Cobordisms of singular maps

PhD Dissertation

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0.1 Motivation

If the reader was asked to draw a torus, the picture would likely resemble Figure 1. Somehow, for most people, this manifestly one-dimensional picture instantly brings to mind a two-dimensional object (either a torus or an annulus), maybe even an embedding of the torus in the three-dimensional space. This seems to suggest that a map from the torus to the plane is sufficiently described by the combinatorial structure of its set of singular points, which are generically either folds (where the singularity curve is smooth) or cusps (where the singularity curve itself has singularities); on the smooth part, which makes up the most of the torus from the measure-theoretic point of view, nothing really interesting happens. So, it is natural to ask, when one substitutes a map with its singularity set, how much geometrical information is really lost? It is immediately obvious that to get a meaningful answer to that question, one has to restrict the mapping under investigation to avoid trivial ways of losing geometric information about the mapping: for example, whatever surface we choose, any mapping of that surface into the plane is homotopic to the constant mapping, and that is not a very interesting mapping to study. Also, when asking the same question in another choice of dimensions of the source and target manifolds, we stumble into the problem of not being able to understand the manifolds themselves up to homeomorphism (or even homotopical equivalence), so the problem of recognizing mappings between them, being at least as hard, is pretty much hopeless. The way of modifying our question to be able to get meaningful results that we choose here is to restrict ourselves only to generic mappings with singularities chosen from a given set and to look at maps up to cobordism or bordism, that is, we consider two maps equivalent when they together form the boundary of a mapping that may only have singularities from the mentioned set. In this setting, the question can sometimes be answered completely and constructively.

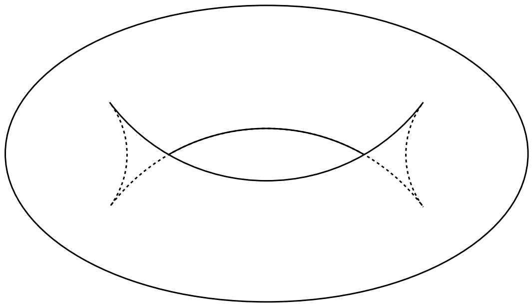


Figure 1: Just a bunch of planar curves

0.2 Introduction

Let τ be a set of stable isolated (multi)singularities of maps of codimension k. We will call a smooth map $f: M \to P$ a τ-map, if for all points $x \in P$ the germ $f_{f^{-1}(x)}$ is equivalent to a suspension of an element of τ. A τ-cobordism between τ-maps $f_0 : M_0 \to P$ and $f_1 : M_1 \to P$ is a cobordism W between M_0 and M_1 together with a τ-map $F : W \to P \times [0, 1]$ that restricts to f_0 and f_1 over $P \times \{0\}$ and $P \times \{1\}$, respectively. The generalized Pontryagin-Thom construction [30] defines such a space X_τ that a 1-to-1 correspondence exists between τ-cobordism classes of τ-maps to P and the homotopy classes $[P, X_\tau]$. In this paper we will investigate the question of calculating these τ-cobordism groups and related constructions in some concrete instances.

In general, the homotopy type of X_τ is intractable (even the problem of classifying all embeddings up to cobordism is not solved). However, when the set τ contains only local restrictions in the sense that it contains all the multisingularities composed out of a fixed set of monosingularities, there is a construction of X_τ conjectured by Endre Szabó and proved by Szűcs [45] that builds X_τ as the total space of successive fibrations. This approach makes our computations possible.

To follow through with the calculations, we need to know how to detect maps that are abstractly cobordant (by abstract cobordism we mean a cobordism in the class of all smooth maps) to τ-maps and how to quantify the amount of non-τ singularities that a null-cobordism of a given τ-map has regardless of the choice of that null-cobordism. Szűcs [44] proved that if η is a top monosingularity in τ (and τ contains only local restrictions in the sense defined above), then the precise obstruction to the existence of an η-free representative in the τ-cobordism class of a given τ-map f is the cobordism class of the η-points of f equipped with a naturally arising normal structure. As a consequence, it turns out that while sometimes the vanishing of the Thom polynomial of η is sufficient to allow the elimination of η-points, this is not the case in general, and in Chapter 2 we construct an explicit counterexample.

In Chapter 1 we will briefly outline some concepts and techniques relevant to these problems. In particular we recall the constructions of the classifying spaces of Kazarian and Szűcs. These will be utilised in Chapter 2 to address the concrete problems that arise during calculation of the singular cobordism groups. The first two non-trivial cases of such calculations are sketched, a new proof of the key fibration [45, Definition 109] is demonstrated and a negative-codimensional application is given. Chapter 3 discusses some less geometric ways of using the classifying space construction. Namely, restrictions on the characteristic classes of maps having only the simplest singularities are given – the so-called avoiding ideal [16] is computed for cusps. The bordism groups of a class of singular mappings are calculated, and bounds on the codimension of fold mappings of real projective spaces into Euclidean spaces are obtained.

Chapter 1

Singularities and classifying spaces

1.1 Basic definitions

In order to talk about the local behaviour of smooth mappings between smooth manifolds, we need the following definition:

Definition 1. *Let M and P be two smooth manifolds with marked points $x \in M$ and $y \in P$. An $(M, x) \to (P, y)$ (mapping) germ is an equivalence class of continuous mappings $f : U \to P$ such that $U \subseteq M$ is an open set with $x \in U$ and $f(x) = y$. Two mappings $f_j : U_j \to P$, $f_j(x) = y$, $j = 0, 1$, are considered to be equivalent if and only if there is an open set $U \subseteq U_0 \cap U_1$ with $x \in U$ on which f_0 and f_1 agree: $f_0(x') = f_1(x')$ for all $x' \in U$.*

For a finite set $W \subseteq M$, an $(M, W) \to (P, y)$ (mapping) multigerm is an equivalence class of continuous mappings $f : U \to P$ such that $U \subseteq M$ is an open set with $W \subset U$ and $f|_W \equiv y$. Two mappings $f_j : U_j \to P$, $f_j|_W = y$, $j = 0, 1$, are considered to be equivalent if and only if there is an open set $U \subseteq U_0 \cap U_1$ with $W \subset U$ on which f_0 and f_1 agree: $f_0(x') = f_1(x')$ for all $x' \in U$.

For a mapping $f : M \to P$ its germ at $x \in M$ is the equivalence class of

f as an $(M,x) \to (P, f(x))$ mapping germ. It is denoted by f_x. Analogously, the multigerm of f at a set $W \subseteq M$ with $f|_W = y \in P$ is the equivalence class of f as an $(M,W) \to (P,y)$ multigerm, and it is denoted by f_W.

It is obvious that locally defined properties of functions can be applied to germs, so in particular we can talk about smooth germs and smooth $(M,x) \to (P,y)$ germs have well-defined derivatives at x. If we fix charts $\phi : (U,x) \to (\mathbb{R}^n, 0)$ and $\psi : (V,y) \to (\mathbb{R}^{n+k}, 0)$ around x and y respectively, we can identify $(M,x) \to (P,y)$ germs with $(\mathbb{R}^n,0) \to (\mathbb{R}^{n+k},0)$ germs by equating the germs $[f]$ and $[\psi \circ f \circ \phi^{-1}]$ for all continuous mappings $f : (U',x) \to (P,y)$ with $\operatorname{dom} f \subseteq U$ and $\operatorname{ran} f \subseteq V$. Choosing different charts $\phi' = h^\phi \circ \phi$ and $\psi' = h^\psi \circ \psi$ from the same smooth atlases, the identification will send $[f]$ to $[\psi' \circ f \circ \phi'^{-1}] = [h^\psi \circ (\psi \circ f \circ \phi^{-1}) \circ h^{\phi^{-1}}]$, so from the point of view of local behaviour of smooth mappings between manifolds we will not be able to distinguish germs that after identifying them with $(\mathbb{R}^n,0) \to (\mathbb{R}^{n+k},0)$ germs differ only in a transformation $[g] \to [v \circ g \circ u^{-1}]$ for some origin-fixing diffeomorphisms $u \in \mathrm{Diffeo}(\mathbb{R}^n)$ and $v \in \mathrm{Diffeo}(\mathbb{R}^{n+k})$.

Definition 2. *Smooth $(\mathbb{R}^n,0) \to (\mathbb{R}^{n+k},0)$ germs $[f]$ and $[g]$ are $\mathcal{A}$-equivalent (left-rigth equivalent), if there are diffeomorphisms $u \in \mathrm{Diffeo}(\mathbb{R}^n)$ and $v \in \mathrm{Diffeo}(\mathbb{R}^{n+k})$ such that $u(0) = 0$, $v(0) = 0$ and $[f] = [v \circ g \circ u^{-1}]$.*

Smooth $(M^n,W) \to (\mathbb{R}^{n+k},0)$ multigerms $[f]$ and $[g]$ are $\mathcal{A}$-equivalent, if for some open neighbourhood U of W in M there are diffeomorphisms $u \in \mathrm{Diffeo}(U)$ and $v \in \mathrm{Diffeo}(\mathbb{R}^{n+k})$ such that $u|_W \equiv id_W$, $v(0) = 0$ and $[f] = [v \circ g \circ u^{-1}]$.

The set of all $\mathcal{A}$-equivalence classes of mapping germs is very hard to oversee. However, many such classes contain mappings that can be eliminated by an arbitrarily small homotopy, and it is definitely desirable not to have small deformations changing the equivalence class of a mapping. Hence, we will only allow germs that are *stable*:

Definition 3. *A collection θ of $\bigsqcup_m (\mathbb{R}^n,0) \to (\mathbb{R}^{n+k},0)$ multigerms is (locally)* *stable, if for any representative f and for every $\epsilon > 0$ there is a $\delta > 0$ such that for any $g : \bigsqcup_m \mathbb{R}^n \to \mathbb{R}^{n+k}$ for which $d_{\mathcal{C}^\infty}(f,g) < \delta$ there is a point $O \in \mathbb{R}^{n+k}$ with $[g_{g^{-1}(O)}] \in \theta$ and for all preimages $x \in g^{-1}(O)$ we have $\|x\| < \varepsilon$.*

That is, a multigerm is stable if it cannot be locally eliminated by any sufficiently small deformation. Classification even of stable mapping germs up to $\mathcal{A}$-equivalence is not known in general, only when n is not large compared to k.

From now on, the terms *(mono)singularity* and *multisingularity* will refer to $\mathcal{A}$-equivalence classes of germs and multigerms, respectively, with the additional identification of every (multi)germ $\theta = f_W$ with its suspension $S\theta = (f \times id_{\mathbb{R}})_{W\times\{0\}}$. In the case of stable germs, this identification does not collapse the previously defined equivalence classes.

Definition 4. *Given a set of multisingularities τ, we call a mapping $f : M \to P$ a τ-map, if for all points $p \in P$ in the target manifold the multisingularity $f_{f^{-1}(p)}$ is in τ. If P has a nonempty boundary ∂P, we additionally require that $f^{-1}(\partial P) = \partial M$ and for all points $p \in \partial P$, there should be a so-called collar coordinate neighbourhood $U = V \times [0, \varepsilon)$ of p such that on $f^{-1}(U)$ the mapping f has the form $f|_{f^{-1}(U)} = g \times id_{[0,\varepsilon)}$ with g a τ-map into V.*

For a fixed $k > 0$, as n is increased, the first monosingularities to appear are the so-called Morin singularities $\Sigma^{1_r,0} = \Sigma^{1,\dots,1,0}$ (r ones). They can be defined by their local forms [54] [25]

$$\begin{aligned}\Sigma^{1_r,0}(x, t_1, \dots, t_{kr+r-1}) = (&t_r x^r + \dots + t_1 x, t_{2r} x^r + \dots + t_{r+1} x, \dots, \\ &t_{kr} x^r + \dots + t_{kr-r+1} x, \\ &x^{r+1} + t_{kr+r-1} x^{r-1} + \dots + t_{kr+1} x, \\ &t_1, \dots, t_{kr+r-1})\end{aligned}$$

or as Thom-Boardman singularities as follows (see [4]). For a sufficiently generic mapping $f : M \to P$, we define the set $\Sigma^i(f) = \{x \in M | \dim \ker df_x = i\}$, on which a natural smooth structure can be defined, and then recursively for every index set $I = (i_1 \geq i_2 \geq \cdots \geq i_l \geq 0)$ we let

$$\Sigma^I(f) = \Sigma^{i_l}\left(f|_{\Sigma^{(i_1,\dots,i_{l-1})}(f)}\right).$$

All Thom-Boardman classes are stable. The classes $\Sigma^{1_r,0}$ are $\mathcal{A}$-orbits, but the more complicated classes can split into infinitely many orbits (see [29] for the case of $\Sigma^{2,0}$). The codimension of Σ^{1_r}-points in the source is $r(k+1)$, and the highest dimensional $\mathcal{A}$-orbit of Σ^2 (traditionally denoted by $III_{2,2}$) has codimension $2k+4$ in the source. Hence the stable multisingularities start to appear in this order when we increase the dimension n while the codimension $k > 0$ is kept fixed:

- Σ^0, for $n \geq 0$;
- $\Sigma^0 + \Sigma^0$, for $n \geq k$;
- $\Sigma^{1,0}$ (fold points), for $n \geq k+1$;
- $\Sigma^0 + \Sigma^0 + \Sigma^0$, for $n \geq 2k$;
- $\Sigma^{1,0} + \Sigma^0$, for $n \geq 2k+1$;
- $\Sigma^{1,1,0}$ (cusp points), for $n \geq 2k+2$;
- $III_{2,2}$, for $n \geq 2k+4$;

and so on. For the applications discussed in this paper this list is sufficient; however, it must be noted that stable $\mathcal{A}$-orbits form a dense set only in a certain range of dimensions (n, k), the so-called "nice dimensions" [23]: for $k \geq 4$, (n, k) is nice if and only if $n \leq 6k+8$, and for $0 \leq k \leq 3$, if and only if $n \leq 6k+9$. Our constructions are not applicable to all cases outside this range.

