When a function $f$ is in $L^p(\Omega)$ in the usual sense, we define the extension $\overline{f}$ on $X$ by $\overline{f}|_\Omega = f$ and $\overline{f}|_{\Omega^c} = 0$. The following result holds (see [4], Theorem 5.3):

**Theorem 4.1.5** *Let $f \in E$ and $t > 0$. Then*

$$S(t)f := \lim_{n\to\infty} (e^{\frac{t}{n}A}P)^n f$$

*exists and $S(t)_{t>0}$ is a continuous degenerate semigroup of positive operators. Furthermore, $S(0) := \lim_{t\to 0} S(t)$ is a projection of the the form $S(0)f = \mathbf{1}_Y f$ where $Y \subset \Omega$ is a measureable set.*

The continuous degenerate semigroup $S(t)_{t>0}$ can also be characterized by the following maximality property (see [4], Theorem 5.1): Let $T(t)_{t>0}$ be any semigroup of positive operators on $L^p(X)$ which maps $L^p(X)$ to $L^p(\Omega)$ and for which $0 \le T(t)f \le e^{tA}f$ for $t > 0$ and $0 \le f \in L^p(X)$. Then $T(t)f \le S(t)f$.

With the notations of Theorem 4.1.5 it can occur that $Y = \emptyset$ and $S(t) = 0$ (see [4], Example 5.4). However, in the following important case $Y = \Omega$ holds (for a detailed discussion of this Example and the following Remark see [4], Section 5 and 7):

**Example** (The Dirichlet Laplacian) Let $p = 2$, $X = \mathbb{R}^n$ (with Lebesgue measure) and $A = \Delta$ the Laplacian on $L^2(\mathbb{R}^n)$. Let $\Omega$ be a bounded open set with Lipschitz boundary. Then (with the notations of Theorem 4.1.5) we have $Y = \Omega$ and $S(t)|_{L^2(\Omega)} = e^{t\Delta_\Omega}$ where $\Delta_\Omega$ is the Dirichlet Laplacian on $L^2(\Omega)$, i.e. $D(\Delta_\Omega) = \{f \in H_0^1(\Omega) : \Delta f \in L^2(\Omega)\}$ and $\Delta_\Omega f = \Delta f$.

**Remark** 2. For general open sets $\Omega$ we still have $Y = \Omega$ and $S(t)|_{L^2(\Omega)} = e^{t\tilde{\Delta}_\Omega}$ where $\tilde{\Delta}_\Omega$ denotes the pseudo-Dirichlet Laplacian on $L^2(\Omega)$, i.e. $\tilde{\Delta}_\Omega$ is associated with the following densely-defined closed positive form $a$ on $L^2(\Omega)$: $D(a) = \{f \in L^2(\Omega) : \overline{f} \in H^1(\mathbb{R}^n)\}$ and $a(f,f) = \int_{\mathbb{R}^n} |\overline{f}|^2 + \sum_{j=1}^n \int_{\mathbb{R}^n} |D_j\overline{f}|^2 = \int_\Omega |f|^2 + \sum_{j=1}^n \int_{\mathbb{R}^n} |D_j\overline{f}|^2$ (this statement is a consequence of Theorem 4.1.2 above). This means that we have $\tilde{\Delta}_\Omega = \Delta_\Omega$ whenever $D(a) = H_0^1(\Omega)$. It is not an aim of this paper to describe sets $\Omega$ where this occurs, but in the Example above we take boundedness and Lipschitz boundary as simple sufficient conditions.

## 4.2 Counterexamples

In view of the results in Section 1 one may conjecture that 4.2 converges in more general settings. In particular, the following conjectures were given in [6]:

**(a)** Let $e^{tA}$ be a contractive $C_0$-semigroup on a Hilbert space $H$, and let $P$ be an orthogonal projection. Then 4.2 should converge.

**(b)** Let $e^{tA}$ be a positive, contractive $C_0$-semigroup on $L^p(X,\Sigma,\mu)$ (where $(X,\Sigma,\mu)$ is a $\sigma$-finite measure space, and $1 < p < \infty$), and let $P$ be a positive, contractive projection. Then 4.2 should converge.

In this section we present two examples which disprove these conjectures. We remark that the case $p = 1$ in conjecture (b) was not included, because a positive, contractive $C_0$-semigroup and a positive, contractive projection on $E = L^1([0,1])$, such that 4.2 fails to converge, was already provided in [6].

### Hilbert space case

Let us first remark that by using the theory of unitary dilations of contractive $C_0$-semigroups in Hilbert spaces (see e.g. [11], Corollary 6.14) one can reduce the first conjecture to the case of unitary $C_0$-semigroups. Indeed, take a unitary dilation $U(t)$ on a Hilbert space $H_0$ of the contractive $C_0$-semigroup $T(t)$ on $H$. Then, for all $x \in H$ we have $(T(\frac{t}{n})P)^n x = Q(U(\frac{t}{n})P_0)^n x$, where $Q$ and $P_0$ denote the orthogonal projections of $H_0$ onto $H$ and $PH$, respectively.

Therefore, we are considering unitary $C_0$-semigroups instead of arbitrary contractive ones. This is a great technical advantage (whether to prove or disprove the conjecture), because unitary semigroups can always be regarded as multiplication semigroups.

We carry out our construction in the space $L^2[0,1]$. As an example of unitary semigroup we take the semigroup of multiplications by $e^{ith}$, where $h$ is a real-valued, measurable function on $[0,1]$, to be specified later. We choose $P$ to be the one-dimensional orthogonal projection onto the space of constant functions, i.e. $Pf = \mathbf{1} \cdot \int_0^1 f(x)dx$. As a test function on which 4.2 will fail for $t = 1$, we take $\mathbf{1}$.

Denoting $c_n = \int_0^1 e^{i\frac{1}{n}h(x)}dx$, the function $\left[e^{\frac{1}{n}A}P\right]^n(\mathbf{1})$ becomes $c_n^{n-1}e^{i\frac{1}{n}h}$. However, by the Lebesgue Dominated Convergence Theorem, $\lim_{n\to\infty} c_n = 1$ as well as $\lim_{n\to\infty} e^{i\frac{1}{n}h} = \mathbf{1}$ in $L_2[0,1]$. So, $\lim_{n\to\infty}\left[e^{\frac{1}{n}A}P\right]^n(\mathbf{1})$ exists in $L^2[0,1]$ if and only if the numerical limit

$$\lim_{n\to\infty} c_n^n \tag{4.3}$$

exists. Now we specify the function $h$, for which we prove that 4.3 diverges. Put $h = \sum_{k=1}^{\infty} \chi_{(1/2^k, 1/2^{k-1}]} 2^k\pi$. Then $c_n = \sum_{k=1}^{\infty} \frac{1}{2^k} e^{i\frac{1}{n}2^k\pi}$. We show the following two inequalities

$$\liminf_{n\to\infty} |c_{2^n}|^{2^n} \geq e^{-(4+\frac{\pi^2}{4})} \tag{4.4}$$

$$\limsup_{n\to\infty} |c_{2^n 3}|^{2^n 3} \leq e^{-(6+\frac{\pi^2}{6}-\frac{\pi^4}{27\cdot 24\cdot 7})} \tag{4.5}$$

Noticing that $4+\frac{\pi^2}{4} < 6+\frac{\pi^2}{6}-\frac{\pi^4}{27\cdot 24\cdot 7}$ we get the desired result.

Let us show 4.4 first. Observe that

$$c_{2^n} = \sum_{k=1}^{n-1} \frac{1}{2^k} e^{i\frac{2^k}{2^n}\pi} - \frac{1}{2^n} + \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \sum_{k=1}^{n-1} \frac{1}{2^k} e^{i\frac{2^k}{2^n}\pi}.$$

Using the inequality $\cos(\alpha) \geq 1 - \frac{\alpha^2}{2}$ we get

$$\begin{aligned} |c_{2^n}| \geq |\mathrm{Re}\ c_{2^n}| &= \sum_{k=1}^{n-2} \frac{1}{2^k}\cos(\frac{2^k}{2^n}\pi) \geq \sum_{k=1}^{n-2} \frac{1}{2^k}(1 - \frac{\pi^2}{2}\frac{4^k}{4^n}) \\ &= 1 - \frac{4}{2^n} - \frac{\pi^2}{2}\frac{1}{4^n}(2^{n-1}-2) = 1 - \frac{1}{2^n}(4+\frac{\pi^2}{4}) + \frac{\pi^2}{4^n}. \end{aligned}$$

Since $\lim_{N\to\infty}(1+\frac{a}{N}+\frac{b}{N^2})^N = e^a$, we obtain 4.4.

To prove 4.5 let us simplify $c_{2^n 3}$. We have

$$\begin{aligned} c_{2^n 3} &= \sum_{k=1}^{n-1} \frac{1}{2^k} e^{i\frac{2^k}{2^n 3}\pi} + \frac{1}{2^n} e^{i\frac{1}{3}\pi} + \sum_{k=n+1}^{\infty} \frac{1}{2^k} e^{i\frac{2^{k-n}}{3}\pi} \\ &= \sum_{k=1}^{n-1} \frac{1}{2^k} e^{i\frac{2^k}{2^n 3}\pi} + \frac{1}{2^n}(\frac{1}{2} + i\frac{\sqrt{3}}{2}) + \frac{1}{2^n}\sum_{k=1}^{\infty} \frac{1}{2^k} e^{i\frac{2^k}{3}\pi}. \end{aligned}$$

Notice that $e^{i\frac{2^k}{3}\pi} = e^{i(-1)^{k+1}\frac{2}{3}\pi} = -\frac{1}{2} + i(-1)^{k+1}\frac{\sqrt{3}}{2}$. Thus, $\sum_{k=1}^{\infty} \frac{1}{2^k} e^{i\frac{2^k}{3}\pi} = -\frac{1}{2} + i\frac{\sqrt{3}}{6}$. After these computations $c_{2^n 3}$ becomes

$$\sum_{k=1}^{n-1} \frac{1}{2^k} e^{i\frac{2^k}{2^n 3}\pi} + i\frac{2\sqrt{3}}{2^n 3}.$$

Now using the inequality $\cos(\alpha) \leq 1 - \frac{\alpha^2}{2} + \frac{\alpha^4}{24}$ we obtain the following estimate

$$|\mathrm{Re}\ c_{2^n 3}| \leq \sum_{k=1}^{n-1} \frac{1}{2^k}\left(1 - \frac{\pi^2}{18}\frac{4^k}{4^n} + \frac{\pi^4}{81\cdot 24}\frac{16^k}{16^n}\right)$$
$$= 1 - \frac{1}{2^{n-1}} - \frac{\pi^2}{18}\frac{2^n-2}{4^n} + \frac{\pi^4}{81\cdot 24}\frac{8^n-8}{16^n 7}$$
$$= 1 - \frac{1}{2^n 3}\left(6 + \frac{\pi^2}{6} - \frac{\pi^4}{27\cdot 24\cdot 7}\right) + \frac{a}{(2^n 3)^2} + \frac{b}{(2^n 3)^4},$$

for some constants $a$ and $b$. Similarly, using $\sin(\alpha) \leq \alpha$, we have

$$|\mathrm{Im}\ c_{2^n 3}| \leq \sum_{k=1}^{n-1} \frac{1}{2^k}\frac{2^k}{2^n 3}\pi + \frac{2\sqrt{3}}{2^n 3} \leq \frac{(n+1)\pi}{2^n 3}.$$

Thus,

$$\begin{aligned} |c_{2^n 3}|^{2^n 3} &= \left(|\mathrm{Re}\ c_{2^n 3}|^2 + |\mathrm{Im}\ c_{2^n 3}|^2\right)^{\frac{2^n 3}{2}} \\ &\leq (1 - \frac{2}{2^n 3}(6 + \frac{\pi^2}{6} - \frac{\pi^4}{27\cdot 24\cdot 7}) + (\frac{2}{2^n 3})^2 (n+1)^2 a_1 \\ &+ (\frac{2}{2^n 3})^2 a_2 + \ldots + (\frac{2}{2^n 3})^8 a_8)^{\frac{2^n 3}{2}}. \end{aligned}$$

Passing to the upper limit as $n \to \infty$, we finally obtain 4.5.

**Remark** 3. The counterexample above was presented at the Autumn School on Evolution Equations, Levico, 2001. Subsequently, in a private communication to the author, G. Metafune proved that the more natural choice $h(x) := \frac{1}{x}$ also provides a counterexample. Interestingly, however, the absolute value of the sequence $c_n^n$ converges in that case.