To be able to describe $\mathcal{A}$-equivalence classes in a finite manner, we will only consider germs that are *finitely determined*:

Definition 5. *An $\mathcal{A}$-equivalence class $\eta = [f]$ is r-determined, if for any smooth germ g the equality of the r-jets $J^r f(0) = J^r g(0)$ implies $[g] = \eta$. The germ f is finitely determined, if $[f]$ is r-determined for some r.*

It is known that the codimension of the union of not r-determined germs in the space of all r-jets grows unboundedly as r tends to infinity for all n and k. Therefore not finitely determined germs are not stable, moreover they can be removed from any finite-dimensional family of mappings with an arbitrarily small homotopy.

For the study of r-determined germs, we will utilize a construction of Kazarian [20]. Consider the space $V = J_0^r(\mathbb{R}^n, \mathbb{R}^{n+k})$ of the r-jets of smooth mappings from $\mathbb{R}^n$ to $\mathbb{R}^{n+k}$ that map the origin to the origin. The group $G = J_0^r \operatorname{Diffeo}(\mathbb{R}^n) \times J_0^r \operatorname{Diffeo}(\mathbb{R}^{n+k}) \sim O(n) \times O(n+k)$ of the r-jets of left-right coordinate transformations acts on V naturally. We can define a bundle $\bar{K}_\infty(n) : (V \times EG)/G \to BG$ associated to the universal G-bundle by this action. Whenever we have a smooth map $f : M \to P$, its r-jet $j^r f : TM \to TP$ is a polynomial map of degree r, and hence corresponds to a section of $\bar{K}_\infty(n)$ over the map $M \to BO(n) \times BO(n+k)$ inducing the bundles TM and f^*TP. Choose a vector bundle ζ over M such that $TM \oplus \zeta$ is a trivial bundle, and consider the trivial extension $j^r f \oplus id_\zeta : TM \oplus \zeta \to f^*TP \oplus \zeta$ of the r-jet $j^r f$. This extension is a polynomial mapping from a trivial vector bundle, so it corresponds to a section of the bundle $K_\infty(n) : (V \times EG')/G' \to BG'$ defined similarly to $\bar{K}_\infty(n)$ with $G' = J_0^r \operatorname{Diffeo}(\mathbb{R}^{n+k})$ acting on V by the natural action on the target. So if we identify polynomial bundle maps that only differ in a trivial extension, we can consider $K_\infty(n)$ instead of $\bar{K}_\infty(n)$. Taking the limit of the spaces $K_\infty(n)$ with respect to the natural embeddings $id_{J_0^r(\mathbb{R}^s, \mathbb{R}^s)} \oplus K_\infty(n) \to K_\infty(n+s)$ gives us the Kazarian space K_∞. By definition, K_∞ is a vector bundle over BO, and whenever there is a mapping $f : M \to P$, its r-jet $j^r f$ can be thought of as a section of K_∞

covering the map inducing ν_f, the virtual normal bundle of f. Now, if η is an r-determined monosingularity, then the set of r-jets $j \in J_0^r(\mathbb{R}^n, \mathbb{R}^{n+k})$ with $[j] = \eta$ is invariant under the action of G and hence defines a subbundle $B\eta(n) \subset K_\infty(n)$. In addition, when η is stable, this subbundle is compatible with the embeddings $K_\infty(n) \to K_\infty(n+s)$, so there is a subbundle $B\eta \subset K_\infty$ as well. It has the property that the η-points of a mapping f correspond to the intersection points of the section $j^r f : M \to K_\infty$ with $B\eta$. For a collection of monosingularities τ the subbundle

$$K_\tau = \bigcup_{\eta \in \tau} B\eta,$$

will be called the Kazarian space for τ-maps since the r-jet of a generic mapping lies in K_τ if and only if f is a τ-map.

$B\eta$ has the homotopy type of BG_η [20], and the neighbourhood of $B\eta$ in K_∞ can be identified with the universal source bundle ξ_η [45]. Hence K_τ can be glued together from the source bundles in a way resembling the construction of the universal source.

Assume that τ is a collection of (r-determined) monosingularities and let $\eta \in \tau$ be an arbitrary stable algebraic monosingularity. Then for any τ-map f the set $\overline{\eta(f)}$ is a closed algebraic set in an appropriate choice of local charts, and therefore defines a $\mathbb{Z}_2$ cohomology class via Poincaré duality. If η is coorientable, then after a choice of coorientation $\overline{\eta(f)}$ will represent an integral cohomology class. With the help of the space K_∞, it is easy to see that this class is the pullback of a universal ($\mathbb{Z}_2$ or $\mathbb{Z}$) cohomology class in $H^*(K_\infty) \cong H^*(BO)$ or $H^*(BSO)$ – the dual of the closure of the singular stratum $B\eta$ – and is hence a polynomial in the characteristic classes of the virtual normal bundle of the mapping f. This polynomial is called the ($\mathbb{Z}_2$, respectively integral) Thom polynomial of the singularity η and is denoted by Tp_η (see also [49] and [11]).

The determination of the Thom polynomial of a given singularity is still an unsolved problem in general, but if Tp_η is known, then it provides an obstruction to the elimination of the η points from a mapping f by a homotopy, for example, since a homotopy does not change the characteristic classes of the virtual normal bundle.

In the same spirit, one can consider not only the dual of $\overline{B\eta}$, but all the cohomology classes supported on $\overline{B\eta}$, that is, the image of the map $H^*(K_\infty, K_\infty \setminus \overline{B\eta}) \to H^*(K_\infty)$. From the long exact sequence of the pair $(K_\infty, K_\infty \setminus \overline{B\eta})$ we can identify this image with the kernel of the map $i^* : H^*(K_\infty) \to H^*(K_\infty \setminus \overline{B\eta})$ induced by the embedding $i : K_\infty \setminus \overline{B\eta} \to K_\infty$. This kernel is called the *avoiding ideal* of η, and in particular the Thom polynomial of η can be recovered from it up to a constant multiple [16].

We will investigate singular maps up to the following equivalence relations:

Definition 6. *Two τ-maps $f_0 : M_0^n \to P^{n+k}$ and $f_1 : M_1^n \to P$ are τ-cobordant, if there exists a smooth manifold W^{n+1} with boundary $\partial W = M_0 \sqcup M_1$ and a τ-map $F : W \to P \times [0,1]$ such that $F^{-1}(P \times \{0\}) = M_0$, $F^{-1}(P \times \{1\}) = M_1$ and $F|_{M_j} = f_j$ for $j = 0, 1$. The set of corresponding equivalence classes is denoted by $Cob_\tau(P, k)$.*

Two oriented τ-maps $f_0 : M_0 \to P$ and $f_1 : M_1 \to P$ are oriented τ-cobordant, if there exists a τ-cobordism $F : W \to P \times [0,1]$ with $\partial W = M_0 \sqcup -M_1$, where $-M$ denotes M with reverse orientation. The set of corresponding equivalence classes is denoted by $Cob_\tau^{or}(P, k)$.

Two τ-maps $f_0 : M_0 \to P_0$ and $f_1 : M_1 \to P_1$ are (left-right) τ-bordant, if there exist smooth manifolds W and Q with boundary $\partial W = M_0 \sqcup M_1$ and $\partial Q = P_0 \sqcup P_1$ as well as a τ-map $F : W \to Q$ such that $F^{-1}(P_0) = M_0$, $F^{-1}(P_1) = M_1$ and $F|_{M_j} = f_j$ for $j = 0, 1$. The set of corresponding equivalence classes is denoted by $Bord_\tau(n, k)$, where $n = \dim M_0$; the set of equivalence classes of oriented τ-bordism is denoted by $Bord_\tau^{or}(n, k)$.

We will only consider sets τ that contain maps of the same codimension k, so k will be omitted from the notation when it is convenient to do so. Also,

$Cob_\tau(n,k)$ and $Cob^{or}_\tau(n,k)$ will be used as abbreviations for $Cob_\tau(\mathbb{S}^{n+k},k)$ and $Cob^{or}_\tau(\mathbb{S}^{n+k},k)$, respectively.

When τ contains all possible singularities, we obtain the unoriented bordism group $\mathfrak{N}_m(P^N) = Cob_\tau(P, N-m)$ and the oriented bordism group $\Omega_m(P^N) = Cob^{or}_\tau(P, N-m)$ [10]. These are abelian groups with the sum being the union of maps and inverse being the same map, with the opposite orientation in the oriented case. If additionally $P = \mathbb{R}^{n+k}$ (or any other contractible manifold), the group $\mathfrak{N}_n = Cob_\tau(\mathbb{R}^{n+k})$ is the well-known cobordism group of n-dimensional manifolds. The sum $\mathfrak{N}_* = \oplus_n \mathfrak{N}_n$, equipped with the product of source manifolds as a multiplication, is a polynomial ring

$$\mathbb{Z}_2\left[Y_j | j \notin \{2^r - 1 | r \geq 1\}\right]$$

with generator Y_j being a j-dimensional manifold [48]. Y_{2j} can be chosen to be the real projective space $\mathbb{R}P^{2j}$, and the rest of the generators can also be given explicitely as done by Dold [12]. The cobordism class of a manifold M is fully determined by its collection of Stiefel-Whitney characteristic numbers

$$w_I[M] = w_{(i_1,\dots,i_l)}[M] = \langle w_{i_1}(TM)\dots w_{i_l}(TM), [M]\rangle$$

with $i_1 + \dots + i_l = \dim M$. In the oriented setting, the analogously defined groups $\Omega_n = Cob^{or}_\tau(point)$ and ring $\Omega_* = \oplus_n \Omega_n$ are similarly well-understood [51]. The torsion part of Ω_*, $\operatorname{Tors}\Omega_* = \{x \in \Omega_* | \exists n > 0 : nx = 0\}$, is the ideal generated by the Dold manifolds $Y_{2j+1}, j \notin \{2^r - 1 | r \geq 0\}$, and consists of order 2 elements. The torsionless factor $\Omega_* / \operatorname{Tors}\Omega_*$ is a polynomial ring $\mathbb{Z}[Y_{4j} | j \geq 0]$, which is rationally generated by the cobordism classes of the even dimensional complex projective spaces: $\Omega_* \otimes \mathbb{Q} \cong \mathbb{Q}[\mathbb{C}P^{2j} | j \geq 0]$. The oriented cobordism class of a manifold is fully determined by its collection of Stiefel-Whitney and Pontryagin characteristic numbers. The problem of providing similar descriptions of at least some of the singular cobordism groups (in a way that can be handled algorithmically) has been investigated in the past for various classes of maps, including embeddings [48] [52], immersions without l-tuple points [50] [42], immersions [37], maps without multiple fold points or ordinary triple points [41] and all smooth maps [39] (note that even finding an inverse element in these groups cannot in general be done by just reversing the orientation, as in the case of abstract cobordism). All of these rely essentially on the study of the corresponding so-called classifying spaces, which we will discuss in the next section.

1.2 Classifying spaces

The abstract notion of a classifying space is probably best described by E.H. Brown's representation theorem, which states the following:

Theorem 1 ([6]). *If there is a contravariant functor $\mathcal{F}$ from the homotopy category of pointed cell complexes to the category of pointed sets that sends bouquets to direct sums and satisfies the Mayer-Vietoris (short) exact sequence, then it is equivalent to a functor $[-, X]$ for a suitable space $X = X(\mathcal{F})$, that is, $\mathcal{F}(H)$ can be identified with $[H, X]$.*

We shall call this space X the classifying space for the functor $\mathcal{F}$.