**Remark** 4. The function $\mathbf{1}$ is not in the domain of the generator $A$ of our semigroup. In fact, we see from Remark 1 above that for any function $f \in D(A)$, $\|f\| = 1$ the formula 4.2 converges and we have

$$\lim_{n\to\infty} (e^{\frac{t}{n}A} P_f)^n f = e^{(Af,f)} \cdot f$$

where $P_f$ denotes the orthogonal projection on the 1-dimensional subspace spanned by $f$.

## $L^p$-case for positive semigroups

Our second example is on the Hilbert space $L^2[0, 2\pi]$, but now for a positive contractive $C_0$-semigroup and positive contractive projection.

We take $e^{tA}f(x) = f(x + 2\pi t)$, regarding $f$ as a $2\pi$-periodic function. Now let $P$ be the orthogonal projection onto the space spanned by the positive norm-one function $g(x) = \frac{1}{\sqrt{34\pi}}\left[4 + \sum_{k=0}^{\infty} \frac{1}{\sqrt{2^k}} \cos 2^k x\right]$. Notice that, like in the previous example, our projection is one-dimensional (see Remark 5 below). Simple substitution shows that 4.2 evaluated at $g$ for $t = 1$ exists if and only if the numerical limit $\lim_{n\to\infty}\left[\int_0^{2\pi} g(x)g(x+\frac{1}{n})dx\right]^n$ exists. Denoting

$$c_n = \int_0^{2\pi} g(x)g(x + \frac{1}{n})dx$$

and using the orthogonality of cosines, we obtain

$$c_n = \frac{16}{17} + \frac{1}{17}\sum_{k=1}^{\infty} \frac{1}{2^k} \cos \frac{2^k}{n}\pi$$

Following the same calculations as for the first example, we obtain inequalities 4.4 and 4.5 with powers doubled on the right hand sides.

This disproves the second conjecture.

**Remark** 5. As we have already noticed, the projections in our examples are one-dimensional. We will examine in Chapter 6 what property of the generator of a $C_0$-semigroup on a Hilbert space is responsible for the existence of 4.2 for all one-dimensional, or more specifically, one-dimensional orthogonal projections.

# Chapter 5

# A similarity result

The last two chapters are based on [21].

The aim of Chapters 5 and 6 is to give a characterization in Hilbert spaces of the generators of $C_0$-semigroups associated with closed, sectorial forms in terms of the convergence of Trotter's product formula for projections. In the course of the proof of the main result (Theorem 6.0.1) we will need a similarity result which is of independent interest: for any unbounded generator $A$ of a $C_0$-semigroup $e^{tA}$ it is possible to introduce an equivalent scalar product on the space, such that $e^{tA}$ becomes non-quasi-contractive with respect to the new scalar product.

The main result of Chapter 6 is then to prove the converse of Kato's result, i.e. that the strong convergence of 4.2 for all orthogonal projections $P$, in fact, characterizes generators $A$ such that $-A$ is associated with a closed sectorial form. To be more precise we recall the following result (see Theorem 4.1.4 and 4.1.2):

**Theorem 5.0.1** *Let $A$ be the generator of a $C_0$-semigroup $e^{tA}$ on a Hilbert space $H$. Consider the following statements:*

*(i) $A$ is bounded.*

*(ii) $-A$ is associated with a densely-defined, closed, sectorial form $a$ on $H$.*

*(iii) The formula $(e^{\frac{t}{n}A}P)^n x$ converges for all projections $P \in \mathcal{B}(H)$, and all $x \in H$ and $t > 0$.*

*(iv) The formula $(e^{\frac{t}{n}A}P)^n x$ converges for all orthogonal projections $P \in \mathcal{B}(H)$, and*

*all $x \in H$ and $t > 0$.*

*The following implications hold: (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv).*

We will show in Chapter 6 that the converse implications also hold. In the course of the proof we will need an auxillary result, given in Theorem 5.1.1 below, which can be regarded as a complement of [9].

## 5.1 Quasi-contractivity and bounded generators

In order to prove our main result [Theorem 6.0.1], first we need to characterize the class of generators $A$ on $H$, such that the $C_0$-semigroup $e^{tA}$ is quasi-contractive for *every* equivalent scalar product $(\ ,\ )_0$ on $H$. The characterization is provided by

**Theorem 5.1.1** *Let $A$ be the generator of a $C_0$-semigroup $e^{tA}$ on a Hilbert space $H$. The following are equivalent:*

*(i) $A$ is bounded.*

*(ii) The semigorup $e^{tA}$ is quasi-contractive for every equivalent scalar product $(\ ,\ )_0$ on $H$.*

*(iii) For every equivalent scalar product $(\ ,\ )_0$ on $H$ there exists $K_0 \in \mathbb{R}$ such that for every vector $x \in D(A)$, $(x,x)_0 = 1$ implies $\mathrm{Re}\ (Ax,x)_0 \le K_0$.*

**Proof.** The implications (ii) $\Leftrightarrow$ (iii) are consequences of the Lumer-Phillips theorem (see e.g. [12], Proposition 3.23.). The implications (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) are trivial. It remains to prove (iii) $\Rightarrow$ (i). We will need the following

**Definition 5.1.2** Let $T \in \mathcal{B}(H)$ be an injective operator, and $x \in H$, $\|x\| = 1$, and $0 < \delta \le 1$. We say that $x$ is a $\delta$-quasi-eigenvector of $T$ if

$$\delta \le \frac{|(x,Tx)|}{\|Tx\|} \le 1 \tag{5.1}$$

Note, that a 1-quasi-eigenvector is, in fact, an eigenvector of $T$.

Now, let $0 < \delta < 1$ be fixed. We prove the implication (iii) $\Rightarrow$ (i) by contradiction. Assume, therefore, that $A \notin \mathcal{B}(H)$, and also, by rescaling, that $A^{-1} =: T \in \mathcal{B}(H)$.

We emphasize that $T$ is an injective operator, and we will use this fact several times. Assume, furthermore, that a sequence $(h_n) \subset H$ is given with the following properties:

(a) $\|h_n\| = 1$ for all $n \geq 1$.

(b) $\{h_k,\ Th_k\} \perp \{h_j,\ Th_j\}$ for all $k \neq j$.

(c) $\lim_{n\to\infty} \|Th_n\| = 0$

(d) For every $n \geq 1$ the vector $h_n$ is *not* a $\delta$-quasi-eginvector of $T$.

We construct an equivalent scalar product $(\ ,\ )_0$ on $H$ with the help of the sequence $h_n$.

Let $H_n = \operatorname{span}\{h_n,\ Th_n\}$. Note, that $H_n$ is 2-dimensional because $h_n$ is not an eigenvector of $T$.

Let $Th_n = c_{1,n}h_n + c_{2,n}h_n^\perp$, where $\|h_n^\perp\| = 1$. Note that

$$\frac{|c_{1,n}|^2}{|c_{1,n}|^2 + |c_{2,n}|^2} < \delta^2 \quad \text{and} \quad \frac{|c_{2,n}|^2}{|c_{1,n}|^2 + |c_{2,n}|^2} > 1 - \delta^2$$

Hence,

$$\frac{|c_{1,n}|}{|c_{2,n}|} < \frac{\delta}{\sqrt{1-\delta^2}} \quad \text{and} \quad \frac{|c_{2,n}|}{\|Th_n\|} > \sqrt{1-\delta^2}$$

Define $Q_n \in \mathcal{B}(H_n)$ by

$$Q_n h_n := h_n + \overline{L}_n h_n^\perp$$
$$Q_n h_n^\perp := L_n h_n + (|L_n|^2 + 1)h_n^\perp$$

where $|L_n| = 2\frac{\delta}{\sqrt{1-\delta^2}}$ and $\overline{L_n c_{2,n}} > 0$ for all $n \geq 1$. It is clear that $Q_n = Q_n^* \geq 0$, $Q_n^{-1} \in \mathcal{B}(H_n)$, and $\|Q_n\|_{H_n} \leq K$, $\|Q_n^{-1}\|_{H_n} \leq K$ for some universal constant $K$ (not depending on $n$). Define $Q \in \mathcal{B}(H)$ by

$$Q := Q_1 \oplus Q_2 \oplus \ldots \bigoplus I_{(H_1 \oplus H_2 \oplus \ldots)^\perp}$$

It is easy to see that $Q$ is well-defined, $Q \in \mathcal{B}(H)$, $Q = Q^* \geq 0$, and $Q^{-1} \in \mathcal{B}(H)$. This means that $Q$ defines an equivalent scalar product on $H$ by $(x, y)_0 := (x, Qy)$.

Now, let $x_n := \frac{Th_n}{\|Th_n\|}$. Then

$$\text{Re } (Ax_n, x_n)_0 = \frac{1}{\|Th_n\|^2}\text{Re } (h_n, QTh_n) = \frac{1}{\|Th_n\|^2}\text{Re } (h_n, c_{1,n}h_n + c_{2,n}L_n h_n) =$$
$$= \frac{1}{\|Th_n\|^2}(\text{Re } c_{1,n} + \overline{c_{2,n}L_n}) \geq \frac{1}{\|Th_n\|^2}\frac{\delta}{\sqrt{1-\delta^2}}|c_{2,n}| \geq \frac{1}{\|Th_n\|}\delta \to +\infty$$

Let $y_n := \frac{x_n}{\|x_n\|_0}$. Then Re $(Ay_n, y_n)_0 \to +\infty$ still holds due to the equivalence of the scalar products $(\ ,\ )$ and $(\ ,\ )_0$.

In order to complete the proof of the theorem it remains to construct the sequence $h_n$ with the required properties. The construction is carried out in several steps.

*Step 1.* We construct an orthonormal sequence $(e_n) \subset H$, such that $\lim_{n\to\infty} \|Te_n\| = 0$.

Take the polar decomposition $T = UT_1$ of $T$, where $U$ is unitary and $T_1 = T_1^* \geq 0$. It is clear from the spectral theorem that there exists an orthonormal sequence $(e_n) \subset H$ such that $\lim_{n\to\infty} \|T_1 e_n\| = 0$ (otherwise $T_1$ and $T$ would be invertible, contrary to our assumption). Note, also, that $\|T_1 e_n\| = \|Te_n\|$ for all $n \in \mathbb{N}$, therefore $\lim_{n\to\infty} \|Te_n\| = 0$ as required.

*Step 2.* We construct an orthonormal sequence $(f_n) \subset H$ such that $\lim_{n\to\infty} \|Tf_n\| = 0$ and $f_{n+1} \perp \{f_1, Tf_1, ...f_n, Tf_n\}$.

We obtain the sequence $(f_n)$ by induction, with the help of the sequence $(e_n)$. Take an index $i_1$ such that $\|Te_{i_1}\| \leq 1$, and let $f_1 := e_{i_1}$. Assume now that $f_1, f_2, \ldots, f_n$ are already given such that

$$\|f_j\| = 1, \quad f_j \perp \{f_k, Tf_k\}, \quad \|Tf_j\| \leq \frac{1}{\sqrt{j}}$$

and $f_j \in \text{span}\{e_1, e_2, \ldots, e_{l_n}\}$, for all $1 \leq j, k \leq n$, $k < j$, and $l_n$ is an index depending on $n$ only.

Let $H_n := \text{span}\{Tf_1, Tf_2, \ldots, Tf_n\}$. Take indices $j_1, \ldots j_{n+1}$ such that $j_k > l_n$ and $\|Te_{j_k}\| \leq \frac{1}{n+1}$ for all $1 \leq k \leq n+1$. The subspace $H_n$ is at most $n$-dimensional, therefore there exists a non-trivial linear combination

$$f_{n+1} := \sum_{k=1}^{n+1} \lambda_k e_{j_k}$$

such that $\|f_{n+1}\| = 1$ and $f_{n+1} \perp H_n$.

It is clear, by construction, that $f_{n+1} \perp \{f_1, Tf_1, ...f_n, Tf_n\}$. Furthermore,

$$\|Tf_{n+1}\| \leq \frac{1}{n+1} \sum_{k=1}^{n+1} |\lambda_k| \leq \sqrt{\frac{\sum_{k=1}^{n+1} |\lambda_k|^2}{n+1}} = \frac{1}{\sqrt{n+1}}$$

*Step 3.* We construct an orthonormal sequence $(g_n) \subset H$ such that $\lim_{n\to\infty} \|Tg_n\| = 0$ and $\{g_j, Tg_j\} \perp \{g_k, Tg_k\}$ for all $j \neq k$.

We obtain the sequence $(g_n)$ by induction, with the help of the sequence $(f_n)$.