In some sense, this theorem states that (nice) contravariant functors, when thought of as structures on their arguments, admit universal models. A very common illustration of this theorem is the case of the functor $H^n(-; \mathbb{Z})$, whose classifying space is the Eilenberg-MacLane space $K(\mathbb{Z}, n)$. The fundamental class $u \in H^n(K(\mathbb{Z}, n); \mathbb{Z})$ is a "universal" n-dimensional integral cohomology class, since every n-dimensional integral cohomology class can be pulled back from it in a homotopically unique way. As a result, we can think of integral cohomology classes and homotopy classes of maps to $K(\mathbb{Z}, n)$ interchangeably, making some calculations more transparent. Another example, if G is a topological group, then the functor of assigning to a space P the set of principal G-bundles over P up to isomorphism can be canonically identified with the set $[P, BG]$, and we usually think of BG as being the base of a universal principal G-bundle, from which every other principal G-bundle can be pulled back (again, in a homotopically unique way).

The case that we will focus on is a generalization of the Pontryagin-Thom construction. When a linear group $G \leq O(k)$ is given, we can consider embeddings of codimension k with the structure group of the normal bundle reduced to G. The classical Pontryagin-Thom construction provides a 1-to-1 correspondence between the set of cobordism classes of such embeddings into a target manifold P and the homotopy classes $[P, MG]$, where MG is the Thom space of the vector bundle associated to the universal G-bundle via the given embedding $G \leq O(k)$.

Definition 7. *Given a vector bundle ξ over a base B, a ξ-structure on a vector bundle η over a base N is a homotopy class $[f] \in [N, B]$ of a mapping f together with an isomorphism $f^*\xi \cong \eta$. The set of cobordism classes of embeddings of n-manifolds into P^{n+k} equipped with a ξ-structure of the normal bundle will be denoted by $Emb^\xi(P)$ (here $n = \dim P - \operatorname{rank} \xi$).*

Immersions with a ξ-structure on the normal bundle will be called ξ-immersions.

The corresponding result stating that $Emb^\xi(P) \cong [P, T\xi]$ is reminescent of Brown's representability theorem. Note, however, that the functor $Emb^\xi(-)$ does not extend obviously to all cell complexes in a homotopically invariant way, so the application of Brown's theorem is not immediate. We will nevertheless call $T\xi$ the classifying space of embeddings with a normal ξ-structure, under the implicit understanding that the set of cobordism classes that is being classified may only be defined for manifolds (possibly with boundary).

Choosing $G = O(k)$ (equivalently, setting $\xi = \gamma_k$, the universal vector k-bundle), we obtain exactly the cobordism group of all embeddings, since a γ_k-structure exists and is unique on any normal bundle. Hence $\pi_{n+k}(MO(k))$

is the cobordism group of embeddings of n-manifolds into $\mathbb{R}^{n+k}$; by choosing $k > n$ a careful analysis of the space $MO(k)$ made by Thom gave the description of the cobordism ring $\mathfrak{N}_*$ quoted above.

1.3 Immersions and singular maps

Regardless of the particular constructions, we can obtain some classifying spaces from others based on general considerations. For example, there is a natural isomorphism

$$[P, X_1 \times X_2] \cong [P, X_1] \times [P, X_2],$$

and consequently if X_1 and X_2 classify structures $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively, then $X_1 \times X_2$ classifies the union of $\mathcal{F}_1$ and $\mathcal{F}_2$ structures, independent of each other. Similarly, the natural isomorphism

$$[P, \Omega X] \cong [SP, X]$$

shows that if X classifies structure $\mathcal{F}$, then ΩX classifies lifts to $\mathcal{F}$-structures on the suspension of the space. As an application, we can obtain another extension of the Pontryagin-Thom construction based on the so-called Multi-Compression Theorem [34, Theorem 4.5], which essentially says that for $m < q$ the embeddings of a manifold M^m into $Q^q \times \mathbb{R}^r$ with an r-frame (that is, r linearly independent normal vector fields) can be transformed by an ambient isotopy into a state with the frame in a standard position (parallel to the axis of $\mathbb{R}^r$) everywhere. Denote by ε^N the trivial bundle (over a point). The compression theorem implies that classifying immersions into P with a ξ-structure is the same as classifying embeddings into $P \times \mathbb{R}^N$ with a $\xi \oplus \varepsilon^N$-structure for a sufficiently large N. Indeed, if we have an immersion $j : M \to P$ and take a lift of j to an embedding $\overline{j} : M \to P \times \mathbb{R}^N$, we can equip $\overline{j}$ with the standard N-framing (parallel to the axis of $\mathbb{R}^N$). In the other direction, assume we have an embedding with its normal bundle split into a ξ-structure and an N-frame. A C^0-small isotopy deforms the frame into the standard position. Projecting the resulting embedding into P gives an immersion with the same ξ-structure on its normal bundle. Consequently for a sufficiently large N there is an isomorphism

$$\begin{aligned} Imm^{\xi}(P) &\cong Emb^{\xi+\varepsilon^N}(P \times \mathbb{R}^N) \cong Emb^{\xi+\varepsilon^N}(S^N P) \\ &\cong [S^N P, T(\xi+\varepsilon^N)] \cong [P, \Omega^N S^N T\xi] \end{aligned}$$

where Imm^{ξ} is the cobordism group of ξ-immersions defined analogously to Emb^{ξ}. The classifying space for ξ-immersions is therefore $\lim_{N\to\infty} \Omega^N S^N T\xi$. For example, using the universal bundle γ_k for ξ, the stable homotopy groups $\pi^S_{n+k}(MO(k))$ are isomorphic to the cobordism groups of immersions of n-manifolds into $\mathbb{R}^{n+k}$.

This result is more geometric than it may seem at first: the space $\Gamma T\xi = \lim_{N\to\infty} \Omega^N S^N T\xi$, although it is not a finite dimensional cell complex, can be homotopically approximated by a series of finite dimensional cell complexes

$$\Gamma T\xi \sim \ldots ((T\xi \underset{\partial D_2}{\cup} D_2) \underset{\partial D_3}{\cup} D_3) \cup \ldots,$$

where the spaces D_r are disk bundles mimicking the local behaviour of ξ-immersions at r-tuple points [40] [42]: D_r is obtained by factorizing the bundle $\underset{r}{\times}\xi$ by the action of the symmetric group permuting the indices of the constituent copies of ξ. The space $\Gamma_{r-1}T\xi = T\xi \cup \cdots \cup D_{r-1}$ is the classifying space of ξ-immersions without r-tuple points, and the gluing map $\partial D_r \to \Gamma_{r-1}T\xi$ corresponds to the ξ-immersion with at most $r-1$-tuple points arising naturally on the boundary ∂D_r.

Rimányi and Szűcs [41] [30] developed an extension of this way of constructing a classifying space for the case of cobordisms with a prescribed set of allowed singularities. They have shown that for any (finite) collection τ of stable multisingularities of maps of codimension k, there is a mapping $f_\tau : Y_\tau \to X_\tau$ that is universal for τ-maps up to τ-cobordism in the following sense. For any smooth manifold P there is an isomorphism $Cob_\tau(P) \cong [P, X_\tau]$, and this correspondence is established via the pullback of f_τ. That is, for any τ-map $f : M \to P$, one can choose (homotopically unique) maps $\sigma : M \to Y_\tau$ and $\kappa : P \to X_\tau$ so that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\sigma} & Y_\tau \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_\tau} \\ P & \xrightarrow{\kappa} & X_\tau \end{array}$$

Definition 8. *The map $\kappa = \kappa(f)$ is called the classifying map of f.*

The spaces X_τ and Y_τ are successively glued together from "blocks" $\tilde{D}_\theta$ and D_θ, respectively, which are disc bundles describing the behaviour of θ-points for all $\theta \in \tau$ corresponding to the more and more complicated multisingularities of τ. The local behaviour of θ points is described by G_θ, the maximal compact subgroup of the symmetry group of θ in the terminology of Jänich [18] [53] [30]; it is conjugate to a linear group, so in an appropriately chosen local coordinate system the action of G_θ is via representations $\lambda_\theta : G \to S_m \wr O(c)$ in the source (for an m-tuple point) and $\tilde{\lambda}_\theta : G \to O(c+k)$ in the target. Define $\tilde{\xi}_\theta$ to be associated to the universal G_θ-bundle over BG_θ by the means of the target representation $\tilde{\lambda}_\theta$, and let $\tilde{D}_\theta$ be its disc bundle. Similarly, let ξ_θ be the $\underset{m}{\sqcup}\mathbb{R}^c$-bundle over BG_θ associated to the source representation λ_θ and denote by D_θ its m-disc bundle. Then there is a natural fiber-preserving map $f_\theta : D_\theta \to \tilde{D}_\theta$ that restricts to a copy of an isolated θ singularity at the origin on every fiber, since both bundles can be considered to have a structure group under which an isolated θ-singularity is invariant. The gluing map $\partial\tilde{D}_\theta \to \underset{\theta'<\theta}{\cup}\tilde{D}_{\theta'}$ arises as the classifying map of the restriction of f_θ to the preimage of $\partial\tilde{D}_\theta$ (it only has multisingularities that are less complicated than θ). The same pullback diagram gives us the gluing map $\partial D_\theta \to \underset{\theta'<\theta}{\cup} D_{\theta'}$ as well.

Since the spaces Y_τ and X_τ are not finite dimensional smooth manifolds, the map f_τ is not a τ-map. By abuse of notation, we will use the term "τ-map" to include mappings that are direct limits of τ-maps with respect to some direct system of embeddings. It is easy to check that the universality of f_τ in the sense of satisfying [30, Theorem 1] is not compromised by expanding the class of τ-maps in this way. We briefly recall the essence of the proof.

In case A of [30, Theorem 1], we are given a τ-map $f : M \to \partial P$ that is already pulled back from f_τ together with an extension $g : P \to X_\tau$ of the classifying map $g|_{\partial P}$ of f, and we have to construct an extension of f that would be pulled back from f_τ by a deformation of g. Making g transversal to all the strata $BG_\theta \subset \tilde{D}_\theta$, $\theta \in \tau$, ensures that the pullback of f_τ by g is a τ-map. Finally, note that there is no need to change g near ∂P where it is already transversal to all the strata since it pulls back a τ-map.

In case B, the τ-mapping $f : N \to P$ is given together with a classifying map for $f|_{\partial N}$ and we need to extend it to a classifying map of the entire f. This will be done by induction, the starting claim is that mapping P to a point (with the empty mapping being induced) can be extended. Assume we have already proven the statement for $\tau' = \tau \setminus \theta$ with θ a maximal multisingularity in τ. First, we extend the classifying map to a small neighbourhood of the $f(\theta(f))$ points.

Lemma 2. *Let ξ be a bundle associated to the universal G-bundle by means of a faithful representation $G \leq O(N)$, and let ψ be a vector bundle of rank N over the base B such that ψ can be induced from ξ. Assume we are also given a mapping $\kappa' : B' \to BG$ for some closed subset $B' \subseteq B$ such that $\kappa'^*\xi \cong \psi|_{B'}$. Then there is an extension $\kappa : B \to BG$ of κ' such that $\kappa^*\xi \cong \psi$.*

Proof. Let $\chi : B \to BG$ be an arbitrarily chosen classifying map of ψ. Since $\chi|_{B'}$ and κ' both pull back $\psi|_{B'}$ and the representation of G is faithful, $\chi|_{B'}$ is homotopic to κ' via some homotopy $h : B' \times [0,1] \to BG$. There is an extension H of h to the whole $B \times [0,1]$; then $H(-,1)$ is an extension of κ' with the required properties. $\square$

Our given classifying map pulls back a relative closed part of a small neighbourhood of $f(\theta(f))$ from $\tilde{D}_\theta$, and the entire neighbourhood can be pulled back from $\tilde{D}_\theta$. Also, the representation $\tilde{\lambda}$ is faithful, since the codimension k is positive, so we can apply the lemma and obtain an extension of the given classifying mapping to a small neighbourhood U of $f(\theta(f))$. But over the remaining part $P \setminus U$, the map f is a τ'-map and hence the existence of an extension of a classifying mapping of $\partial U \Delta \partial P$ to the whole $P \setminus U$ is guaranteed by the inductive hypothesis.