Let $g_1 = f_1$. Assume now that $g_1, g_2, \ldots, g_n$ are already given such that

$$\|g_j\| = 1, \quad \{g_j, Tg_j\} \perp \{g_k, Tg_k\}, \quad \|Tg_j\| \leq \frac{1}{\sqrt{2j-1}}$$

and $g_j \in \operatorname{span}\{f_1, f_2, \ldots, f_{b_n}\}$, for all $1 \leq j \neq k \leq n$, and $b_n$ is an index depending on $n$ only.

Let $G_n := \operatorname{span}\{g_1, Tg_1, g_2, Tg_2, \ldots, g_n, Tg_n\}$. Take indices $m_1, \ldots m_{2n+1}$ such that $m_k > b_n$ and $\|Tf_{m_k}\| \leq \frac{1}{2n+1}$ for all $1 \leq k \leq 2n+1$. The subspace $G_n$ is at most $2n$-dimensional, therefore there exists a non-trivial linear combination

$$g_{n+1} := \sum_{k=1}^{2n+1} \mu_k f_{m_k}$$

such that $\|g_{n+1}\| = 1$ and $Tg_{n+1} \perp G_n$.

It is clear, by construction, that $\{g_{n+1}, Tg_{n+1}\} \perp \{g_1, Tg_1, ...g_n, Tg_n\}$. Furthermore,

$$\|Tg_{n+1}\| \leq \frac{1}{2n+1} \sum_{k=1}^{2n+1} |\mu_k| \leq \sqrt{\frac{\sum_{k=1}^{2n+1} |\mu_k|^2}{2n+1}} = \frac{1}{\sqrt{2(n+1)-1}}$$

*Step 4.* We construct the orthonormal sequence $(h_n)$ with the properties stated at the beginning of the proof.

We obtain the sequence $(h_n)$ by induction, with the help of the sequence $(g_n)$.

Take an index $r_1$ such that $\|Tg_{r_1}\| \leq \frac{\delta^2}{10}\|Tg_1\|$. Let

$$h_1 := \frac{\delta}{2} g_1 + \sqrt{1 - \frac{\delta^2}{4}}\, g_{r_1}$$

We need to prove that $h_1$ is not a $\delta$-quasi-eigenvector of $T$. It is clear that

$$1 \geq \|Th_1\| \geq \left( \frac{\delta}{2} - \frac{\delta^2}{10}\sqrt{1 - \frac{\delta^2}{4}} \right) \|Tg_1\|$$

Also,

$$|(h_1, Th_1)| = \left| \frac{\delta^2}{4}(g_1, Tg_1) + (1 - \frac{\delta^2}{4})(g_{r_1}, Tg_{r_1}) \right| \leq \left( \frac{\delta^2}{4} + (1 - \frac{\delta^2}{4})\frac{\delta^2}{10} \right) \|Tg_1\|$$

Combining these two ineqalities a simple calculation shows that $\frac{|(h_1,Th_1)|}{\|Th_1\|} < \delta$, as required.

Assume now that vectors $h_1, \ldots, h_n$ are already given, such that $h_j$ is not a $\delta$-quasi-eigenvector of $T$,

$$\|h_j\| = 1, \quad \{h_j, Th_j\} \perp \{h_k, Th_k\}, \quad \|Th_j\| \leq \frac{1}{\sqrt{j}}$$

and $h_j \in \operatorname{span}\{g_1, g_2, \ldots, g_{a_n}\}$, for all $1 \leq j \neq k \leq n$, and $a_n$ is an index depending on $n$ only. Take indices $p_1$, $p_2$, such that $p_1, p_2 > a_n$ and $\|Tg_{p_1}\| \leq \frac{1}{\sqrt{n+1}}$, and $\|Tg_{p_2}\| \leq \frac{\delta^2}{10}\|Tg_{p_1}\|$. Let

$$h_{n+1} := \frac{\delta}{2} g_{p_1} + \sqrt{1 - \frac{\delta^2}{4}} g_{p_2}$$

It is clear that $\|Th_{n+1}\| \leq \frac{1}{\sqrt{n+1}}$, and it can be shown as above that $h_{n+1}$ is not a $\delta$-quasi-eigenvector of $T$. Hence, the sequence $(h_n)$ satisfies all requirements, and the proof is complete. $\square$

We see that the proof above exploits heavily the geometric structure of Hilbert spaces.

**Problem 5.1.3** *The author conjectures that a result corresponding to Theorem 5.1.1 holds also in Banach spaces. Namely, whenever $A$ is not bounded it should be possible to introduce an equivalent norm on the space such that $e^{tA}$ is not quasi-contractive with respect to the new norm. This problem, however, remains open.*

# Chapter 6

# The convergence of Trotter's formula

Now we present the main result concerning the convergence of Trotter's product formula for projections. We remark that the first part of Theorem 6.0.1 gives a result in the spirit of [10] Chapter 6: the universally 'nice' generators are necesssarily bounded.

**Theorem 6.0.1** *Let $A$ be the generator of a $C_0$-semigroup $e^{tA}$ on a Hilbert space $H$. Consider the following statements.*

*(i) $A$ is bounded.*

*(ii) $-A$ is associated with a densely-defined, closed, sectorial form $a$ on $H$.*

*(iii) The formula $(e^{\frac{t}{n}A}P)^n x$ converges for all projections $P \in \mathcal{B}(H)$, and all $x \in H$ and $t > 0$.*

*(iv) The formula $(e^{\frac{t}{n}A}P)^n x$ converges for all orthogonal projections $P \in \mathcal{B}(H)$, and all $x \in H$ and $t > 0$.*

*The following implications hold: (i) $\Leftrightarrow$ (iii), (ii) $\Leftrightarrow$ (iv).*

**Proof.** The implication (i) $\Rightarrow$ (iii) was proved in [22], while the implication (ii) $\Rightarrow$ (iv) is a consequence of [18], Addendum (see also [22], Theorem 4).

We prove the implication (iii) $\Rightarrow$ (i) by contradiction.

Assume first that the semigroup $e^{tA}$ is not quasi-contractive. By the Lumer-Phillips theorem this is equivalent to the fact that the numerical range of $A$ is not contained in any left half-plane.

We construct an element $g \in H$ such that $\|g\| = 1$, and

$$\lim_{n\to\infty} (e^{\frac{1}{n}A} P_g)^n g$$

does not exist, where $P_g$ denotes the one-dimensional projection onto the subspace spanned by $g$. The vector $g$ will be given as

$$g := \frac{\lim_{k\to\infty} g_k}{\| \lim_{k\to\infty} g_k \|}$$

where $(g_k)$ denotes a convergent sequence in $H$ to be constructed in the sequel.

Let $g_1 \in D(A)$, such that $\|g_1\| = 1$. First, we show that

$$\lim_{n\to\infty} (e^{\frac{1}{n}A} P_{g_1})^n g_1 = e^{(Ag_1, g_1)} g_1$$

(Note, that this result follows from the proof of Theorem 4.1.4 as mentioned in Remark 4. in Chapter 4. However, we give a more elementary proof here.)

Indeed,

$$(e^{\frac{1}{n}A} P_{g_1})^n g_1 = e^{\frac{1}{n}A} (P_{g_1} e^{\frac{1}{n}A} P_{g_1})^{n-1} g_1 = e^{\frac{1}{n}A} (P_{g_1} e^{\frac{1}{n}A} P_{g_1} g_1,\ g_1)^{n-1} g_1$$

and

$$\lim_{n\to\infty} (P_{g_1} e^{\frac{1}{n}A} P_{g_1} g_1,\ g_1)^{n-1} = e^{(Ag_1,\ g_1)}$$

because

$$\lim_{n\to\infty} \frac{(P_{g_1} e^{\frac{1}{n}A} P_{g_1} g_1,\ g_1) - 1}{1/n} = \lim_{n\to\infty} \left( \frac{(e^{\frac{1}{n}A} - I) g_1}{1/n},\ g_1 \right) = (Ag_1,\ g_1)$$

Now, choose $g_1$ such that Re $(Ag_1, g_1) \geq 1$ holds also.

Let $\varepsilon > 0$ be fixed. Take an index $n_1$ so large that

$$\| \left( e^{\frac{1}{n_1}A} P_{g_1} \right)^{n_1} g_1 - e^{(Ag_1, g_1)} g_1 \| < \varepsilon$$

It is clear from standard continuity arguments that there exists a $\delta_1 > 0$, such that for all $h \in B(g_1, \delta_1)$ we have

$$\| \left( e^{\frac{1}{n_1}A} P_{\frac{h}{\|h\|}} \right)^{n_1} \frac{h}{\|h\|} - e^{(Ag_1, g_1)} g_1 \| < 2\varepsilon$$

Without loss of generality we can assume that $\delta_1 < \frac{1}{2}$.

Now assume, that vectors $g_1, g_2, \ldots, g_k$, and positive numbers $\delta_1, \delta_2, \ldots, \delta_k$, and indices $n_1, n_2, \ldots, n_k$ are already given with the properties that:

$$g_j \in D(A), \quad \text{Re } (Ag_j, g_j) \geq j$$

and

$$\| \left( e^{\frac{1}{n_j}A} P_{\frac{h}{\|h\|}} \right)^{n_j} \frac{h}{\|h\|} - e^{(A\frac{g_j}{\|g_j\|}, \frac{g_j}{\|g_j\|})} \frac{g_j}{\|g_j\|} \| < 2\varepsilon$$

for all $1 \leq j \leq k$ and all $h \in B(g_j, \delta_j)$. Assume, furthermore, that

$$\|g_{j+1} - g_j\| < \min \left\{ \frac{\delta_1}{2^j}, \frac{\delta_2}{2^{j-1}}, \ldots \frac{\delta_j}{2} \right\}$$

for all $1 \leq j \leq k-1$.

The numerical range of $A$ is not bounded from the right, hence there exists a vector $f \in D(A)$ such that

$$\|f\| < \min \left\{ \frac{1}{\|Ag_k\|}, \frac{\delta_1}{2^k}, \frac{\delta_2}{2^{k-1}}, \ldots \frac{\delta_k}{2} \right\}$$

and Re $(Af, f) \geq 2$. Let $f_k := e^{i\alpha} f$ with suitable $\alpha$ such that Re $(Af_k, g_k) \geq 0$. Let

$$g_{k+1} := g_k + f_k$$

Then

$$\begin{aligned} \text{Re } (Ag_{k+1}, g_{k+1}) &= \text{Re } (Ag_k, g_k) + \text{Re } (Ag_k, f_k) + \\ + \text{Re } (Af_k, g_k) + \text{Re } (Af_k, f_k) &\geq k + (-1) + 0 + 2 = k + 1 \end{aligned}$$

Furthermore, we have

$$\lim_{n \to \infty} (e^{\frac{1}{n}A} P_{\frac{g_{k+1}}{\|g_{k+1}\|}})^n \frac{g_{k+1}}{\|g_{k+1}\|} = e^{(A\frac{g_{k+1}}{\|g_{k+1}\|}, \frac{g_{k+1}}{\|g_{k+1}\|})} \frac{g_{k+1}}{\|g_{k+1}\|}$$

Take an index $n_{k+1}$ so large that $n_{k+1} > n_k$ and

$$\| \left( e^{\frac{1}{n_{k+1}}A} P_{\frac{g_{k+1}}{\|g_{k+1}\|}} \right)^{n_{k+1}} \frac{g_{k+1}}{\|g_{k+1}\|} - e^{(A\frac{g_{k+1}}{\|g_{k+1}\|}, \frac{g_{k+1}}{\|g_{k+1}\|})} \frac{g_{k+1}}{\|g_{k+1}\|} \| < \varepsilon$$

It is clear from standard continuity arguments that there exists a $\delta_{k+1} > 0$, such that for all $h \in B(g_{k+1}, \delta_{k+1})$ we have

$$\| \left( e^{\frac{1}{n_{k+1}}A} P_{\frac{h}{\|h\|}} \right)^{n_{k+1}} \frac{h}{\|h\|} - e^{(A\frac{g_{k+1}}{\|g_{k+1}\|}, \frac{g_{k+1}}{\|g_{k+1}\|})} \frac{g_{k+1}}{\|g_{k+1}\|} \| < 2\varepsilon$$

It is clear, by construction, that the sequence $g_k$ converges in $H$. Let

$$h := \lim_{k\to\infty} g_k \quad \text{and} \quad g := \frac{h}{\|h\|}$$

Recall, that $\|g_1\| = 1$ and $\delta_1 < \frac{1}{2}$, therefore $\frac{1}{2} < \|g_k\| < \frac{3}{2}$ for all $k \geq 1$. It is also clear, by construction, that $h \in B(g_k, \delta_k)$ for all $k \geq 1$. Hence, for all $k \geq 1$ we have

$$\| \left( e^{\frac{1}{n_k}A} P_g \right)^{n_k} g - e^{(A\frac{g_k}{\|g_k\|}, \frac{g_k}{\|g_k\|})} \frac{g_k}{\|g_k\|} \| < 2\varepsilon$$

Notice, that

$$\|e^{(A\frac{g_k}{\|g_k\|}, \frac{g_k}{\|g_k\|})} \frac{g_k}{\|g_k\|}\| = e^{\frac{1}{\|g_k\|^2}\text{Re } (Ag_k, g_k)} > e^{\frac{1}{4}k}$$

This means that (the norm of) the sequence $(e^{\frac{1}{n}A}P_g)^n g$ does not converge.