Remark: If k is not positive, the representation $\tilde{\lambda}$ need not be faithful. However, we can consider the direct sum $\lambda \oplus \tilde{\lambda}$, which is faithful by definition, and the Whitney sum of the normal bundles in the source and the target has a canonical corresponding normal structure given by f. This implies that the universal map f_τ can be constructed in the nonpositive codimensional setting as well.[1]

This construction of the universal τ-mapping only needs the fact that all the singular strata have a (global) neighbourhood that can be pulled back from a universal one uniquely up to homotopy. Let us consider mappings with a given orientation of the virtual normal bundle; we call such an orientation a *coorientation* of the underlying map. This restriction can be incorporated into the construction by replacing G_θ with its subgroup $G_\theta^{SO} = \ker(\det \lambda_\theta \cdot \det \tilde{\lambda}_\theta)$ and using restrictions of the previously mentioned representations in the source and in the target. The classifying space we get in this way, X_τ^{SO}, classifies by construction τ-maps equipped with a coorientation. Hence if we consider its homotopy groups $\pi_{n+k}(X_\tau^{SO})$ or oriented bordism groups $\Omega_{n+k}(X_\tau^{SO})$, they will contain equivalence classes of mappings that classify τ-maps into oriented manifolds (in the first case to $\mathbb{S}^{n+k}$) with a

[1]For $k \leq 0$ a monosingularity is a germ at a connected component of the whole preimage of a point, up to left-right equivalence

given coorientation. Given an orientation on the target space, an orientation of the virtual normal bundle is equivalent to a compatible orientation on the source manifold, so this construction gives us the oriented τ-cobordism and τ-bordism groups of maps of codimension k:

$$Cob_\tau^{or}(\mathbb{S}^{n+k}) \cong \pi_{n+k}(X_\tau^{SO}),$$
$$Bord_\tau^{or}(n) \cong \Omega_{n+k}(X_\tau^{SO}).$$

The reason for choosing such a peculiar way of specifying mappings between oriented manifolds is that the orientation of all disc bundles ξ_θ, $\theta \in \tau$, does not provide an orientation for the source manifold. In fact, no amount of additional structure can give such an orientation unless the choice of possible target manifolds is also restricted – any classifying map $\kappa : P \to X_\mathcal{F}$ composed with the projection map $pr : P \times N \to P$ for an unorientable manifold N provides an example of a map pulled back from X_τ with an unorientable source manifold, so in particular we cannot construct a classifying space for the mappings of oriented manifolds into unoriented manifolds. When regarding the orientation of the virtual normal bundle, however, this problem does not arise as the virtual normal bundle of the map f, classified by a map $\kappa : P \to X_\tau$, is the pullback of the virtual normal bundle of the universal τ-map f_τ (or, rather, a finite-dimensional approximation thereof) by κ. Throughout this paper, we will use the superscript ^{SO} to indicate objects corresponding to cooriented mappings.

We will need one more generally applicable construction of a classifying space. Assume $X_\mathcal{F}$ classifies some kind of singular maps of codimension k into the target space P, denote the class of such mappings by $\mathcal{F}$, and let $f_\mathcal{F} : Y_\mathcal{F} \to X_\mathcal{F}$ be the corresponding universal map. Then any map into $SX_\mathcal{F} = X_\mathcal{F} \wedge \mathbb{S}^1$, $\kappa : P \to SX_\mathcal{F}$ say, is the composition of a map $\bar{\kappa} : \bar{P} \to X_\mathcal{F} \times \mathbb{S}^1$ with $\bar{P}/\partial\bar{P} = P$ that sends $\partial\bar{P}$ to $X_\mathcal{F} \vee \mathbb{S}^1$ and the pinching of the boundary $\partial\bar{P}$. Maps into $\mathbb{S}^1$ correspond to cooriented hypersurfaces, and maps into $X_{\mathcal{F}}$ correspond to $\mathcal{F}$-maps, so maps into the product $X_{\mathcal{F}} \times \mathbb{S}^1$ classify the union of the two structures. The factorization by the bouquet $X_{\mathcal{F}} \vee \mathbb{S}^1$ corresponds to forgetting the structures away from the intersection of the images of the two maps classified by $\bar{\kappa}$, in particular on the boundary $\partial \bar{P}$. After a small perturbation, we can assume that the hypersurface is transverse to all singular strata, so the $\mathcal{F}$-mapping in a neighbourhood of the hypersurface is uniquely determined by its restriction to the hypersurface. Hence, the structure classified by $\kappa : P \to SX_{\mathcal{F}}$ is a mapping into P such that its image lies in the germ of a cooriented hypersurface and is an $\mathcal{F}$-map when considered as a map into that hypersurface. The coorientation of the hypersurface can be given by a normal vector, so we will call such maps *framed $\mathcal{F}$-maps*;[2] iterating this construction r times, the resulting maps will be called r-framed $\mathcal{F}$-maps. It is easy to see that the universal r-framed $\mathcal{F}$-map can be obtained as $f_{\mathcal{F}} : Y_{\mathcal{F}} \to S^r X_{\mathcal{F}}$, where $S^r X_{\mathcal{F}}$ is considered to be the quotient of $X_{\mathcal{F}} \times \mathbb{R}^r$ and the framing is pulled back from a basis of $\mathbb{R}^r$. Framed τ-maps play an important role in finding the inverse element in the group $Cob_\tau(P)$ for general τ and P, see [45, Proposition 13].

[2]The definition in [45] is slightly different, as there the hypersurface germ is required to exist locally, and the hypersurfaces of the preimages of the same point need not to coincide

Chapter 2

Geometric line of attack

2.1 CHP

The construction of the classifying space X_τ mentioned in the previous chapter, while very intuitive and geometric, has the serious drawback of being composed of blocks glued together, and hence the homotopy groups of X_τ, which would give the cobordism groups $Cob_\tau(n, k)$, are hard to compute directly. In the situation when the set τ contains only local restrictions, that is, τ contains all multisingularities composed from a given set of monosingularities, Szűcs [45] circumvents this problem by proving a result conjectured by Endre Szabó, that there is a construction of X_τ based on subsequent fibrations instead of gluings. In this section, we will reprove the key statement of this latter construction in a way that is extensible to certain situations with negative codimensional mappings.

In what follows, η will be an isolated monosingularity and τ' will be a collection of all multisingularities composed from a fixed set of monosingularities such that $\partial\eta \subseteq \tau'$ and $\eta \notin \tau'$. The set of multisingularities of form $\theta + l\eta$, $\theta \in \tau'$, $l \leq r$ will be denoted by τ_r, and we define $\tau = \cup_{r\geq 0}\tau_r$. Denote by X_r and Γ_r, $r \geq 0$, the classifying spaces of τ_r-maps and immersions with a normal $\tilde{\xi}_\eta$-structure and at most r-tuple points, respectively. All mappings in τ are of codimension k.

The statement of [45, Proposition 108] is as follows:

Theorem 3. *Assume $k > 0$. There is a Serre fibration*

$$X_{\tau'} \to X_\tau \xrightarrow{\kappa_I} \Gamma T\tilde{\xi},$$

where $\tilde{\xi} = \tilde{\xi}_\eta$ and κ_I is the classifying mapping of $f_\tau|_{\overline{\eta[f_\tau]}}$ as an immersion equipped with a normal $\tilde{\xi}$-structure given by f_τ.

Remark: [45, Definition 7] allows κ_I to be defined abstractly as a map inducing the natural transformation

$$[P, X_\tau] \ni [f] \mapsto [\eta(f)] \in [P, \Gamma T\tilde{\xi}],$$

which sends the τ-map f to its η-set regarded as an immersion equipped with a normal $\tilde{\xi}$-structure.

Proof. The map κ_I restricts to an $X_r \to \Gamma_r$ map naturally, the fiber at the selected point of Γ_r is homotopically equivalent to $X_{\tau'}$, and we will show that this restriction satisfies the covering homotopy property. This will ensure that κ_I, which is the direct limit of the restrictions $\kappa_r = \kappa_I|_{X_r}$, is a Serre fibration itself.

We will check the covering homotopy property of the map κ_r directly, so we need to show that for any given finite cell complex P and given maps $h : P \times [0,1] \to \Gamma_r$ and $H_0 : P \times \{0\} \to X_r$ such that $\kappa_r H_0 = h|_{P\times\{0\}}$ there is an extension $H : P \times [0,1] \to X_r$ of H_0 satisfying the equality $\kappa_r H = h$. Without loss of generality we can assume that P is a manifold with boundary by replacing P with such a neighbourhood of its embedding into a sufficiently high dimensional Euclidean space that P is a deformational retract thereof. We can also assume that both h and H_0 are transverse to the strata of the stratifications of Γ_r and X_r arising from their construction. Indeed, for all N the N-homotopy type of X_r is that of a compact cell complex. Hence

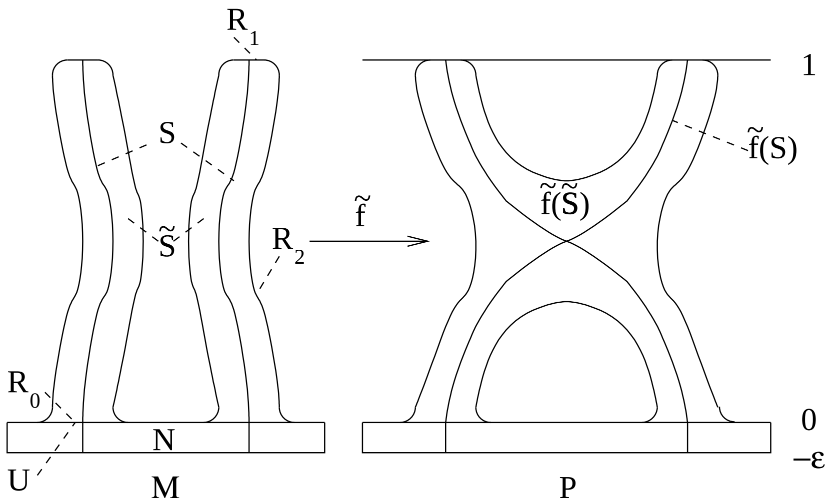


Figure 2.1: Extension of a cobordism of $\eta(f)$

choosing a sequence of approximations of h and H_0 that are transverse to the strata would yield a sequence of lifts, from which we can then choose a convergent subsequence, converging to a lift of the original homotopy h with the given starting position H_0.

Assume we are given a mapping $\hat{f} : M^n \cup S \to P^{n+k} \times [0,1]$, where $\hat{f}|_M$ is a τ_r-map multiplied by the inclusion $\{0\} \to [0,1]$ and $\hat{f}|_S$ is a cobordism of the immersion $\hat{f}|_{\eta(\hat{f}|_M)}$ as an r-immersion with a $\tilde{\xi}$-structure (Figure 2.1). We need to extend $\hat{f}$ to a τ_r-mapping $F : \tilde{M}^{n+1} \to P \times [0,1]$ with $\eta(F) = S$. The normal structure on S allows us to extend the mapping of S to a small neighbourhood $\tilde{S}^{n+1}$ of S glued together from disc bundles associated to the source representation $\lambda_{s\eta}$ over the s-tuple points of $\hat{f}|_S$ for all s such that $0 < s \leq r$. $\tilde{S}$ is a manifold with corners, set $R_0 = \tilde{S} \cap \hat{f}^{-1}(P \times \{0\})$, $R_1 = \tilde{S} \cap \hat{f}^{-1}(P \times \{1\})$ and $R_2 = \overline{\partial\tilde{S} \setminus (R_0 \cup R_1)}$ to denote the smooth parts of its boundary. There is a homeomorphism j between R_0 and the closure of a neighbourhood U of $\eta(\hat{f}|_M)$ such that $\hat{f}|_U \circ j = \hat{f}|_{R_0}$, so $\tilde{S}$ can be assumed to be attached to $P \times \{0\}$ by $\hat{f}$. Note that $\hat{f}|_{R_0}$ does not match $\hat{f}|_M$ in general, only $\hat{f}|_U$. Still, we can extend $\hat{f}$ from $M \times \{0\}$ as a direct product with $id : [-\varepsilon, 0] \to [-\varepsilon, 0]$ to a product collar of M. By doing this, we obtain an extension of $\hat{f}$ to a map $\tilde{f} : M \times [-\varepsilon, 0] \cup \tilde{S} \to P \times [-\varepsilon, 1]$ such that both $\tilde{f}|_{\tilde{f}^{-1}(P \setminus \tilde{f}(U))}$ and $\tilde{f}|_{R_2}$ are τ'-maps. Additionally, this extension is almost a τ_r-map, except that it does not map the boundary of the source manifold to the boundary of the target manifold.

The next step is noticing that we may assume that the image of the tangent mapping of $\tilde{f}|_{R_2}$ does not contain the "vertical" tangent vector of $P \times [0, 1]$.[1] Indeed, the image of $\tilde{f}|_{R_2}$ is a cell complex of codimension at least 2, so if we denote by v the image of a nowhere zero normal vector field of R_2 in $\tilde{S}$ under the differential $T\tilde{f}$ (note that v is nowhere zero since $\partial\tilde{S}$ is transverse to all the singular strata of $\tilde{f}$), then v can be "straightened" by an arbitrarily small isotopy of the ambient space $P \times [0, 1]$ according to the compression theorem (see [34] and the extension to the case of stratified manifolds in [45]). The corners of $M \times [-\varepsilon, 0] \underset{R_0}{\cup} \tilde{S}$ can be smoothed in such a way that $\tilde{f}$ still has the property of never mapping the normal vector of the boundary in the entire source manifold to a horizontal vector in $T(P \times [0, 1]) \cong TP \times T[0, 1]$ (cutting back convex corners and filling up concave corners as in Figure 2.1). Such a smoothing of $M \times [-\varepsilon, 0] \underset{R_0}{\cup} \tilde{S}$ will be denoted by N, and $\tilde{f}$ can be considered to be an $N \to P \times [-\varepsilon, 1]$ map.