Now, assume only that $A \notin \mathcal{B}(H)$. Introduce, by Theorem 5.1.1, an equivalent scalar product $(x, y)_0 := (x, Qy)$ on $H$, such that the semigroup $e^{tA}$ is not quasi-contractive with respect to $( \ , \ )_0$. Take an orthogonal projection $P_g$ (with respect to the scalar product $( \ , \ )_0$ ), such that $(e^{\frac{1}{n}A}P_g)^n g$ does not converge. Then, $P_g$ is a bounded (possibly non-orthogonal) projection with respect to the original scalar product $( \ , \ )$, such that $(e^{\frac{1}{n}A}P_g)^n g$ does not converge. This proves the implication (iii) $\Rightarrow$ (i).

The implication (iv) $\Rightarrow$ (ii) is also proved by contradiction.

Asume, that the numerical range of $A$ is not contained in any sector

$$\Sigma_{\phi,\omega} := \{z \in \mathbb{C} : \frac{\pi}{2} + \phi < \arg\,(z - \omega) < \frac{3}{2}\pi - \phi\}$$

with $\omega \in \mathbb{R}$, $\phi \in (0, \frac{\pi}{2})$. There are two cases to consider.

If the semigroup $e^{tA}$ is not quasi-contractive , then, by the arguments above, there exists a vector $g \in H$, such that $\|g\| = 1$ and $(e^{\frac{1}{n}A}P_g)^n g$ does not converge.

If the semigroup $e^{tA}$ is quasi-contractive then, by rescaling, we can assume that Re $(Ax, x) \leq -1$ for all $x \in D(A)$, $\|x\| = 1$.

We construct an element $g \in H$ such that $\|g\| = 1$, and $\lim_{n\to\infty}(e^{\frac{1}{n}A}P_g)^n g$ does not exist, where $P_g$ denotes the one-dimensional projection onto the subspace spanned by $g$. The vector $g$ will be given as

$$g := \frac{\lim_{k\to\infty} g_k}{\|\lim_{k\to\infty} g_k\|}$$

where $(g_k)$ denotes a convergent sequence in $H$ to be constructed in the sequel.

Take an arbitrary vector $g_1 \in D(A)$, $\|g_1\| = 1$. Let $(Ag_1, g_1) =: a_1 + b_1 i$. We know that

$$\lim_{n\to\infty} \left(e^{\frac{1}{n}A}P_{g_1}\right)^n g_1 = e^{(Ag_1,g_1)}g_1$$

Let $\varepsilon > 0$, and $\rho > 0$ be fixed. Take an index $n_1$ so large that

$$\| \left(e^{\frac{1}{n_1}A}P_{g_1}\right)^{n_1} g_1 - e^{(Ag_1,g_1)}g_1\| < \varepsilon$$

It is clear from standard continuity arguments that there exists a $\delta_1 > 0$, such that for all $h \in B(g_1, \delta_1)$ we have

$$\| \left(e^{\frac{1}{n_1}A}P_{\frac{h}{\|h\|}}\right)^{n_1} \frac{h}{\|h\|} - e^{(Ag_1,g_1)}g_1\| < 2\varepsilon$$

Without loss of generality we can assume that $\delta_1 < \frac{1}{2}$.

Now assume, that vectors $g_1, g_2, \ldots, g_k$, and positive numbers $\delta_1, \delta_2, \ldots, \delta_k$, real numbers $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k$, and indices $n_1, n_2, \ldots, n_k$ are already given with the following properties: for all $1 \le j \le k$ we have $|\varepsilon_j| < \rho$,

$$g_j \in D(A), \quad (A\frac{g_j}{\|g_j\|}, \frac{g_j}{\|g_j\|}) = a_j + (\varepsilon_j + b_1 + (j-1)\pi)i$$

(note that $\varepsilon_1 = 0$), where $a_1 - 1 < a_j \le -1$, and

$$\| \left(e^{\frac{1}{n_j}A}P_{\frac{h}{\|h\|}}\right)^{n_j} \frac{h}{\|h\|} - e^{(A\frac{g_j}{\|g_j\|},\frac{g_j}{\|g_j\|})}\frac{g_j}{\|g_j\|}\| < 2\varepsilon$$

for all $h \in B(g_j, \delta_j)$. Assume, furthermore, that

$$\|g_{j+1} - g_j\| < \min \left\{\frac{\delta_1}{2^j}, \frac{\delta_2}{2^{j-1}}, \ldots \frac{\delta_j}{2}\right\}$$

for all $1 \le j \le k-1$.

Now, we construct the vector $g_{k+1}$. The numerical range of $A$ is not contained in any sector, therefore there exists a sequence $(x_j) \subset D(A)$ such that, $\lim_{j\to\infty} \|x_j\| = 0$ and

$$\text{Im}\,\frac{(Ax_j, x_j)}{\|g_k\|^2} = \pi \quad \text{and} \quad \frac{\text{Re}\,(Ax_j, x_j)}{\|g_k\|^2} < \frac{a_k - (a_1 - 1)}{2}$$

Take $y_j := e^{i\alpha_j} x_j$ with suitable $\alpha_j$ such that $(Ay_j, g_k) \geq 0$ real. Then

$$\frac{(A(g_k + y_j), g_k + y_j)}{\|g_k\|^2} = \frac{(Ag_k, g_k)}{\|g_k\|^2} + \frac{(Ag_k, y_j)}{\|g_k\|^2} + \\ + \frac{(Ay_j, g_k)}{\|g_k\|^2} + \frac{(Ay_j, y_j)}{\|g_k\|^2} =: c_j + d_j i$$

The real part $c_j$ of this expression satisfies

$$c_j > (a_1 - 1) + (\frac{a_k - (a_1 - 1)}{2}) - \frac{|(Ag_k, y_j)|}{\|g_k\|^2}$$

for all $j \geq 1$. For the imaginary part $d_j$, we have

$$\lim_{j\to\infty} d_j = \varepsilon_k + b_1 + k\pi$$

This means that for large $j$ we have $\|y_j\| < \min\,\{\frac{\delta_1}{2^k}, \frac{\delta_2}{2^{k-1}}, \ldots \frac{\delta_k}{2}\}$, and

$$\frac{\text{Re}(A(g_k + y_j), g_k + y_j)}{\|g_k + y_j\|^2} > a_1 - 1$$

and

$$\frac{\text{Im}(A(g_k + y_j), g_k + y_j)}{\|g_k + y_j\|^2} = \varepsilon_{k+1} + b_1 + k\pi$$

where $|\varepsilon_{k+1}| < \rho$. Take such an index $j$, and define

$$g_{k+1} := g_k + y_j$$

Again, standard continuity arguments show that there exist a positive number $\delta_{k+1}$ and an index $n_{k+1}$ such that

$$\| \left(e^{\frac{1}{n_{k+1}}A} P_{\frac{h}{\|h\|}}\right)^{n_{k+1}} \frac{h}{\|h\|} - e^{(A\frac{g_{k+1}}{\|g_{k+1}\|}, \frac{g_{k+1}}{\|g_{k+1}\|})} \frac{g_{k+1}}{\|g_{k+1}\|} \| < 2\varepsilon$$

for all $h \in B(g_{k+1}, \delta_{k+1})$.

It is clear, by construction, that the sequence $g_k$ converges. Let

$$h := \lim_{k\to\infty} g_k \quad \text{and} \quad g := \frac{h}{\|h\|}$$

Recall, that $\|g_1\| = 1$ and $\delta_1 < \frac{1}{2}$, therefore $\frac{1}{2} < \|g_k\| < \frac{3}{2}$ for all $k \geq 1$. It is also clear, by construction, that $h \in B(g_k, \delta_k)$ for all $k \geq 1$. Hence, for all $k \geq 1$ we have

$$\| \left(e^{\frac{1}{n_k}A} P_g\right)^{n_k} g - e^{(A\frac{g_k}{\|g_k\|}, \frac{g_k}{\|g_k\|})} \frac{g_k}{\|g_k\|} \| < 2\varepsilon$$

Notice, furthermore that

$$\begin{aligned}
\|e^{(A\frac{g_{2k+1}}{\|g_{2k+1}\|}, \frac{g_{2k+1}}{\|g_{2k+1}\|})} \frac{g_{2k+1}}{\|g_{2k+1}\|} - e^{(A\frac{g_{2k}}{\|g_{2k}\|}, \frac{g_{2k}}{\|g_{2k}\|})} \frac{g_{2k}}{\|g_{2k}\|}\| = \\
\|e^{a_{2k+1}} e^{(\varepsilon_{2k+1}+b_1+2k\pi)i} \frac{g_{2k+1}}{\|g_{2k+1}\|} - e^{a_{2k}} e^{(\varepsilon_{2k}+b_1+(2k-1)\pi)i} \frac{g_{2k}}{\|g_{2k}\|}\| \geq \\
\|e^{a_{2k+1}+b_1 i} g_1 - e^{a_{2k}+(b_1-\pi)i} g_1\| - \|e^{a_{2k+1}+b_1 i}(e^{\varepsilon_{2k+1}} \frac{g_{2k+1}}{\|g_{2k+1}\|} - g_1)\| - \\
-\|e^{a_{2k}+(b_1-\pi)i}(e^{\varepsilon_{2k}} \frac{g_{2k}}{\|g_{2k}\|} - g_1)\| \geq \\
2e^{a_1-1} - \|e^{a_{2k+1}+b_1 i}(e^{\varepsilon_{2k+1}} \frac{g_{2k+1}}{\|g_{2k+1}\|} - g_1)\| - \|e^{a_{2k}+(b_1-\pi)i}(e^{\varepsilon_{2k}} \frac{g_{2k}}{\|g_{2k}\|} - g_1)\|
\end{aligned}$$

We can now choose the values of $\varepsilon, \delta_1, \rho$ so small that

$$\|e^{a_{2k+1}+b_1 i}(e^{\varepsilon_{2k+1}} \frac{g_{2k+1}}{\|g_{2k+1}\|} - g_1)\| + \|e^{a_{2k}+(b_1-\pi)i}(e^{\varepsilon_{2k}} \frac{g_{2k}}{\|g_{2k}\|} - g_1)\| \leq e^{a_1-1}$$

and $5\varepsilon \leq e^{a_1-1}$

Then we have

$$\|(e^{\frac{1}{n_{2k+1}}A} P_g)^{n_{2k+1}} g - (e^{\frac{1}{n_{2k}}A} P_g)^{n_{2k}} g\| \geq \varepsilon$$

Therefore the sequence $(e^{\frac{1}{n}A} P_g)^n g$ does not converge, and the proof is complete.

We also see from the proof that the set of vectors $g$, such that $\|g\| = 1$ and $(e^{\frac{1}{n}A} P_g)^n g$ does not converge, is dense on the unit sphere. □

As a last remark we note the following:

The specific counterexamples in Chapter 4 show that even the norm of $(e^{\frac{1}{n}A} P_g)^n g$ might not converge. The proof of the general case above, however, relies on the 'change of direction' of $(e^{\frac{1}{n}A} P_g)^n g$ for a particular $g$.

**Problem 6.0.2** *Assume that the generator $A$ of a $C_0$-semigroup is not associated with a closed form. Is it possible to choose a vector $g$ of norm 1, such that the norm of the sequence $(e^{\frac{1}{n}A}P_g)^n g$ does not converge?*

Also, it is natural to expect that the first part of Theorem 6.0.1 holds in arbitrary Banach spaces.

**Problem 6.0.3** *Let $e^{tA}$ be a $C_0$-semigroup on a Banach space $X$. Is it true that if $(e^{\frac{t}{n}A}P)^n x$ converges for all $x \in X,\ t > 0$ and all projections $P \in \mathcal{B}(X)$ then $A$ must be bounded?*

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# Summary

Closed sectorial forms provide a convenient way to define $m$-sectorial (and, in particular, semibounded self-adjoint) operators. The Friedrichs extension and the form sum are two basic manifestations of this idea. The form sum is related to Trotter's product formula by a result of Kato. This dissertation presents the author's results in this circle of ideas.