If we can suitably extend $\tilde{f}$ composed with the isotopy provided by the compression theorem, then applying the inverse of the isotopy will give a suitable extension F of the original $\tilde{f}$. The source manifold of F will be $\tilde{M} = N \cup \{\{x\} \times [\alpha, 1] | \tilde{f}(x) = (p, \alpha), p \in P, \alpha \in [0, 1]\}$ and the extension of $\tilde{f}$ on each segment $\{x\} \times [\alpha, 1]$ of the collar added to N is defined to be a parametrization of the vertical segment connecting $\tilde{f}(x) = (p, \alpha)$ to $(p, 1)$ matching $\tilde{f}$ at x. No new η-points are created, since none were present on $\partial N \setminus (R_1 \cup f^{-1}(P \times \{-\varepsilon\}))$ (which consists of $M \setminus U$ together with R_2 glued to it via a trivial collar; on both these parts $\tilde{f}$ is a τ'-map). □

[1] Here, "vertical" means "parallel to the constituent of the cylinder $P \times [0, 1]$", and "horizontal" means perpendicularity to that direction

It is worth noticing that in this proof, the restriction $k > 0$ was only used at the point where we needed to make the vector field v vertical. The only problem is that when $k \leq 0$, the image of $\tilde{f}|_{R_2}$, in general, may be of codimension 1 in $P \times [0,1]$ and thus the compression theorem cannot be applied. Hence if the image of the mapping in question was concentrated in a small neighbourhood of the image of the singular set, which is of codimension at least 2 in the target, there would be no obstruction to proceeding in the same way.

It turns out that when $k = -1$, we indeed get this positive-codimensional behaviour in the sense of Theorem 4 below. Under the additional assumption that τ' contains the equivalence class of the definite fold, [19, Proposition 3.4] shows the following:

Theorem 4. *Let τ be a collection of multisingularities of maps of $n+1$-manifolds into n-manifolds. Assume that τ is composed from all monosingularities of a fixed set, and the definite fold*

$$(x_0, x_1, ..., x_n) \mapsto (x_0^2 + x_1^2, x_2, ..., x_n)$$

is a τ-map. Then every τ-map $f : M^{n+1} \to P^n$ is τ-cobordant to one whose image is contained in an arbitrarily small neighbourhood of the singular set $f(S(f)) = \{f(x)|x \in M, \ \operatorname{rank} df < n\}$.

Proof. We start with the trivial τ-cobordism $f \times id_{[0,1]} : M \times [0,1] \to P \times [0,1]$. Let $d : P \to [0,2]$ be a smooth function such that $d^{-1}(2) = f(S(f))$, $d^{-1}(0) = \emptyset$, 1 is a regular value of d and $d^{-1}([1,2])$ is contained in a prescribed neighbourhood of $f(S(f))$. We define the manifold (with boundary) $N = \{(x,t) \in M \times [0,1] | d(f(x)) \geq t\}$. Denote by $\tilde{\partial} N$ the part of boundary of N lying in the interior of $M \times [0,1]$:

$$\tilde{\partial} N = \overline{\{(x,t) \in M \times [0,1) | d(f(x)) - t = 0\}}.$$

This is a smooth manifold with boundary as the preimage of a regular value of a smooth function: the restriction of f to $\{x \in M | d(f(x)) < 2\}$ is a submersion. The map f maps $\tilde{\partial} N$ to the level set $Q = \{(y,t) \in P \times [0,1] | d(y) - t = 0\}$, which is a smooth manifold with boundary on $P \times \{1\}$ since 1 was a regular value of d. For the same regularity reason the preimage by $f \times id_{[0,1]}$ of any point $(y,t) \in Q$ consists of disjoint circles embedded into $\tilde{\partial} N$, giving a representation of $\tilde{\partial} N$ as the circle bundle of some disc bundle $p : D\xi \to B\xi$. We now define $\hat{N}$ to be the union of N and $D\xi$ glued along $\partial D\xi = \tilde{\partial} N$, and we define $F : \hat{N} \to P \times [0,1]$ on N to be $(f \times id_{[0,1]})|_N$. In order to extend F to the entire $\hat{N}$ we choose a sufficiently small normal vector field of Q pointing into the domain of negative values of $d(y) - t$ and perturb it slightly to still be nowhere tangent to Q while being horizontal on the boundary $\partial Q = Q \cap P \times \{1\}$; denote this perturbed vector field by v. We can now define F on $D\xi$ to be

$$F(x) = f(\overline{x}) + (1 - \langle x, x \rangle) v(f(\overline{x})),$$

where $\overline{x}$ is an arbitrary point in the boundary of the fiber $p^{-1}(p(x))$ (f maps all those points to the same point in $P \times [0,1]$) and $\langle , \rangle$ is a smooth Riemannian metric on ξ according to which ξ is the unit disc bundle. After smoothing F around Q, it will have the same singularities as $f \times id_{[0,1]}$ and definite folds along the translate of Q by v, so it is a τ-map. On $M \times \{0\}$, the preimage of $P \times \{0\}$, F is the same as f, and on $M \times \{1\}$, the preimage of $P \times \{1\}$, the image of F is within the enlargement of $d^{-1}([1,2])$ by v, which can be chosen to be still within the prescribed neighbourhood of $f(S(f))$. Thus $F|_{M \times \{1\}}$ is a τ-mapping τ-cobordant to f with the required property. □

So, under the circumstances described above, every map is cobordant to one with its image contained in a small neighbourhood of the image of the (original) singular set, hence the proof of Theorem 3 can be applied.

For $k \leq -2$ Theorem 4 is not true in general: for $k = -2$, say, the natural projection $M \times \mathbb{R}P^2 \to M$ has only regular points, but none of its fibers is null-cobordant and therefore all maps cobordant to it are surjective (regardless of the amount of the allowed singularities). However, if we restrict the set τ to only contain equivalence classes of regular fibers that bound singularities present in τ as follows, a slight weakening of Theorem 4 will still be true:

Theorem 5. *Suppose that τ is such a set of multisingularities of maps of codimension $k < 0$ that whenever F^{-k} is a regular fiber of a germ $[F \to \{point\}]$ from τ, there is a τ-mapping $g_F : (W, \partial W) \to ([0,1), \{0\})$ such that ∂W is diffeomorphic to F. Then for any τ-map $f : M^n \to P^{n+k}$ there is an $(n+k-1)$-dimensional subcomplex K in P such that for any neighbourhood U of K in P there is a τ-map τ-cobordant to f with its image contained in U.*

Proof. We choose a simplicial subdivision of $\operatorname{im} f$ compatible with its stratification according to the singularities of f, and fix K to be the $n+k-1$-skeleton of this subdivision. Set $d : P \to [0,2]$ to be a smooth function such that $d^{-1}(2) = K$, $d^{-1}(0) = \emptyset$ and 1 is a regular value of d, finally $d^{-1}([1,2])$ is contained in the given neighbourhood U of K. With this d, we can repeat the proof of Theorem 4 word by word, only replacing the extensions by definite folds in direction v. In our setup, the components of Q are contained in $n+k$-cells of the subdivision, so the mapping f is diffeomorphic to a projection $D^{n+k} \times F^{-k} \to D^{n+k}$ on the whole cell, in particular, on the selected component of Q. On such a component, we can extend the mapping on Q by mappings g_F in the direction of v, obtaining the desired cobordism between f and a mapping satisfying the claim of the theorem. □

Hence under these conditions the proof of Theorem 3 proceeds in the same way as before.

2.2 Removal of top monosingularity stratum by cobordism

An immediate consequence of the fibration $X_{\tau'} \xrightarrow{i} X_\tau \xrightarrow{\kappa_I} \Gamma T\tilde{\xi}_\eta$ is the criterion of [45, Theorem 8] (which itself is a generalization of a previous result [44]):

Theorem 6. *If the sets τ' and τ satisfy the conclusion of Theorem 3, that is, there is a Serre fibration $X_{\tau'} \xrightarrow{i} X_\tau \xrightarrow{\kappa_I} \Gamma T\tilde{\xi}_\eta$, then a τ-map $f : M \to P$ is τ-cobordant to a τ'-map if and only if the singular set $\overline{\eta[f]}$ is null-cobordant as an immersion with a normal $\tilde{\xi}_\eta$-structure.*

Proof. Applying the functor $[P, -]$ to the assumed fibration yields a short exact sequence:

$$[P, X_{\tau'}] \stackrel{[P,i]}{\rightarrow} [P, X_\tau] \stackrel{[P,\kappa_I]}{\rightarrow} [P, \Gamma T\tilde{\xi}_\eta].$$

The homotopy class of the classifying map υ of f lies in $[P, X_\tau]$, and is mapped by $[P, \kappa_I]$ to that of the classifying mapping of $\overline{\eta[f]}$. If f is τ-cobordant to a τ'-map via a cobordism F, then the restriction of F to $\overline{\eta[F]}$ is a null-cobordism of the restriction of f to $\overline{\eta[f]}$ as an immersion with a normal $\tilde{\xi}_\eta$-structure. Conversely, if the image $[P, \kappa_I]([\upsilon])$ is null, then $[\upsilon]$ lies in the image of $[P, i]$ and thus υ is homotopic to the classifying map of a τ'-map, which means that f is cobordant to that τ'-map. □

Remark: The same proof applied to the fibration $X_{\tau'} \to X_r \to \tilde{\Gamma}_r$ shows that the statement of Theorem 6 holds true if we restrict the multiplicity of η, so f is τ_r-cobordant to a τ'-map if and only if the restriction of f to the singular set $\overline{\eta[f]}$ is null-cobordant as an r-immersion with a normal $\tilde{\xi}_\eta$-structure.

In other words, the cobordism class of $\eta[f]$ as a $\tilde{\xi}_\eta$-immersion is the precise obstruction to the removal of the η-stratum by a τ-cobordism. One may hope that this obstruction can be calculated algebraically, and, indeed, in several cases it can be expressed as an evaluation of a characteristic class of the virtual normal bundle of the investigated mapping. In the first such case, Theorem 7, the obstruction turns out to be identical to the Thom polynomial, and one may be tempted to conjecture that the vanishing of the Thom polynomial is a sufficient condition for the removability of the top singular stratum by τ-cobordism. However, this is not the case, as will be shown by the proof of Theorem 15.

2.3 Removal of singularities from cobordisms only

A natural starting point for the investigation of singular cobordism groups $Cob_\tau(P)$ is the case when τ is least restrictive – when all singularities that arise in a generic mapping or a generic cobordism are in τ. We are only considering the case of nice dimensions, so this means that any arising mapping can be deformed into a τ-map and we can hence drop the requirement on the mappings. This assumption implies that $Cob_\tau(n,k) \cong \mathfrak{N}_n$ and $Cob_\tau^{or}(n,k) \cong \Omega_n$, with the isomorphism given by the abstract (oriented) cobordism class of the source manifold.

The next simplest case is when τ-maps are dense for maps of n-manifolds, but not for those of $(n+1)$-manifolds. We can then use Theorem 3 in a way similar to Theorem 6 to gain information about $Cob_\tau(n,k)$:

Theorem 7. *Assume τ is a collection of multisingularities of maps (respectively, of cooriented maps) of codimension k such that all generic maps of n-manifolds into $(n+k)$-manifolds are τ-maps, and there is a monosingularity η of codimension $n+1$ in the source, such that all generic maps from $n+1$-dimensional manifolds into $(n+k+1)$-dimensional manifolds are $\tau\cup\{\eta\}$-maps. Additionally, if we consider non-cooriented mappings, $\tilde{\xi}_\eta$ is required to be non-orientable. Let $G \leq \mathbb{Z}_2$ be the range of the $\mathbb{Z}_2$ Thom polynomial of η evaluated on all closed $(n+1)$-manifolds (all oriented $(n+1)$-manifolds in the* *case of cooriented maps); define* $G \leq \mathbb{Z}$ *to be the range of the integral Thom polynomial of* η *evaluated on all closed oriented* $(n+1)$*-manifolds in the case of a coorientable* η *and cooriented mappings. Then the forgetful mappings*

$$r : Cob_\tau(\mathbb{S}^{n+k}) \to \mathfrak{N}_n(\mathbb{S}^{n+k}) \cong \mathfrak{N}_n$$
$$r^{SO} : Cob_\tau^{SO}(\mathbb{S}^{n+k}) \to \Omega_n(\mathbb{S}^{n+k}) \cong \Omega_n$$

are clearly onto. Their kernels are

$$\ker r \cong \mathbb{Z}_2/G$$

and

$$\ker r^{SO} = \begin{cases} \mathbb{Z}/G & \text{if } \tilde{\xi}_\eta \text{ is orientable,} \\ \mathbb{Z}_2/G & \text{if } \tilde{\xi}_\eta \text{ is not orientable.} \end{cases}$$

Proof. Consider the homotopy long exact sequence of the fibration $X_\tau \to X_{\tau\cup\{\eta\}} \to \Gamma T\tilde{\xi}_\eta$ (Theorem 3):

$$\cdots \to \pi_{n+k+1}(X_{\tau\cup\{\eta\}}) \xrightarrow{pr} \pi_{n+k+1}(\Gamma T\tilde{\xi}) \to$$
$$\to \pi_{n+k}(X_\tau) \to \pi_{n+k}(X_{\tau\cup\{\eta\}}) \to \pi_{n+k}(\Gamma T\tilde{\xi}) \to \dots$$

By definition, $\pi_{n+k+q}(\Gamma T\tilde{\xi}) = \pi_{N+n+k+q}(S^N T\tilde{\xi}) = \pi_{N+n+k+q}(T(\tilde{\xi}+\varepsilon^N))$ for N large enough. In particular, since $\tilde{\xi}+\varepsilon^N$ has rank $N+n+k+1$, $\pi_{n+k}(\Gamma T\tilde{\xi}) = \pi_{N+n+k}(T(\tilde{\xi}+\varepsilon^N)) = 0$. The following lemma deals with the case $q = 1$.