In Chapter 1 we describe a factorization argument for positive self-adjoint operators. This argument establishes a connection between form methods and operator methods. Applications of this factorization are included.

In Chapter 2 we apply the construction of Chapter 1 to the addition problem of positive, symmetric operators. We arrive at a generalized notion of the form sum construction. We prove a commutation property of this construction. We also describe some pathological phenomena concerning the addition of positive self-adjoint operators.

Chapter 3 considers closed, positive forms on reflexive Banach spaces. We examine which of the Hilbert space results can be carried over to this case.

In Chapter 4 we recall Kato's result concerning closed forms and Trotter's formula. We apply this result in the case when one of the semigroups is replaced by an orthogonal projection. The convergence of Trotter's formula for projections is then further investigated. Some convergence results and non-trivial counterexamples are given.

Chapter 5 describes a similarity result which will be needed in the characterization of the convergence of Trotter's formula for projections. We prove that if the generator of a $C_0$-semigroup on a Hilbert space is unbounded then it is possible to introduce an equivalent scalar product such that the semigroup becomes non-quasi-contractive.

In Chapter 6 we prove the converse of Kato's result: if Trotter's formula converges for all orthogonal projections then the generator must be associated to a closed form.

# Magyar nyelvű összefoglalás

Alulról korlátos önadjungált (és általánosabban $m$-szektoriális) operátorok definiálása gyakori zárt szektoriális formák segítségével. Két egyszerű példa erre a Friedrichs kiterjesztés és a formaösszeg. Kato egyik eredménye kapcsolatot létesít a formaösszeg és a Trotter formula között. Ez a disszertáció a szerző ilyen irányú eredményeit tartalmazza.

Az első fejezetben egy faktorizációs tételt bizonyítunk pozitív önadjungált operátorokra. Tárgyaljuk a tétel néhány alkalmazását.

A második fejezetben a formaösszeg fogalmának egy lehetséges átalánosítását definiáljuk az első fejezetben látott konstrukció segítségével. Bebizonyítjuk konstrukciónknak egy kommutációs tulajdonságát. Vizsgáljuk a pozitív önadjungált operátorok összegére adható különböző konstrukciók közötti kapcsolatot.

A harmadik fejezetben definiáljuk a pozitív zárt forma fogalmát reflexív Banach terekben. Megvizsgáljuk, hogy a Hilbert terek elméletéből ismert eredmények közül melyek vihetők át erre az esetre.

A negyedik fejezetben felidézzük Kato eredményét a zárt formák és a Trotter formula kapcsolatáról. Megemlítjük azt az esetet, amikor az egyik félcsoportot egy ortogonális projekcióval helyettesítjük. Ezután tovább vizsgáljuk a Trotter formula ezen változatát, és a konvergencia eredmények mellett két érdekes ellenpéldát is bemutatunk.

Az ötödik fejezetben egy hasonlósági eredményt bizonyítunk, amelyre a projekciós Trotter formula konvergenciájának karakterizációjakor lesz szükség: ha egy $C_0$-félcsoport generátora nem korlátos, akkor be lehet vezetni egy olyan ekvivalens skalár szorzatot, amelyre nézve a félcsoport nem kvázi-kontraktív.

A hatodik fejezetben kiegészítjük Kato eredményét: ha a Trotter formula minden ortogonális projekcióra konvergens, akkor a generátor egy zárt formából származik.

# Theses

of the PhD Thesis

## CLOSED FORMS AND TROTTER'S PRODUCT FORMULA

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Self-adjoint operators play a central role in the theory of Hilbert space operators. In the bounded case self-adjoint operators and symmetric sesquilinear forms are, in fact, the same. The generalization of this fact to the semibounded case is the representation theorem, which provides a convenient way to define semibounded self-adjoint operators via closed semibounded forms. Two basic examples of this idea are manifested in the Friedrichs extension of a positive symmetric operator, and the form sum construction of two positive self-adjoint operators. The theory of positive self-adjoint extensions was later significantly developed by Krein, while the form sum was distinguished among other possible extensions of the operator sum by a famous result of Kato on the convergence of the Trotter product formula.

My dissertation presents a collection of my results in this direction, based on the papers [1], [2], [3], and [4]. In the sequel chapters and theorems are numbered as in the dissertation.

## 1 Factorization of positive operators

This introductory chapter describes a factorization argument, due to Z. Sebestyén, which plays a central role in Chapters 2 and 3.

Given a subspace $D \subset H$ and a positive operator $a : D \to H$ the new scalar product $[ax, ay] := (ax, y)$ is well defined on $\operatorname{ran} a$ because $a$ is symmetric. It is also positive definite if we assume that $\mathrm{D}_*(a) := \{y \in H : \sup\{|(ax, y)|^2 : x \in D, (ax, x) \leq 1\} < \infty\}$ is dense in $H$. The completion of the space $(\operatorname{ran} a, [\ ,\ ])$ is denoted by $H_a$. Define $J : H_a \to H$ by: $\operatorname{dom} J := \operatorname{ran} a$, and $Jx = x$. It is easy check that the positive, self-adjoint operator $a_K := J^{**}J^*$ is a positive self-adjoint extension of $a$. In particular, if $a$ is self-adjoint then $a = J^{**}J^*$ holds.

Also, we see from the factorization that

$$\operatorname{dom} a_K{}^{\frac{1}{2}} = \operatorname{dom} J^* = \mathrm{D}_*(a),$$

$$\|a_K{}^{\frac{1}{2}}y\|^2 = \|J^*y\|^2 = \sup\ \{|(ax, y)|^2 : x \in \operatorname{dom} a, (ax, x) \leq 1\}$$

holds. Therefore we can identify the closed form corresponding to $a_K$.

The extended operator $a_K$ is called the Krein-von Neumann extension of $a$. Slight modifications of the same argument provide the Friedrichs extension $a_F$ and, in general, all extremal extensions of $a$.

At the end of the chapter we illustrate the previous notions by particular examples of different extensions of a positive symmetric operator $a$.

## 2 Form sum constructions

As the factorization argument of Chapter 1 establishes a link between 'form methods' and 'operator methods', it is natural to try to apply a similar approach to construct the form sum of positive operators. This is the theme of Chapter 2.

Let $a$ and $b$ be two positive, symmetric operators, and suppose that $\mathrm{D}_*(a) \cap \mathrm{D}_*(b)$ is dense in $H$. This implies, a fortiori, that $\mathrm{D}_*(a)$ and $\mathrm{D}_*(b)$ are dense, so that the auxillary Hilbert spaces $H_a$, $H_b$ are possible to construct, and the Krein-von Neumann extensions $a_K$ and $b_K$ exist. Consider the space $H_a \oplus H_b$, and the operator

$$J : H_a \oplus H_b \to H, \text{ with } \operatorname{dom} J = \operatorname{ran} a \oplus \operatorname{ran} b, \quad J(ax \oplus by) = ax + by.$$

It is easy to prove that $J^{**}J^*$ is a positive self-adjoint extension of $a+b$. The next theorem implies that $J^{**}J^*$ is, in fact, a generalization of the form sum construction.

**Theorem 2.1.2** *Let $a$ and $b$ be positive, symmetric operators such that $\mathrm{D}_*(a) \cap \mathrm{D}_*(b)$ is dense in $H$, and let $J$ be as above. Then the form sum of $a_K$ and $b_K$ is $J^{**}J^*$, i.e.*

$$a_K \dotplus b_K = J^{**}J^*.$$

The main result of Chapter 2 describes a commutation property of the form sum.

**Theorem 2.2.4** *Let $a, b$ be positive, symmetric operators with $\mathrm{D}_*(a) \cap \mathrm{D}_*(b)$ dense in $H$, and suppose that $E, F \in \mathcal{B}(H)$, such that both $E$ and $F$ leave $\operatorname{dom} a$ and $\operatorname{dom} b$ invariant, and for all $x \in \operatorname{dom} a$ and $y \in \operatorname{dom} b$*

$$E^*ax = aFx, \quad F^*ax = aEx, \quad E^*by = bFy, \quad F^*by = bEy.$$

*Then*

$$E^*(a \dotplus b) \subseteq (a \dotplus b)F \quad \text{and} \quad F^*(a \dotplus b) \subseteq (a \dotplus b)E.$$

As an interesting result we mention that the Friedrichs extension $(a+b)_F$ of the operator sum is also possible to construct in a similar way. Define

$$Q : H \to H_a \oplus H_b, \text{ with } \operatorname{dom} Q = \operatorname{dom} a \cap \operatorname{dom} b, \ \ Qx = ax \oplus bx.$$

**Theorem 2.3.1** *Suppose that $a$ and $b$ are positive, symmetric operators, and* $\operatorname{dom} a \cap \operatorname{dom} b$ *is dense in $H$. Then $Q^*Q^{**} = (a+b)_F$.*

At the end of the chapter we show that the extensions $(a+b)_K$, $a \dot{+} b$, and $(a+b)_F$ of the operator sum $a+b$ are, in general, different from each other.

The content of this chapter can be found in [1].

## 3 Positive forms on Banach spaces

It is natural to try to generalize the results of Chapter 2 to reflexive Banach spaces.

Let $X$ denote a reflexive complex Banach space, and $X^*$ its conjugate dual space (i.e. the space of all continuous, conjugate linear functionals over $X$). Let $D \subseteq X$ be a dense subspace, and let $t : D \times D \to \mathbb{C}$ be a sesquilinear form on $D$ (where $t$ is linear in the first variable and conjugate linear in the second). Assume that $t$ is positive with positive lower bound, i.e. $t(x,x) \geq \gamma \|x\|^2$, $\gamma > 0$. Assume also that $t$ is "closed" in the sense that $(D, t(\cdot,\cdot)) =: H$ is a Hilbert space (i.e. it is complete). In this case, the injection $i : H \to X$ is continuous, so $H$ can be regarded as a subspace of $X$. For brevity we will use the notation $[\cdot,\cdot]$ for $t(\cdot,\cdot)$. An operator $A$ from $X$ to $X^*$ can be associated to the form $t$ in a natural way: let $x \in D$ and take the functional $[x,y]$, $y \in D$; if this functional is continuous in the norm of $X$ then there is an element $z$ in $X^*$ for which $[x,y] = z(y) =: (z,y)$, in this case, let $Ax := z$.

**Theorem 3.1.1** *With notations as above the operator $A : X \to X^*$ is a positive, self-adjoint operator.*

Naturally, in this setting it is harder to establish self-adjointness of an operator. The following lemma can be used:

**Lemma 3.1.2** *If $B : X^* \to X$ is a bounded, injective, self-adjoint operator then $A := B^{-1}$ is also a self-adjoint operator from $X$ to $X^*$.*

With the help of Theorem 3.1.1 we are able to prove the existence of the Friedrichs extension in the strictly positiv case.

**Theorem 3.2.1** *Let $a : X \to X^*$ be a positive, densely defined operator with positive lower bound, $(ax, x) \geq \gamma \|x\|^2$, $\gamma > 0$ for every $x \in \operatorname{dom} a$. Then $a$ admits a positve self-adjoint extension with the same lower bound.*

The other way to show that an operator is self-adjoint is to prove that it is a symmetric extension of a given self-adjoint operator. This is the core of the argument in the following

**Lemma 3.2.2** *Let $A$ be a positive self-adjoint operator from $X$ to $X^*$ (it is not necessary that $A$ has positive lower bound). Then there exists an auxiliary Hilbert space $H$ and an operator $J : H \to X^*$ such that $A = JJ^*$.*

It is possible to introduce a more general notion of positive, closed forms (in order to include forms with lower bound 0). A positive form $t : D \times D \to \mathbb{C}$ will be called closed if whenever $x_n \subseteq D$ and $x_n \to x$ in $X$ and $t(x_n - x_m, x_n - x_m) \to 0$ then $x \in D$ and $t(x_n - x, x_n - x) \to 0$ (notice that when $t$ has positive lower bound then this definition agrees with the previous one). A particular example of a positive, closed form is the 'covariance form' of an $X$-valued random variable.

Consider a probability measure space $\langle \Omega, \mathcal{A}, \mu \rangle$, and let $\xi : \Omega \to X$ a random variable i.e. a weakly measurable function. Suppose that $\xi$ possesses a weak expectation, in other words

$$\int_\Omega f(\xi)\, \mathrm{d}\mu$$

exists for all $f \in X^*$. Further, we make assumptions on the second moments, and suppose that the set

$$D = \left\{ f : f \in X^*, \int_\Omega |f(\xi)|^2\, \mathrm{d}\mu < +\infty \right\}$$

is dense in $X^*$. We do not require that $D = X^*$.

As an example, take $X = \ell_2$, $\Omega = \{\omega_n : n = 1, 2, \ldots\}$ and $\mu(\{\omega_n\}) = ce^{-(3/2)n}$ with a suitable constant $c$. Setting $\xi(\omega_n)_k = n^k/k!$, it is easy to compute that, in this case, $D \neq X^*$ is dense.

We assume that $\mathrm{E}\,\xi = 0$, since we could take $\xi - \mathrm{E}\,\xi$ instead of $\xi$.

Define the sesquilinear form

$$t(f, g) = \mathrm{E}\ (f(\xi)\bar{g}(\xi))$$

for $f, g \in D$. We call $t$ the covariance form of $\xi$.

**Theorem 3.3.1** *$t$ is a positive, closed, sesquilinear form on $D \times D$.*

Positive closed forms on reflexive Banach spaces also appear in partial differential equations.

Take $X = L_p(\Omega), 1 \leq p < +\infty$ where $\Omega$ is a bounded domain with smooth boundary in $\mathbb{R}^n$. Define the operator $A$ from $L_p(\Omega)$ to $L_q(\Omega)$ by dom $A = C_0^\infty(\Omega)$ and

$$Af = -\sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}(a_{ik}\frac{\partial f}{\partial x_k}) + bf$$

where $a_{ik} \in C^1(\Omega)$, $b \in L^1_{loc}(\Omega)$, $b \geq 0$ and

$$\sum_{i,k=1}^{n} a_{ik}(x)\beta_i\overline{\beta_k} \geq \gamma \sum_{i}^{n} |\beta_i|^2, \gamma > 0$$

everywhere in $\Omega$ (uniform ellipticity). In this case we have

$$(Af, f) = \int_\Omega \left( -\sum_{i,k=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ik} \frac{\partial f}{\partial x_k} \right) + bf \right) \overline{f} \, \mathrm{d}x =$$

$$\int_\Omega \left( \sum_{i,k=1}^{n} a_{ik} \frac{\partial f}{\partial x_i} \frac{\overline{\partial f}}{\partial x_k} + b|f|^2 \right) \mathrm{d}x \geq \gamma \int_\Omega \sum_{i=1}^{n} \left| \frac{\partial f}{\partial x_i} \right|^2 \mathrm{d}x.$$

Now, for $p \leq 2n/(n-2)$ we have

$$\int_\Omega \sum_{i=1}^{n} \left| \frac{\partial f}{\partial x_i} \right|^2 \mathrm{d}x \geq c\|f\|_p^2, \quad c > 0$$

by the Sobolev imbedding theorem. Thus $A$ has positive lower bound. The Friedrichs extension of $A$ is surjective, and this means that the equation

$$-\sum_{i,k=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ik} \frac{\partial f}{\partial x_k} \right) + bf = g$$

has a weak solution for every $g \in L_q(\Omega)$ whenever $q \geq 2n/(n+2)$.

The content of this chapter is based on [2].

## 4 Trotter's formula for projections

In the Hilbert space setting positive closed forms are related to the convergence of Trotter's product formula by a famous result of Kato. We describe the basic notions briefly:

Let $H$ be a Hilbert space and let

$$a : D(a) \times D(a) \to \mathbb{C}$$

be a sesquilinear mapping where $D(a)$, the domain of $a$, is a subspace of $H$. We assume that $a$ is semibounded, i.e. that there exists $\lambda \in \mathbb{R}$ such that

$$\|u\|_a^2 := \text{Re } a(u,u) + \lambda(u,u)_H > 0$$

for all $u \in D(a)$, $u \neq 0$. Moreover, we assume that $a + \lambda$ is sectorial and closed, i.e., that $|\text{Im } a(u,u)| \leq M(\text{Re } a(u,u) + \lambda(u,u)_H)$ and $(D(a), \|\cdot\|_a)$ is complete. In short, we will call $a$ a *closed form*. Let $K = \overline{D(a)}$ be the closure of $D(a)$ in $H$. Denote by $A$ the operator on $K$ associated with $a$, i.e.

$$D(A) = \{u \in D(a) : \exists v \in K \text{ such that } a(u,\phi) = (v,\phi)_H \text{ for all } \phi \in D(a)\}$$

and $Au = v$. Then $-A$ generates a $C_0$-semigroup $e^{-tA}$ on $K$. Denote by $Q$ the orthogonal projection on $K$. Now, define the operator $e^{-ta}$ on $H$ by

$$e^{-ta}x = e^{-tA}Qx, \quad x \in H, \ t \geq 0$$

Then $e^{-ta}$ is a continuous degenerate semigroup on $H$. We call it the *degenerate semigroup generated by $a$ on $H$.*

Now, let $b$ be a second closed form on $H$. Define $a+b$ on $H$ by $D(a+b) = D(a) \cap D(b)$ and $(a+b)(u,v) = a(u,v) + b(u,v)$. Then it is easy to see that $a+b$ is a closed form again. Now, by a result of Kato, the following product formula holds:

**Theorem 4.1.1** *Let $x \in H$. Then*

$$e^{-t(a+b)}x = \lim_{n\to\infty} (e^{-\frac{t}{n}a}e^{-\frac{t}{n}b})^n x$$

*for all $t > 0$.*

We apply this result in a particular situaion. Let $P$ be an orthogonal projection. Define the form $b$ by $D(b) = PH$ and $b(u,v) = 0$ for all $u, v \in PH$. Then $e^{-tb} = P$ for all $t \geq 0$. Therefore, as a corollary of Theorem 4.1.1 we have

**Theorem 4.1.2** *For any orthogonal projection $P$ and closed form $a$, the limit*

$$S(t)x = \lim_{n\to\infty} (e^{-\frac{t}{n}a}P)^n x$$

*exists for all $x \in H$ and $t > 0$, and $S(t)_{t>0}$ is the continuous degenerate semigroup generated by the form $a|_{PH}$.*

A particularly interesting example of this theorem is the following:

**Example** (The Dirichlet Laplacian) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, and let $\Delta$ denote the Laplacian on $L^2(\mathbb{R}^n)$. Let $Pf := \mathbf{1}_\Omega f$. Then, for all $f \in L^2(\Omega)$ we have $\lim_{n\to\infty}(e^{\frac{t}{n}\Delta}P)^n f = e^{t\Delta_\Omega} f$ where $\Delta_\Omega$ is the Dirichlet Laplacian on $L^2(\Omega)$, i.e. $D(\Delta_\Omega) = \{f \in H_0^1(\Omega) : \Delta f \in L^2(\Omega)\}$ and $\Delta_\Omega f = \Delta f$.

Kato's result and this interesting example gives motivation to study the convergence of Trotter's formula for projections.

Further convergence results are possible to prove if the generator is bounded, or the semigroup is positive and the projection is of a particular form. These results are summarized in the next two theorems.

**Theorem 4.1.4** *Let $A \in \mathcal{B}(E)$ be the generator of a $C_0$-semigroup $(e^{tA})_{t\geq 0}$ and let $P \in \mathcal{B}(E)$ be a projection. Then*

$$\lim_{n\to\infty} (e^{\frac{t}{n}A}P)^n x = e^{PAPt}Px$$

*for all $x \in E$ and uniformly for $t \in [0,T]$.*

**Theorem 4.1.5** *Let $(X, \Sigma, \mu)$ be $\sigma$-finite measure space and let $(e^{tA})_{t\geq 0}$ be a positive $C_0$-semigroup on $E = L^p(X)$ where $1 \leq p < \infty$. Let $\Omega \subset X$ be measureable and let $Pf := \mathbf{1}_\Omega f$. Then for all $f \in E$ and $t > 0$*

$$S(t)f := \lim_{n\to\infty} (e^{\frac{t}{n}A}P)^n f$$

*exists and $S(t)_{t>0}$ is a continuous degenerate semigroup of positive operators. Furthermore, $S(0) := \lim_{t\to 0} S(t)$ is a projection of the the form $S(0)f = \mathbf{1}_Y f$ where $Y \subset \Omega$ is a measureable set.*

The last theorem is due to W. Arendt, and C. Batty.

In view of these convergence results one may conjecture that 4.2 converges in more general settings. In particular, the following conjectures were unsolved since 1997:

**(a)** Let $e^{tA}$ be a contractive $C_0$-semigroup on a Hilbert space $H$, and let $P$ be an orthogonal projection. Then $\lim_{n\to\infty}(e^{\frac{t}{n}A}P)^n x$ should converge for all $x \in H$ and $t > 0$.

**(b)** Let $e^{tA}$ be a positive, contractive $C_0$-semigroup on $L^p(X, \Sigma, \mu)$ (where $(X, \Sigma, \mu)$ is a $\sigma$-finite measure space, and $1 < p < \infty$), and let $P$ be a positive, contractive projection. Then $\lim_{n\to\infty}(e^{\frac{t}{n}A}P)^n x$ should converge for all $x \in H$ and $t > 0$.

The main result of Chapter 4 is to disprove these conjectures by providing two counterexamples.

In the first example we take $H = L^2[0, 1]$. We take the (unitary) multiplication semigroup $e^{ith}$ on $H$, where $h = \sum_{k=1}^{\infty} \chi_{(1/2^k, 1/2^{k-1}]} 2^k \pi$, and we take $Pf = \mathbf{1} \cdot \int_0^1 f(x)dx$.

In the second example we take $H = L^2[0, 2\pi]$. We take $e^{tA}f(x) = f(x + 2\pi t)$, regarding $f$ as a $2\pi$-periodic function, and we let $P$ be the orthogonal projection onto the space spanned by the positive norm-one function $g(x) = \frac{1}{\sqrt{34\pi}}\left[4 + \sum_{k=0}^{\infty} \frac{1}{\sqrt{2^k}} \cos 2^k x\right]$.

In both examples non-trivial calculations show that the norm of the sequence $(e^{\frac{t}{n}A}P)^n f$ does not converge for $f = \mathbf{1}$ and $f = g$, respectively.

Chapter 4 is based on [3].

## 5 A similarity result

The result of this chapter will be used in the characterization of the convergence of Trotter's formula for projections in Chapter 6. However, this similarity result can also be of independent interest. We remark that the corresponding result is not known for arbitrary Banach spaces.

**Theorem 5.1.1** *Let $A$ be the generator of a $C_0$-semigroup $e^{tA}$ on a Hilbert space $H$. The following are equivalent:*

*(i) $A$ is bounded.*

*(ii) The semigorup $e^{tA}$ is quasi-contractive for every equivalent scalar product $(\ ,\ )_0$ on $H$.*

*(iii) For every equivalent scalar product $(\ ,\ )_0$ on $H$ there exists $K_0 \in \mathbb{R}$ such that for every vector $x \in D(A)$, $(x, x)_0 = 1$ implies* $\mathrm{Re}\ (Ax, x)_0 \leq K_0$.

## 6 The convergence of Trotter's formula

In this chapter we give a characterization of the convergence of Trotter's formula for projections in terms of properties of the generator. The second part of the result proves, in a sense, the converse of Kato's Theorem.

**Theorem 6.0.1** *Let $A$ be the generator of a $C_0$-semigroup $e^{tA}$ on a Hilbert space $H$. Consider the following statements.*
*(i) $A$ is bounded.*
*(ii) $-A$ is associated with a densely-defined, closed, sectorial form $a$ on $H$.*
*(iii) The formula $(e^{\frac{t}{n}A}P)^n x$ converges for all projections $P \in \mathcal{B}(H)$, and all $x \in H$ and $t > 0$.*
*(iv) The formula $(e^{\frac{t}{n}A}P)^n x$ converges for all orthogonal projections $P \in \mathcal{B}(H)$, and all $x \in H$ and $t > 0$.*
*The following implications hold: (i) $\Leftrightarrow$ (iii), (ii) $\Leftrightarrow$ (iv).*

The results of Chapters 5 and 6 can be found in [4].

## References

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# Tézisek

a

**CLOSED FORMS AND TROTTER'S PRODUCT FORMULA**

**(Zárt formák és a Trotter formula)**

**című PhD értekezéshez**

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Az önadjungált operátorok központi szerepet játszanak a Hilbert terek operátorainak elméletében. Korlátos esetben az önadjungált operátorok és a szimmetrikus szeszkvilineáris formák között természetes bijekció van. Ennek az állításnak a megfelelője a félig korlátos esetben a reprezentációs tétel, amelynek segítségével félig korlátos önadjungált operátorokat gyakran definiálunk félig korlátos zárt formák által. Két egyszerű példa erre a pozitív szimmetrikus operátorok Friedrichs kiterjesztése illetve két pozitív önadjungált operátor formaösszegének konstrukciója. A pozitív önadjungált kiterjesztések elméletét később Krein fejlesztette tovább, míg a formaösszeg és a Trotter formula közötti kapcsolatra Kato egyik híres eredménye mutatott rá.