Lemma 8. *Given a vector bundle* ξ *of rank* $m \geq 1$ *over a connected base* B*,*

$$\pi_m(T\xi) = \begin{cases} \mathbb{Z} & \text{if } \xi \text{ is orientable} \\ \mathbb{Z}_2 & \text{if } \xi \text{ is not orientable} \end{cases} \tag{2.1}$$

and the mapping $[f] \to [f \cap B\xi]$ *is an isomorphism. Here* $[f \cap B\xi]$ *denotes the number of intersection points of* $B\xi$ *and the image of* f*, taken with sign if* ξ *is oriented, after a small perturbation to make* f *transversal to* $B\xi$*.*

Proof. Since $T\xi$ is $(m-1)$-connected, $\pi_m(T\xi) \cong H_m(T\xi;\mathbb{Z})$. This group is generated by the class of the fiber, which is a free generator if ξ is orientable and has order 2 if ξ is not orientable. The mapping $[f] \to [f \cap B\xi]$ is the evaluation of the Thom class (with $\mathbb{Z}_2$ coefficients if ξ is not orientable) on the image of $[f]$ under the Hurewicz homomorphism, hence it is an isomorphism. □

In the case of the space $\Gamma T\tilde{\xi}$, which classifies immersions with normal $\tilde{\xi}$-structure, we get that the isomorphism of the lemma counts η-points modulo 2 in general and with sign if $\tilde{\xi}$ is orientable, implying the coorientability of η under the assumptions of the theorem. Given a generic mapping $f : N^{n+1} \to \mathbb{S}^{n+k+1}$ classified by $\kappa : \mathbb{S}^{n+k+1} \to X_{\tau\cup\{\eta\}}$, the image $pr([\kappa])$ will be represented by a classifying mapping of the restriction of f to its η-points (with the normal structure defined by f). Therefore the composition of pr with the isomorphism of the lemma counts η-points in f (with sign if $\tilde{\xi}_\eta$ is orientable and modulo 2 otherwise), evaluating the Thom polynomial of η on the normal bundle of f, which is the same virtual bundle as the virtual normal bundle of N since $\mathbb{S}^{n+k+1}$ is stably parallelizable. Hence the image of pr is isomorphic to G, and substituting it into the long exact sequence above we get the short exact sequence

$$\begin{array}{ccccc} 0 \to \mathbb{Z}_2/G(\text{resp. } \mathbb{Z}/G) & \longrightarrow & \pi_{n+k}(X_\tau) & \longrightarrow & \pi_{n+k}(X_{\tau+\eta}) \to 0 \\ & & \downarrow\cong & & \downarrow\cong \\ & & Cob_\tau(\mathbb{S}^{n+k}) & & \mathfrak{N}_n(\text{resp. } \Omega_n) \end{array}$$

which is equivalent to the statement of Theorem 7. □

2.4 Cobordisms of fold maps in the nearly generic dimensions

Armed with Theorem 7, we can now calculate a singular cobordism group. The dimensions n and $n+k$ for which the conditions of Theorem 7 hold, that is, η singularities generically appear only in the cobordisms but not in the maps themselves will be called *nearly generic*. Set τ to contain all multisingularities formed from regular and $\Sigma^{1,0}$ points. In this case, a τ-map will be called a *fold map*.

Theorem 9. *The cobordism groups of fold maps in the nearly generic dimensions are as follows.*

(a) *For any $k \geq 1$*

$$Cob_\tau(2k+1, k) \cong \mathfrak{N}_{2k+1},$$

(b_1)

$$Cob_\tau^{SO}(5,2) \cong \Omega_5 \oplus \mathbb{Z}_2 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

(b_*) *For any $m \geq 2$*

$$Cob_\tau^{SO}(4m+1, 2m) \cong \Omega_{4m+1}.$$

(c) *For any $m \geq 1$*

$$Cob_\tau^{SO}(4m-1, 2m-1) \cong \Omega_{4m-1} \oplus \mathbb{Z}_{3^t},$$

where $t = \min\{j \mid \alpha_3(2m+j) \leq 3j\}$ and $\alpha_3(x)$ denotes the sum of digits of the integer x in triadic system. For example, $Cob_\tau^{SO}(3,1) \cong \mathbb{Z}_3$ and $Cob_\tau^{SO}(33,15) \cong \Omega_{33} \oplus \mathbb{Z}_9$.

Proof. The settings satisfy the conditions of Theorem 7. Indeed, from all generic singularities in the given dimensions we exclude only the cusps, which have codimension $2k+2$ in the source. The target bundle $\tilde{\xi}_{\Sigma^{1,1,0}}$ is not orientable in the non-cooriented case: the source and target representations of the group $G_{\Sigma^{1,1,0}} \cong \mathbb{Z}_2 \times O(k)$ are (see [30])

$$\lambda(\varepsilon, A) = \begin{pmatrix} \varepsilon & & & \\ & 1 & & \\ & & A & \\ & & & \varepsilon A \end{pmatrix} \text{ and } \tilde{\lambda}(\varepsilon, A) = \begin{pmatrix} \varepsilon & & & & \\ & 1 & & & \\ & & A & & \\ & & & \varepsilon A & \\ & & & & A \end{pmatrix}, \quad (2.2)$$

so $\det \tilde{\lambda} = \varepsilon^{k+1} \det A$ is never identically 1 (for non-cooriented maps).

We need to calculate the range of the Thom polynomial of the cusp, which is $w_{k+1}^2 + w_{k+2}w_k$ over $\mathbb{Z}_2$ and $p_{\frac{k+1}{2}}$ over $\mathbb{Z}$ in the oriented setting and k odd ([33], or [31] in combination with [5]). This latter range has been calculated by Stong [38], and it consists exactly of the numbers of form $s3^t$, $s \in \mathbb{Z}$ with t defined as in (c).

For $w_{k+1}^2 + w_{k+2}w_k$, we first consider the manifold $W = (\mathbb{R}P^2)^{k+1}$. Its total normal Stiefel-Whitney class is the product of the total normal Stiefel-Whitney classes of the constituent projective planes and can be expressed in terms of the generator $a \in H^1(\mathbb{R}P^2)$ as

$$\bar{w}(W) = (1 + a) \otimes \cdots \otimes (1 + a),$$

hence $\bar{w}_{k+2}(W) = 0$, $\bar{w}_{k+1}(W) = a \otimes \cdots \otimes a$ and therefore

$$(\bar{w}_{k+1}^2 + \bar{w}_{k+2}\bar{w}_k)[W] = (a \otimes \cdots \otimes a)^2[W] = 1, \quad (2.3)$$

demonstrating that the $\mathbb{Z}_2$ Thom polynomial is surjective in the unoriented case.

In the oriented setting we separate the case $m = 1$ as the oriented cobordism group Ω_6 vanishes, consequently the range of any characteristic number, in particular the Thom polynomial of the cusp, is $\{0\}$. However, when

$m \geq 2$, we let Y be the Dold manifold $(\mathbb{C}P^2 \times S^1)/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by complex conjugation on $\mathbb{C}P^2$ and by multiplication by -1 on S^1. This manifold is orientable and generates $\Omega_5 \cong \mathbb{Z}_2$ (see [24]). Denote by N the product $Y \times (\mathbb{R}P^2)^{m-2}$. Since the manifold $N \times N$ is a square, there exists an orientable manifold cobordant to it [51] [10]; let W be such a manifold. Applying the product formula for normal Stiefel-Whitney classes, we obtain

$$\begin{aligned}
\bar{w}_{2m+1}^2[N \times N] &= \sum_{i_1+j_1=2m+1} \sum_{i_2+j_2=2m+1} (\bar{w}_{i_1} \times \bar{w}_{j_1})(\bar{w}_{i_2} \times \bar{w}_{j_2})[N \times N] = \\
&= \sum_{i_1+j_1=2k+1} \sum_{i_2+j_2=2m+1} (\bar{w}_{i_1}\,\bar{w}_{i_2})[N](\bar{w}_{j_1}\,\bar{w}_{j_2})[N] = \\
&= 2 \sum_{0 \leq i \leq k} (\bar{w}_i\,\bar{w}_{2k+m-i})[N](\bar{w}_{2m+1-i}\,\bar{w}_i)[N] = 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\bar{w}_{2m+2}\bar{w}_{2m}[N \times N] &= \sum_{i_1+j_1=2m+2} \sum_{i_2+j_2=2m} (\bar{w}_{i_1} \times \bar{w}_{j_1})(\bar{w}_{i_2} \times \bar{w}_{j_2})[N \times N] = \\
&= \sum_{i_1+j_1=2m+2} \sum_{i_2+j_2=2m} (\bar{w}_{i_1}\,\bar{w}_{i_2})[N](\bar{w}_{j_1}\,\bar{w}_{j_2})[N] = \\
&= \sum_{0 \leq i \leq 2m} (\bar{w}_{2m+1-i}\,\bar{w}_i)[N](\bar{w}_{i+1}\,\bar{w}_{2m-i})[N] = \\
&= \big(\bar{w}_{m+1}\,\bar{w}_m[N]\big)^2 = \bar{w}_{m+1}\,\bar{w}_m[N],
\end{aligned}$$

and, since $\bar{w}_i\,\bar{w}_j[\mathbb{R}P^2] = 1$ if and only if $i = j = 1$, we find

$$\begin{aligned}
&\bar{w}_{m+1}\,\bar{w}_m[N] = \\
&= \sum_{\substack{i_0+\dots+i_{m-2}=m+1\\ j_0+\dots+j_{m-2}=m}} (\bar{w}_{i_0}\times\dots\times\bar{w}_{i_{m-2}})(\bar{w}_{j_0}\times\dots\times\bar{w}_{j_{m-2}})[Y\times(\mathbb{R}P^2)^{m-2}] = \\
&= \sum_{\substack{i_0+\dots+i_{m-2}=m+1\\ j_0+\dots+j_{m-2}=m}} \big(\bar{w}_{i_0}\,\bar{w}_{j_0}[Y]\big)\big(\bar{w}_{i_1}\,\bar{w}_{j_1}[\mathbb{R}P^2]\big)\dots\big(\bar{w}_{i_{m-2}}\,\bar{w}_{j_{m-2}}[\mathbb{R}P^2]\big) = \\
&= \bar{w}_3\,\bar{w}_2[Y] = 1.
\end{aligned}$$

To see that the last equality holds, note that $\bar{w}_5(Y)$ vanishes as Y has an embedding in $\mathbb{R}^{10}$ according to Whitney's theorem. Since Y is orientable $w_1(Y) = \bar{w}_1(Y) = 0$. The manifold Y generates $\Omega_5 \cong \mathfrak{N}_5$ and thus it is not null-cobordant. Therefore at least one of its normal Stiefel-Whitney numbers must be non-zero and by the two conditions mentioned above the only possibility is $\bar{w}_2\bar{w}_3[Y]$. Thus $\bar{w}_2\bar{w}_3[Y] = 1$ indeed, and our manifold W demonstrates that the $\mathbb{Z}_2$ Thom polynomial of the cusp is surjective in this case as well.