Ez a disszertáció a szerző ilyen irányú eredményeit mutatja be az [1], [2], [3] és [4] publikációkra építve. A továbbiakban a fejezeteket és tételeket a disszertációnak megfelelően számozom.

## 1 Pozitív operátorok faktorizációja

Ez a bevezető jellegű fejezet vázolja Z. Sebestyén faktorizációs eljárását, amely kulcsfontosságú lesz a 2. és 3. fejezetben.

Legyen adott egy $D \subset H$ altéren értelmzett $a : D \to H$ pozitív lineáris operátor. Az $a$ operátor $\operatorname{ran} a$ képterén bevezetjük az $[ax, ay] := (ax, y)$ új skalár szorzatot, amely jól definiált, mert $a$ szimmetrikus. Ha még feltesszük, hogy $\mathrm{D}_*\,(a) := \{y \in H : \sup\{|(ax, y)|^2 : x \in D, (ax, x) \leq 1\} < \infty\}$ sűrű, akkor az új skalár szorzat pozitív definit is. A $(\operatorname{ran} a, [\ ,\ ])$ tér teljessé tételét $H_a$-val jelöljük. Definiáljuk a $J : H_a \to H$ operátort a $\operatorname{dom} J := \operatorname{ran} a$ és $Jx = x$ összefüggésekkel. Könnyű látni, hogy a pozitív önadjungált $a_K := J^{**}J^*$ operátor kiterjesztése $a$-nak. Speciálisan, ha $a$ maga is önadjungált, akkor $a = J^{**}J^*$ teljesül.

A faktorizációból következik, hogy

$$\operatorname{dom} a_K{}^{\frac{1}{2}} = \operatorname{dom} J^* = \mathrm{D}_*\,(a),$$

$$\|a_K{}^{\frac{1}{2}}y\|^2 = \|J^*y\|^2 = \sup\,\{|(ax, y)|^2 : x \in \operatorname{dom} a, (ax, x) \leq 1\}.$$

Tehát az $a_K$-hoz tartozó zárt formát megadhatjuk a $J^*$ operátor segítségével.

Az $a_K$ operátort az $a$ operátor Krein-von Neumann kiterjesztésének nevezzük. A fenti faktorizációs eljárás kis változtatásával előállíthatjuk az $a$ operátor $a_F$ Friedrichs kiterjesztését is, illetve általában az összes úgynevezett extremális kiterjesztést.

A fejezet végén egy konkrét példán mutatjuk be a bevezetett fogalmakat.

## 2 A formaösszeg

Az első fejezet faktorizációs eljárása kapcsolatot teremt a 'forma módszer' és az 'operátor módszer' között, ezért természetes gondolat, hogy két pozitív operátor formaösszegét is hasonló faktorizációval próbáljuk előállítani. Erről szól a második fejezet.

Legyen $a$ és $b$ két pozitív szimmetrikus operátor, és tegyük fel, hogy $\mathrm{D}_*(a) \cap \mathrm{D}_*(b)$ sűrű $H$-ban. Ekkor $\mathrm{D}_*(a)$ és $\mathrm{D}_*(b)$ szintén sűrűek, így a $H_a$ és $H_b$ terek, valamint az $a_K$ és $b_K$ operátorok léteznek. Tekintsük a $H_a \oplus H_b$ teret, és a

$$J : H_a \oplus H_b \to H, \mathrm{dom}\, J = \mathrm{ran}\, a \oplus \mathrm{ran}\, b, \quad J(ax \oplus by) = ax + by$$

operátort.

Könnyű megmutatni, hogy $J^{**}J^*$ pozitív önadjungált kiterjesztése $a + b$-nek. A következő tétel azt mutatja, hogy $J^{**}J^*$ a formaösszeg általánosításának tekinthető.

**2.1.2 Tétel** *Legyen $a$ és $b$ két pozitív szimmetrikus operátor, amelyekre $\mathrm{D}_*(a) \cap \mathrm{D}_*(b)$ sűrű $H$-ban, és legyen $J$ a fent definiált operátor. Ekkor*

$$a_K \dot{+} b_K = J^{**}J^*.$$

A második fejezet fő eredménye a formaösszeg egy kommutációs tulajdonságát bizonyítja.

**2.2.4 Tétel** *Legyen $a$ és $b$ két pozitív szimmetrikus operátor, amelyekre $\mathrm{D}_*(a) \cap \mathrm{D}_*(b)$ sűrű $H$-ban, és tegyük fel, hogy $E, F \in \mathcal{B}(H)$, olyan operátorok, hogy $E$ és $F$ $\mathrm{dom}\, a$-t és $\mathrm{dom}\, b$-t invariánsan hagyják, és minden $x \in \mathrm{dom}\, a$-re és $y \in \mathrm{dom}\, b$-re fennáll, hogy*

$$E^*ax = aFx, \quad F^*ax = aEx, \quad E^*by = bFy, \quad F^*by = bEy.$$

*Ekkor*

$$E^*(a \dot{+} b) \subseteq (a \dot{+} b)F \quad \text{és} \quad F^*(a \dot{+} b) \subseteq (a \dot{+} b)E.$$

Érdekességkent megemlítjük, hogy az operátor összeg Friedrichs kiterjesztése (ha létezik) szintén előállítható hasonló faktorizációs eljárással. Ehhez definiáljuk a $Q$ operátort a

$$Q : H \to H_a \oplus H_b, \ \operatorname{dom} Q = \operatorname{dom} a \cap \operatorname{dom} b, \ \ Qx = ax \oplus bx$$

összefüggésekkel.

**2.3.1 Tétel** *Legyen $a$ és $b$ két pozitív szimmetrikus operátor, amelyekre* $\operatorname{dom} a \cap \operatorname{dom} b$ *sűrű $H$-ban. Ekkor $Q^*Q^{**} = (a+b)_F$.*

A fejezet végén megmutatjuk, hogy az $(a+b)_K$, $a \dot{+} b$, és $(a+b)_F$ kiterjesztések általában különböznek egymástól.

Ezt a fejezetet lényegében tartalmazza az [1] publikáció.

# 3 Pozitív formák Banach tereken

Természetes ötlet megpróbálni a második fejezet eredményeit reflexív Banach terekre álataláносítani.

Legyen $X$ reflexív Banach tér, és $X^*$ a konjugált duális tere (azaz az $X$-en értelmezett folytonos, konjugáltan lineáris funkcionálok tere). Legyen $D \subset X$ sűrű altér, és legyen $t : D \times D \to \mathbb{C}$ szeszkvilineáris forma $D$-n (megállapodás szerint $t$ az első változóban lineáris, és a másodikban konjugáltan lineáris). Tegyük fel, hogy $t(x,x) \geq \gamma \|x\|^2$ valamilyen $\gamma > 0$-ra. Tegyük fel továbbá, hogy $t$ 'zárt' olyan értelemben, hogy $(D, t(\cdot,\cdot)) =: H$ Hilbert teret alkot. Az $i : H \to X$ beágyazás folytonos, így $H$-t az $X$ tér egy alterének tekinthetjük. A rövidség kedvéért a $[\cdot,\cdot] := t(\cdot,\cdot)$ jelölést fogjuk alkalmazni. A $t$ formához természetes módon asszociálható egy $A : X \to X^*$ operátor: legyen $x \in D$ és tekintsük az $[x,y]$, $y \in D$ funkcionált; ha ez folytonos $X$ felett, akkor létezik egy $z \in X^*$, amelyre $[x,y] = z(y) =: (z,y)$ teljesül. Legyen $Ax := z$.

**3.1.1 Tétel** *A fenti $A : X \to X^*$ operátor pozitív és önadjungált.*

Természetesen most nehezebb egy operátor önadjungáltságát megmutatni, mint Hilbert terek esetében. A fenti tétel bizonyítása a következő lemmát használja:

**3.1.2 Lemma** *Ha $B : X^* \to X$ injektív korlátos önadjungált operátor, akkor $A := B^{-1}$ szintén önadjungált.*

A 3.1.1 Tétel segítségével könnyű bebizonyítani a Friedrichs kiterjesztés létezését a szigorúan pozitív esetben.

**3.2.1 Tétel** *Legyen $a : X \to X^*$ pozitív sűrűn definiált operátor, amelyre $(ax, x) \geq \gamma \|x\|^2$, $\gamma > 0$. Ekkor a-nak létezik pozitív önadjungált kiterjesztése ugyanazzal az alsó korláttal.*

Egy operátor önadjungáltságának bizonyítása történhet még úgy is, hogy megmutatjuk, hogy az operátorunk egy adott önadjungált operátor szimmetrikus kiterjesztése. Ez az érvelés érvényesül a következő lemmában.

**3.2.2 Lemma** *Legyen $A : X \to X^*$ pozitív önadjungált operátor (itt nem szükséges, hogy A-nak pozitív alsó korlátja legyen). Ekkor létezik egy H Hilbert tér, és egy $J : H \to X^*$ operátor, amelyre $A = JJ^*$ teljesül.*

A pozitív zárt formáknak egy általánosabb definíciója is lehetséges (annak érdekében, hogy az olyan formákra is kiterjesszük a definíciót, amelyeknek alsó korlátja 0). Egy $t : D \times D \to \mathbb{C}$ pozitív formát zártnak nevezünk, ha abból, hogy $x_n \subseteq D$ és $x_n \to x$ $X$-ben és $t(x_n - x_m, x_n - x_m) \to 0$ következik, hogy $x \in D$ és $t(x_n - x, x_n - x) \to 0$ (megjegyezzük, hogy a fenti definíció ekvivalens a korábbival, ha $t$-nek pozitív az alsó korlátja). Egy $X$ értékű valószínűségi változó 'kovariancia formája' konkrét példát szolgáltat pozitív zárt formákra.

Legyen $\langle \Omega, \mathcal{A}, \mu \rangle$ egy valószínűségi mététktér, és legyen $\xi : \Omega \to X$ egy valószínűségi vátozó (azaz egy gyengén mérhető függvény). Tegyük fel, hogy

$$\int_\Omega f(\xi)\, \mathrm{d}\mu$$

létezik minden $f \in X^*$ esetén. Tegyük fel továbbá, hogy a

$$D = \left\{ f : f \in X^*, \int_\Omega |f(\xi)|^2\, \mathrm{d}\mu < +\infty \right\}$$

halmaz sűrű $X^*$-ban. (Azt azonban nem szükséges kikötni, hogy $D = X^*$ teljesüljön.)

Lássunk egy konkrét példát: legyen $X = \ell_2$, $\Omega = \{\omega_n : n = 1, 2, \ldots\}$ és $\mu(\{\omega_n\}) = ce^{-(3/2)n}$ egy megfelelő $c$ konstanssal. Legyen $\xi(\omega_n)_k = n^k/k!$. Könnyű belátni, hogy ekkor $D \neq X^*$ sűrű altér.

Feltesszük, hogy $\mathrm{E}\, \xi = 0$, hiszen különben tekinthetjük $(\xi - \mathrm{E}\, \xi)$-t $\xi$ helyett.

Legyen

$$t(f,g) = \mathrm{E}\ (f(\xi)\bar{g}(\xi))$$

minden $f, g \in D$-re. A $t$ formát a $\xi$ valószínűségi változó kovariancia formájának nevezzük.

**3.3.1 Tétel** *A fenti $t$ forma pozitív zárt forma $D \times D$-n.*

A reflexív Banach tereken definiált pozitív zárt formák a parciális differenciálegyenletekben is szerepet kapnak.

Legyen $X = L_p(\Omega),\ 1 \le p < +\infty$, ahol $\Omega$ egy sima peremű korlátos tartomány $\mathbb{R}^n$-ben. Definiáljuk az $A : L_p(\Omega) \to L_q(\Omega)$ operátort a következőképpen: $\operatorname{dom} A = C_0^\infty(\Omega)$ és

$$Af = -\sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}(a_{ik}\frac{\partial f}{\partial x_k}) + bf$$

ahol $a_{ik} \in C^1(\Omega)$, $b \in L^1_{loc}(\Omega)$, $b \ge 0$ és

$$\sum_{i,k=1}^{n} a_{ik}(x)\beta_i\overline{\beta_k} \ge \gamma \sum_{i}^{n} |\beta_i|^2, \gamma > 0$$

mindenütt $\Omega$-n (uniform ellipticitás). Ilyenkor fennáll, hogy

$$(Af,f) = \int_\Omega \left( -\sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ik}\frac{\partial f}{\partial x_k}\right) + bf \right) \overline{f}\,\mathrm{d}x =$$

$$\int_\Omega \left( \sum_{i,k=1}^{n} a_{ik}\frac{\partial f}{\partial x_i}\frac{\overline{\partial f}}{\partial x_k} + b|f|^2 \right)\,\mathrm{d}x \ge \gamma \int_\Omega \sum_{i=1}^{n} \left|\frac{\partial f}{\partial x_i}\right|^2 \mathrm{d}x.$$

Továbbá $p \le 2n/(n-2)$ esetén

$$\int_\Omega \sum_{i=1}^{n} \left|\frac{\partial f}{\partial x_i}\right|^2 \mathrm{d}x \ge c\|f\|_p^2, \quad c > 0$$

teljesül a Sobolev-féle beágyazási tétel szerint. Tehát az $A$ operátor alsó korlátja pozitív. Az $A$ operátor Friedrichs kiterjesztése szürjektív, és ez azt jelenti, hogy a

$$-\sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ik}\frac{\partial f}{\partial x_k}\right) + bf = g$$

egyenletnek minden $g \in L_q(\Omega)$-ra létezik gyenge megoldása, ha $q \geq 2n/(n+2)$.

A harmadik fejezet eredményei a [2] publikációban találhatók meg.

## 4 A Trotter formula projekciókra

A Hibert tereken értelmezett zárt formák és a formaösszeg konstrukciója Kato egyik érdekes tétele szerint kapcsolatban állnak a Trotter formula konvergenciájával. Röviden ismertetjük a fogalmakat:

Legyen a $H$ Hilbert téren adva egy

$$a : D(a) \times D(a) \to \mathbb{C}$$

szeszkvilineáris forma (ahol $D(a)$ a $H$ tér egy altere). Tegyük fel, hogy $a$ alulról korlátos, azaz létezik olyan $\lambda \in \mathbb{R}$, amelyre

$$\|u\|_a^2 := \text{Re } a(u,u) + \lambda(u,u)_H > 0$$

minden $u \in D(a)$, $u \neq 0$ esetén. Feltesszük továbbá, hogy $a + \lambda$ szektoriális és zárt, azaz $|\text{Im } a(u,u)| \leq M(\text{Re } a(u,u) + \lambda(u,u)_H)$ és a $(D(a), \|\cdot\|_a)$ tér teljes. Röviden $a$-t *zárt formának* nevezzük. Legyen $K = \overline{D(a)}$ a $D(a)$ altér lezártja $H$-ban. Jelöljük $A$-val az $a$ formához asszociált operátort a $K$ téren, azaz

$$D(A) = \{u \in D(a) : \exists v \in K \text{ amelyre } a(u,\phi) = (v,\phi)_H \text{ for all } \phi \in D(a)\}$$

és $Au = v$. Ekkor $-A$ egy $e^{-tA}$ $C_0$-félcsoportot generál a $K$ téren. Jelöljük $Q$-val a $K$ altérre való ortogonális projekciót. Definiáljuk most az $e^{-ta}$ operátort a $H$ téren az

$$e^{-ta}x = e^{-tA}Qx, \quad x \in H, \ t \geq 0$$

formulával. Az $e^{-ta}$ folytonos degenerált félcsoportot az *a forma által generált félcsoportnak* hívjuk.

Tegyük most fel, hogy adott egy további $b$ zárt forma is $H$-n, és definiáljuk az $a+b$ formát $D(a+b) := D(a) \cap D(b)$-n az $(a+b)(u,v) = a(u,v) + b(u,v)$ formulával. Könnyű belátni, hogy $a+b$ ismét csak zárt forma. Kato eredménye szerint a következő Trotter formula teljesül:

**4.1.1 Tétel** *Legyen $x \in H$. Ekkor*

$$e^{-t(a+b)}x = \lim_{n\to\infty}(e^{-\frac{t}{n}a}e^{-\frac{t}{n}b})^n x$$

*minden $t > 0$-ra.*

Most ezt a formulát egy speciális esetben alkalmazzuk. Legyen $P$ tetszőleges ortogonális projekció. Definiáljuk a $b$ formát a következőképpen: $D(b) = PH$ és $b(u,v) = 0$ minden $u, v \in PH$-ra. Ekkor $e^{-tb} = P$ teljesül minden $t \geq 0$-ra. Tehát a 4.1.1 Tétel következményeként kapjuk:

**4.1.2 Tétel** *Legyen $P$ ortogonális projekció, és legyen a zárt forma. Ekkor*

$$S(t)x = \lim_{n\to\infty}(e^{-\frac{t}{n}a}P)^n x$$

*létezik minden $x \in H$-ra, és $t > 0$-ra, és $S(t)_{t>0}$ az $a|_{PH}$ forma által generált folytonos félcsoport.*

Ennek az eredménynek egy érdekes alkalmazása a következő:

**Példa** (Dirichlet-féle Laplace operátor) Legyen $\Omega \subset \mathbb{R}^n$ korlátos nyílt halmaz Lipschitz peremmel, és jelölje $\Delta$ a Laplace operátort $L^2(\mathbb{R}^n)$-en. Legyen $Pf := \mathbf{1}_\Omega f$. Ekkor minden $f \in L^2(\Omega)$-ra igaz, hogy $\lim_{n\to\infty}(e^{\frac{t}{n}\Delta}P)^n f = e^{t\Delta_\Omega}f$ ahol $\Delta_\Omega$ jelöli a Dirichlet peremfeltételű Laplace operátort $L^2(\Omega)$-n, azaz $D(\Delta_\Omega) = \{f \in H_0^1(\Omega) : \Delta f \in L^2(\Omega)\}$ and $\Delta_\Omega f = \Delta f$.

Kato fenti eredménye és ez az érdekes alkalmazás adta a motivációt a projekciós Trotter formula további vizsgálatára.

A következő két tétel további olyan eseteket tárgyal, amikor konvergencia teljesül.

**4.1.4 Tétel** *Legyen $A \in \mathcal{B}(E)$ az $(e^{tA})_{t\geq 0}$ $C_0$-félcsoport generátora, és legyen $P \in \mathcal{B}(E)$ tetszőleges projekció. Ekkor*

$$\lim_{n\to\infty}(e^{\frac{t}{n}A}P)^n x = e^{PAPt}Px$$

*minden $x \in E$-re és a konvergencia egyenletes $t \in [0,T]$-re.*

**4.1.5 Tétel** *Legyen $(X, \Sigma, \mu)$ egy $\sigma$-véges mértéktér, és legyen $(e^{tA})_{t\geq 0}$ egy pozitív $C_0$-félcsoport $E = L^p(X)$-en, ahol $1 \leq p < \infty$. Legyen $\Omega \subset X$ egy mérhető halmaz, és* *legyen $Pf := \mathbf{1}_\Omega f$. Ekkor minden $f \in E$-re és $t > 0$-ra*

$$S(t)f := \lim_{n\to\infty} (e^{\frac{t}{n}A}P)^n f$$

*létezik, és $S(t)_{t>0}$ pozitív operátoroknak egy folytonos degenerált félcsoportja. Továbbá, az $S(0) := \lim_{t\to 0} S(t)$ projekció a következő alakú: $S(0)f = \mathbf{1}_Y f$, ahol $Y \subset \Omega$ egy mérhető halmaz.*

Ez a tétel W. Arendt-től és C. Batty-től származik.

A fenti eredmények alapján W. Arendt a következő sejtéseket fogalmazta meg 1997-ben:

**(a)** Legen $e^{tA}$ kontraktív $C_0$-félcsoport egy $H$ Hilbert téren, és legyen $P$ ortogonális projekció. Ekkor $\lim_{n\to\infty}(e^{\frac{t}{n}A}P)^n x$ konvergens minden $x \in H$-ra és $t > 0$-ra.

**(b)** Legyen $e^{tA}$ egy pozitív, kontraktív $C_0$-félcsoport $L^p(X, \Sigma, \mu)$-n (ahol $(X, \Sigma, \mu)$ egy $\sigma$-véges mértéktér, és $1 < p < \infty$), és legyen $P$ pozitív kontraktív projekció. Ekkor $\lim_{n\to\infty}(e^{\frac{t}{n}A}P)^n x$ konvergens minden $x \in H$-ra és $t > 0$-ra.

A negyedik fejezet fő eredménye ezeknek a sejtéseknek a megcáfolása egy-egy ellenpélda konstruálásával.

Az első példában $H = L^2[0,1]$. Tekintjük az (unitér) $e^{ith}$ szorzás félcsoportot $H$-n, ahol $h = \sum_{k=1}^{\infty} \chi_{(1/2^k, 1/2^{k-1}]} 2^k \pi$. A $P$ projekciót pedig a $Pf = \mathbf{1} \cdot \int_0^1 f(x)dx$ formulával definiáljuk.

A második példában $H = L^2[0, 2\pi]$. Tekintjük az $e^{tA}f(x) = f(x + 2\pi t)$ eltolás félcsoportot, ahol $f$-et $2\pi$-periódusú függvénynek tekintjük. Tekintjük továbbá a $g(x) = \frac{1}{\sqrt{34\pi}}\left[4 + \sum_{k=0}^{\infty} \frac{1}{\sqrt{2^k}} \cos 2^k x\right]$ pozitív, 1 normájú függvényt, és az általa kifeszített 1-dimenziós altérre való ortogonális projekciót jelöljük $P$-vel.

Mindkét esetben nem-triviális számítások igazolják, hogy az $(e^{\frac{t}{n}A}P)^n f$ sorozat (normája) nem konvergens az $f = \mathbf{1}$ és $f = g$ választás mellett.

A negyedik fejezet a [3] publikációra épül.

## 5 Egy hasonlósági eredmény

Ennek a fejezetnek az eredményére a hatodik fejezetben lesz szükség a projekciós Trotter formula konvergenciájának karakterizációjánál. Az eredmény azonban önmagában is érdekes. Megjegyezzük, hogy az eredmény megfelelője Banach terekben nem ismert.

**5.1.1 Tétel** *Legyen A az $e^{tA}$ $C_0$-félcsoport generátora a H Hilbert téren. A következők ekvivalensek:*
*(i) A korlátos.*
*(ii) Az $e^{tA}$ félcsoport quázi-kontraktív minden H-beli ekvivalens $(\ ,\ )_0$ skalárszozat esetén.*
*(iii) Minden H-beli ekvivalens $(\ ,\ )_0$ skalárszorzat esetén létezik olyan $K_0 \in \mathbb{R}$, amelyre teljesül, hogy minden $x \in D(A)$-ra $(x,x)_0 = 1$ esetén fennáll, hogy* $\mathrm{Re}\,(Ax,x)_0 \leq K_0$.

# 6 A Trotter formula konvergenciája

Ebben a fejezetben a generátor bizonyos tulajdonságaival jellemezzük a projekciós Trotter formula konvergenciáját. A következő tétel második része felfogható Kato eredményének megfordításaként.

**6.0.1Tétel** *Legyen A az $e^{tA}$ $C_0$-félcsoport generátora a H Hilbert téren. Tekintsük a következő állításokat:*
*(i) A korlátos.*
*(ii) $-A$ egy sűrűn definiált zárt szektoriális a formához asszociált operátor.*
*(iii) Az $(e^{\frac{t}{n}A}P)^n x$ formula konvergál minden $P \in \mathcal{B}(H)$ projekcióra, és minden $x \in H$-ra, és $t > 0$-ra.*
*(iv) Az $(e^{\frac{t}{n}A}P)^n x$ konvergál minden ortogonális $P \in \mathcal{B}(H)$ projekcióra és minden $x \in H$-ra és $t > 0$-ra.*
*A következő implikációk teljesülnek: (i) $\Leftrightarrow$ (iii), (ii) $\Leftrightarrow$ (iv).*

Az ötödik és hatodik fejezet eredményeit a [4] publikáció tartalmazza.

# Irodalom


[1] B. Farkas, M. Matolcsi: Commutation properties of the form sum of positive, symmetric operators, *Acta Sci. Math. (Szeged)*, (2001).
[2] B. Farkas, M. Matolcsi: Positive forms on Banach spaces, *Acta Math. Hung.,* (megjelenés alatt).
[3] M. Matolcsi, R. Shvidkoy: Trotter's product formula for projections, *Arch. Math.*, (megjelenés alatt).
[4] M. Matolcsi: On the relation of closed forms and Trotter's product formula, *(közlésre benyújtva).*