So Theorem 7 gives us short exact sequences

$$0 \to 0 \to Cob_\tau(2k+1,k) \to \mathfrak{N}_{2k+1} \to 0 \text{ for all } k \geq 1,$$
$$0 \to \mathbb{Z}_2 \to Cob_\tau^{SO}(5,2) \to \Omega_5 \cong \mathbb{Z}_2 \to 0,$$
$$0 \to 0 \to Cob_\tau^{SO}(4m+1,2m) \to \Omega_{4m+1} \to 0 \text{ for all } m \geq 2,$$
$$0 \to \mathbb{Z}_{3^t} \to Cob_\tau^{SO}(4m-1,2m-1) \to \Omega_{4m-1} \to 0 \text{ for all } m \geq 1.$$

The first and third sequences give us immediately claims (a) and (b_*). Since Ω_{2k+1} is a direct sum of cyclic groups of order 2 ([51] gives the exact number of summands), the fourth sequence corresponds to the trivial extension and is equivalent to claim (c). To finish the proof, we need to prove that the second sequence corresponds to the trivial extension $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and not to $\mathbb{Z}_4$.

If the group $Cob_\tau^{SO}(5,2)$ was isomorphic to $\mathbb{Z}_4$, then all its elements not in the kernel of the forgetful map to Ω_5 would have order 4. Denote by $[f, M] \in Cob_\tau^{SO}(5, 2)$ the fold cobordism class represented by the stable map $f : M \to \mathbb{R}^7$. It is therefore enough to prove the existence of a stable map $f : Y \to \mathbb{R}^7$ such that Y is not null-cobordant (in the oriented sense) and $[f, Y]$ has order 2. To this end, let Y be the 5-dimensional Dold manifold with some fixed orientation. By [9] there exists an immersion $\tilde{f} : Y \to \mathbb{R}^8$. We show that if $f = \pi \circ \tilde{f} : Y \to \mathbb{R}^7$, where $\pi : \mathbb{R}^8 \to \mathbb{R}^7$ is a generically chosen projection then $[f, Y]$ is an element of order 2. The proof has three steps.

Step 1. *Let $g : Y \to \mathbb{R}^7$ be any fold map. Let $H \subset \mathbb{R}^7$ be a hyperplane disjoint from $g(Y)$, let $r : \mathbb{R}^7 \to \mathbb{R}^7$ be the reflection in H, and let $-Y$ denote Y with the opposite orientation. Then $[g, Y] + [r \circ g, -Y] = 0$.*

This follows from the standard construction of a cobordism inverse: in a coordinate system where the coordinate x_1 is perpendicular to H and centered on H, if $g = (g_1, g_+) \in \mathbb{R} \times H$, then $G : Y \times [0, \pi] \to \mathbb{R}^8_+$ given by

$$G(\theta, y) = (\sin\theta g_1(y), \cos\theta g_1(y), g_+(y)) \in \mathbb{R}_+ \times \mathbb{R} \times H$$

gives a fold cobordism establishing Step 1.

The manifold Y admits an orientation reversing diffeomorphism $A : Y \to Y$ induced by complex conjugation on the S^1-factor thought of as the unit circle in $\mathbb{C}$. Let X be the mapping torus of A, i.e., $X = (Y \times I)/\sim$, where $(y, 0) \sim (A(y), 1)$. Then X is a non-orientable closed 6-manifold (this is the 6-dimensional Wall manifold, X_6 in the notation of [51]). In [51], the cohomology ring of X and its Stiefel-Whitney classes were computed. This computation in combination with a straightforward calculation imply that the normal Stiefel-Whitney number $\bar{w}_3^2[X] + \bar{w}_2\bar{w}_4[X]$ equals 0. Hence the number of cusps of any stable map $F : X \to \mathbb{R}^8$ is even.

Step 2. *Let $g : Y \to \mathbb{R}^7$ be any fold map then $[g, Y] + [r \circ g, Y] = 0$.*

To see this we first define a map $K : X \to \mathbb{R}^8$ as follows. Consider $X = (Y \times [0,1])/\sim$, where $(y,0) \sim (A(y),1)$. Define $K : Y \times [0, \frac{1}{2}] \to \mathbb{R}^8_+$, $K(t,y) = G(2\pi t, y)$ where G is as in Step 1 and then extend K over $Y \times [\frac{1}{2}, 1]$. Since K does not have any cusps in $Y \times [0, \frac{1}{2}]$ it follows that K has an even number of cusps in $Y \times [\frac{1}{2}, 1]$. Since $\Omega_5 \cong \mathbb{Z}_2$ there exists an orientable 6-manifold W with $\partial W = Y \sqcup Y$. Let $F : W \to \mathbb{R}^8_+$ be a stable map such that the restriction to one boundary component agrees with g and the restriction to the other agrees with $r \circ g$. Gluing the maps F and $K|_{Y \times [\frac{1}{2}, 1]}$ along their common boundary we obtain a map of an orientable 6-manifold into $\mathbb{R}^8$. The parity of the number of cusps of this map is a characteristic number of the underlying manifold, which is null-bordant even as an oriented manifold, so there is an even number of cusps. It follows that F has an even number of cusps. In the short exact sequence under investigation, the number of cusps of F modulo 2 is mapped to the cobordism class of $\partial F = [g, Y] + [r \circ g, Y]$, hence this latter class vanishes.

Step 3. *Let $f : Y \to \mathbb{R}^7$ be the projection of an immersion $\tilde{f} : Y \to \mathbb{R}^8$. Then $[f, Y] + [f, -Y] = 0$.*

The cobordism group of immersions of oriented 5-manifolds into $\mathbb{R}^8$ is isomorphic to the 8^{th} stable homotopy group of the Thom space $MSO(3)$, which is isomorphic to the 9^{th} stable homotopy group of the suspension $\Sigma MSO(3)$, which classifies 1-framed immersions of codimension 4. The involution on this cobordism group induced by changing the orientation on the source corresponds to the involution on the stable homotopy group induced by ι, the reflection of $\Sigma MSO(3)$ in $MSO(3)$ (changing the orientation of the frame). It follows from standard properties of suspensions that if Z is any topological space and $\text{im}(\pi_n(Z))$ denotes the image of the suspension map $\pi_n(Z) \to \pi_{n+1}(\Sigma Z)$, then the map ι_*, induced by the reflection in Z $\iota : \Sigma Z \to \Sigma Z$, agrees with multiplication by -1 on $\text{im}(\pi_n(Z))$. Thus there exists an oriented 6-manifold W with $\partial W = Y \sqcup (-Y)$ and an immersion $\tilde{F} : W \to \mathbb{R}^9_+$ that agrees with $\tilde{f}$ on both boundary components.

Let ν denote the normal bundle of $\tilde{F}$. Consider the manifold $\hat{W} = W \cup (Y \times [0,1])$ obtained by identifying boundary components. Note that $\hat{W}$ is orientable. Moreover, we can extend ν to a bundle over $\hat{W}$ in a canonical way, using the identity transition function at both junctions. Let $\hat{\nu}$ denote this extension. Note that $T\hat{W} \oplus \hat{\nu}$ is a trivial bundle and thus $\hat{\nu}$ is a normal bundle for $\hat{W}$. Consider a unit vector $v \in \mathbb{R}^8$ as a vector field along $\tilde{F}(W)$. Projecting v to ν we get a section n of the bundle ν over W. The section n can be continued in a canonical way along $Y \times [0,1]$ and thus gives a section of $\hat{\nu}$. The zero set Σ of n is dual to $\bar{w}_3(\hat{W})$. Moreover, along Σ, v gives a vector field in the restriction $T\hat{W}|_\Sigma$. A point where v is tangent to $\Sigma \cap W$ corresponds to a cusp of F, where $F = \pi_v \circ \tilde{F}$ and π_v is a projection parallel to v.

For generic v, $\pi_v \circ \tilde{f}$ does not have any cusps. Thus, for such v the corresponding section of $T\hat{W}|_\Sigma$ is nowhere tangent to $\Sigma \cap Y \times [0,1]$. Furthermore the mod 2 number of tangency points is equal to the mod 2 self intersection number $[\Sigma] \cdot [\Sigma]$ of Σ in $\hat{W}$. By Poincaré duality this self-intersection number is

$$[\Sigma] \cdot [\Sigma] = \bar{w}_3^2[\hat{W}] = 0,$$

where the last equality holds since $\hat{W}$ is an orientable 6-manifold and $\Omega_6 = 0$. We conclude that if $F = \pi_v \circ \tilde{F}$ then F is a cobordism with an even number of cusps, implying that $[f, Y] + [f, -Y] = 0$.

We are now in position to finish the proof. By Steps 1–3, we have

$$[f, Y] = -[r \circ f, -Y] = -[f, -Y] = -[f, Y].$$

We conclude that $[f, Y]$ has order 2, hence the extension constructing the group $Cob_\tau^{SO}(5, 2)$ is trivial and (b_1) holds. □

These calculations can also be performed in the case of left-right bordism groups. We will denote by $\mathfrak{N}(n, k)$ (respectively, $\Omega(n, k)$) the left-right bordism group of arbitrary mappings of n-dimensional manifolds into

$(n+k)$-dimensional manifolds (respectively, oriented n-manifolds into oriented $(n+k)$-manifolds). These groups have been computed by Stong [39]: there are isomorphisms $\mathfrak{N}(n,k) \cong \mathfrak{N}_{n+k}(\Omega^\infty MO(k+\infty))$ and $\Omega(n,k) \cong \Omega_{n+k}(\Omega^\infty MSO(k+\infty))$.

Theorem 10. *The left-right bordism groups of fold maps in the almost generic dimensions are as follows.*

(a) *For any* $k \geq 0$

$$Bord_\tau(2k+1,k) \cong \mathfrak{N}(2k+1,k),$$

(b) *For any* $m \geq 1$

$$Bord_\tau^{SO}(4m+1,2m) \cong \Omega(4m+1,2m).$$

(c) *For any* $m \geq 1$*, there is an exact sequence*

$$0 \to \mathbb{Z}_{3^u} \to Bord_\tau^{SO}(4m-1,2m-1) \to \Omega(4m-1,2m-1) \to 0,$$

where the power u *satisfies* $0 \leq u \leq t$*,* $t = \min\{j \mid \alpha_3(2m+j) \leq 3j\}$*, see Theorem 9 (c).*

Proof. Define σ to be the set of all singularities formed from regular, fold and cusp points. In the given dimensions the codimensions of all multi-singularities from τ that are more complicated than the cusp is at least $k+3k+2 > 3k+2$. Hence for the purpose of calculating the bordism group $Bord_\tau(n) \cong \mathfrak{N}_{n+k}(X_\tau)$ or $Bord_\tau^{SO}(n) \cong \Omega_{n+k}(X_\tau^{SO})$ for $n = 2k+1$ we can omit from the construction of the appropriate classifying space all the blocks containing the cusp except for the one corresponding to the single cusp. Therefore, the relevant fragment of the long exact sequence (in the homology theory of oriented or unoriented bordism) of the pair (X_σ, X_τ) can be interpreted as

$$\cdots \to \mathfrak{N}_{3k+2}(X_\sigma) \to \tilde{\mathfrak{N}}_{3k+2}(T\tilde{\xi}) \to \mathfrak{N}_{3k+1}(X_\tau) \to$$
$$\to \mathfrak{N}_{3k+1}(X_\sigma) \to \tilde{\mathfrak{N}}_{3k+1}(T\tilde{\xi}) = 0$$
$$\cdots \to \Omega_{3k+2}(X_\sigma^{SO}) \to \tilde{\Omega}_{3k+2}(T\tilde{\xi}^{SO}) \to \Omega_{3k+1}(X_\tau^{SO}) \to$$
$$\to \Omega_{3k+1}(X_\sigma^{SO}) \to \tilde{\Omega}_{3k+1}(T\tilde{\xi}^{SO}) = 0$$

The final terms vanish because the vector bundles $\tilde{\xi}$ and $\tilde{\xi}^{SO}$ have rank $3k+2$. We also have (using the formula of [10])

$$\tilde{\mathfrak{N}}_{3k+2}(T\tilde{\xi}) \cong \bigoplus_{i=0}^{3k+2} \mathfrak{N}_i \otimes \tilde{H}_{3k+2-i}(T\tilde{\xi};\mathbb{Z}_2) = H_{3k+2}(T\tilde{\xi};\mathbb{Z}_2) \cong H_0(B\tilde{\xi};\mathbb{Z}_2)$$

and similarly

$$\tilde{\Omega}_{3k+2}(T\tilde{\xi}^{SO}) \cong \begin{cases} H_0(B\tilde{\xi}^{SO};\mathbb{Z}) & \text{if } \tilde{\xi}^{SO} \text{ is orientable,} \\ H_0(B\tilde{\xi}^{SO};\mathbb{Z}_2) & \text{if } \tilde{\xi}^{SO} \text{ is not orientable.} \end{cases}$$

Hence according to Lemma 8 we can identify $\tilde{\Omega}_{3k+2}(T\tilde{\xi}^{SO})$ with $\pi_{3k+2}(\Gamma T\tilde{\xi}^{SO})$ and $\tilde{\mathfrak{N}}_{3k+2}(T\tilde{\xi})$ with $\pi_{3k+2}(\Gamma T\tilde{\xi}) \otimes \mathbb{Z}_2$. In the unoriented case the mapping $\mathfrak{N}_{3k+2}(X_\sigma) \to \tilde{\mathfrak{N}}_{3k+2}(T\tilde{\xi})$ contains the image of the mapping $\pi_{3k+2}(X_\sigma) \to \pi_{3k+2}(\Gamma T\tilde{\xi}) \cong \mathfrak{N}_{3k+2}(T\tilde{\xi})$. We proved in Theorem 9 that the latter mapping is surjective, therefore the exact sequence degenerates in the same way:

$$0 \to \mathfrak{N}_{3k+1}(X_\tau) \to \mathfrak{N}_{3k+1}(X_\sigma) \to 0,$$

proving claim a).

For the case b), it is similarly sufficient to demonstrate that the mapping $\Omega_{3k+2}(X_\sigma^{SO}) \to \tilde{\Omega}_{3k+2}(T\tilde{\xi}^{SO})$ is surjective. Since in this case (k is even) the bundle $T\tilde{\xi}^{SO}$ is not orientable, we have $\tilde{\Omega}_{3k+2}(T\tilde{\xi}^{SO}) \cong \mathbb{Z}_2$ and the mapping in question is the evaluation of the Thom polynomial with $\mathbb{Z}_2$ coefficients

$w_{k+1}^2 + w_{k+2}w_k$ on the virtual normal bundle of a representative. For $m \geq 2$, we know from case b$_*$) of Theorem 9 that this Thom polynomial is not identically 0 even on the image of $\pi_{3k+2}(X_\sigma^{SO})$ in $\Omega_{3k+2}(X_\sigma^{SO})$.

The case $m = 1$ requires a separate construction. We shall construct a map $f : M \to \mathbb{C}P^4$ of an oriented 6-manifold M with the desired property

$$\langle (w_3^2 + w_4w_2)(\nu_f), [M] \rangle = 1,$$

where ν_f is the virtual normal bundle of f. To construct such a map $f : M \to \mathbb{C}P^4$ it is sufficient to construct $F : S^L\mathbb{C}P^4 \to MSO(L+2)$ for some sufficiently large L, such that

$$\langle F^*(Uw_3^2 + Uw_4w_2), [S^L\mathbb{C}P^4] \rangle = 1, \tag{2.4}$$

where U is the Thom class of $MSO(L+2)$. Indeed, applying the Pontryagin-Thom construction to F gives an embedding $i : M \to S^L\mathbb{C}P^4$ of an orientable 6-manifold M such that

$$(w_3^2 + w_4w_2)(\nu_i) = (F|_M)^*(w_3^2 + w_4w_2),$$

where the cohomology classes in the left hand side are the Stiefel-Whitney classes of the universal oriented $L+2$-bundle γ_{L+2}^{SO}. Without loss of generality, we may assume that $i(M) \subset \mathbb{R}^L \times \mathbb{C}P^4 \subset S^L\mathbb{C}P^4$. Defining $f = \pi \circ i$, where $\pi : \mathbb{R}^L \times \mathbb{C}P^4 \to \mathbb{C}P^4$ is the natural projection, we obtain $f : M \to \mathbb{C}P^4$ with properties as desired. To see this, note that the virtual normal bundle ν_f of f belongs to the equivalence class of ν_i and hence

$$\begin{aligned}\langle (w_3^2 + w_4w_2)(\nu_f), [M] \rangle &= \langle (w_3^2 + w_4w_2)(\nu_i), [M] \rangle = \\ &= \langle (F|_M)^*(w_3^2 + w_4w_2), [M] \rangle = \\ &= \langle F^*(Uw_3^2 + Uw_4w_2), [S^L\mathbb{C}P^4] \rangle = 1.\end{aligned}$$

To construct the map $F : S^L\mathbb{C}P^4 \to MSO(L+2)$, we note that the dimension of $\mathbb{S}^L\mathbb{C}P^4$ equals $L+8$. Thus we can replace $MSO(L+2)$ with the homotopically $(L+8)$-equivalent product of Eilenberg-MacLane spaces

$$K = K(\mathbb{Z}, L+2) \times K(\mathbb{Z}, L+6) \times K(\mathbb{Z}_2, L+7).$$

A standard calculation shows that the class

$$Uw_3^2 + Uw_4w_2 \in H^{L+8}(MSO(L+2); \mathbb{Z}_2)$$

is identified with the class

$$Sq^4Sq^2l_{L+2} + Sq^2l_{L+6} + Sq^1l'_{L+7} \in H^{(L+8)}(K; \mathbb{Z}_2)$$

under an $(L+8)$-equivalence, where

$$l_N \in H^N(K(\mathbb{Z}, N); \mathbb{Z}_2)$$

and

$$l'_N \in H^N(K(\mathbb{Z}_2, N); \mathbb{Z}_2)$$

are the cohomological fundamental classes, and where Sq^m denotes the degree m Steenrod operation. Define F as the composition $j \circ \kappa$ of the natural inclusion $j : K(\mathbb{Z}, L+2) \to K$ and the map $\kappa : S^L\mathbb{C}P^4 \to K(\mathbb{Z}, L+2)$ that corresponds to the generator S^La of the group $H^{L+2}(S^L\mathbb{C}P^4) \cong H^2(\mathbb{C}P^4) =$

$\mathbb{Z}\langle a\rangle$ (i.e., the map κ is such that $\kappa^* l_{L+2} = S^L a$). The map F then satisfies

$$\begin{aligned}
&\langle F^*(Uw_3^2 + Uw_4 w_2), [S^L\mathbb{C}P^4]\rangle = \\
&\qquad = \langle F^*(Sq^4Sq^2 l_{L+2} + Sq^2 l_{L+6} + Sq^1 l'_{L+7}), [S^L\mathbb{C}P^4]\rangle = \\
&\qquad = \langle \kappa^* j^*(Sq^4Sq^2 l_{L+2} + Sq^2 l_{L+6} + Sq^1 l'_{L+7}), [S^L\mathbb{C}P^4]\rangle = \\
&\qquad = \langle \kappa^* j^*(Sq^4Sq^2 l_{L+2}), [S^L\mathbb{C}P^4]\rangle = \langle Sq^4Sq^2\kappa^* j^* l_{L+2}, [S^L\mathbb{C}P^4]\rangle = \\
&\qquad = \langle Sq^4Sq^2 S^L a, [S^L\mathbb{C}P^4]\rangle = \langle S^L Sq^4Sq^2 a, [S^L\mathbb{C}P^4]\rangle = \\
&\qquad = \langle Sq^4Sq^2 a, [\mathbb{C}P^4]\rangle = \langle Sq^4 a^2, [\mathbb{C}P^4]\rangle = \langle a^4, [\mathbb{C}P^4]\rangle = 1.
\end{aligned}$$

Thus, (2.4) holds as desired.

Finally, the case c) is the immediate consequence of case c) of Theorem 9: the image of the mapping $\Omega_{3k+2}(X_\sigma^{SO}) \to \tilde{\Omega}_{3k+2}(T\tilde{\xi}^{SO})$ contains the image of the mapping $\pi_{3k+2}(X_\sigma^{SO}) \to \pi_{3k+2}(T\tilde{\xi}^{SO}) \cong \tilde{\Omega}_{3k+2}(T\tilde{\xi}^{SO})$, so the cokernel is a quotient of the group $\mathbb{Z}/3^t\mathbb{Z}$. The only quotients of the cyclic group of order 3^t are cyclic groups of order 3^u with $0 \le u \le t$. Hence the long exact sequence from which we started is equivalent to a short exact sequence of the form described in statement c). □

Remark: Similar results can be proved for $k = 0$ as well, but require a different, more geometric technique. In the paper [15], it is shown that pairs of cusps (of different signs, if applicable) can be eliminated by a surgery in a small neighbourhood of a line segment connecting the two cusps. Consequently, the elements in the kernel of the forgetful mapping $Cob_\tau(n) \to \mathfrak{N}_n$ or $Cob_\tau^{SO}(n) \to \Omega_n$ can be identified by the modulo 2 number (respectively, the algebraic number) of cusps in a mapping that has the mapping in question as a boundary. These numbers are well-defined modulo G, where G is the range of the evaluation of the Thom polynomial on all maps of closed (oriented) $(2n+2)$-manifolds into $\mathbb{R}^{3k+2}$.

It is an interesting fact (probably related in some way to the results of [43]) that the Thom polynomials of $III_{2,2}$ for mappings of codimension k are exactly the same as those of $\Sigma^{1,1}$ for mappings of codimension $k+1$ (see [32]), and the orientability assumptions of Theorem 6 can be easily checked to be satisfied using [30], so with exactly the same proof as above we obtain the following theorems.

Theorem 11. *Let τ consist of all multisingularities composed of regular, fold and cusp points.*

(a) *For any $k \geq 1$*

$$Cob_\tau(2k+3, k) \cong \mathfrak{N}_{2k+3},$$

(b$_*$) *For any $m \geq 1$*

$$Cob_\tau^{SO}(4m+3, 2m) \cong \Omega_{4m+3} \oplus \mathbb{Z}_{3^t},$$

where $t = \min\{j \mid \alpha_3(2m+2+j) \leq 3j\}$ and $\alpha_3(x)$ denotes the sum of digits of the integer x in triadic system,

(c) *For any $m \geq 2$*

$$Cob_\tau^{SO}(4m+1, 2m-1) \cong \Omega_{4m+1}.$$

Theorem 12. *Let τ consist of all multisingularities composed of regular, fold and cusp points.*

(a) *For any $k \geq 0$,*

$$Bord_\tau(2k+3, k) \cong \mathfrak{N}(2k+3, k),$$

(b) *For any $m \geq 1$, there is an exact sequence*

$$0 \to \mathbb{Z}_{3^u} \to Bord_\tau^{SO}(4m+3, 2m) \to \Omega(4m+3, 2m) \to 0,$$

where the power u satisfies $0 \leq u \leq t$, $t = \min\{j \mid \alpha_3(2m+2+j) \leq 3j\}$,

(c) *For any* $m \geq 1$

$$Bord_{\tau}^{SO}(4m+1, 2m-1) \cong \Omega(4m+1, 2m-1).$$

2.5 Fold maps from M^{2k+2} to $\mathbb{R}^{3k+2}$

We now attempt to apply the methods of the proof of Theorem 9 to the next possible case, when fold mappings of $(2k+2)$-manifolds into $\mathbb{R}^{3k+2}$ are considered. In this range of dimensions, the forbidden singularities have dimension 0 for the maps and dimension 1 for the cobordisms. We need an analogue of Lemma 8.

Lemma 13. *Let* ξ *be an arbitrary vector bundle of rank* $n \geq 3$ *over a connected base* B*, and set* $U_2 \in H^n(T\xi; \mathbb{Z}_2)$ *to be the Thom class. Denote by* $C : [f] \mapsto [f \cap B]$ *the map assigning to the homotopy class of* $f : \mathbb{S}^{n+1} \to T\xi$ *the class of the transverse intersection of the image of* f *with* B*. Here the class* $[f \cap B]$ *is considered as an element*

- *in* $\Omega_1(B) \cong H_1(B; \mathbb{Z})$*, if* ξ *is orientable,*
- *in* $\{[\gamma] \in \mathfrak{N}_1(B) : \gamma^*\xi$ *is orientable*$\} \cong \ker w_1(\xi) \leq H_1(B; \mathbb{Z}_2)$*, if* ξ *is not orientable.*

Then C *is onto, and its kernel is the following (independently of whether* ξ *is orientable or not):*

$$\ker C \cong \begin{cases} \mathbb{Z}_2 & \text{if } \langle U_2 w_2(\xi), H_{n+2}(T\xi; \mathbb{Z}) \rangle = 0, \\ 0 & \text{if } \exists a \in H_{n+2}(T\xi; \mathbb{Z}) \text{ such that } \langle U_2 w_2(\xi), a \rangle \neq 0. \end{cases}$$

Proof. We shall use Serre's method of killing spaces. Let $T\xi|_n$ be the n^{th} killing space. This means that we construct a fibration $K(\pi_n(T\xi), n-1) \to T\xi|_n \to T\xi$ with an n-connected $T\xi|_n$ in the usual way [26], by pulling back the path space fibration $K(\pi_n(T\xi), n-1) \to PK(\pi_n(T\xi), n) \to K(\pi_n(T\xi), n)$

with the classifying map of the generator of $H^n(T\xi;\pi_n(T\xi))$, the integral Thom class U in the case of an orientable ξ and the Thom class U_2 in the case of a nonorientable ξ. This way the group $\pi_{n+1}(T\xi) \cong \pi_{n+1}(T\xi|_n) \cong H_{n+1}(T\xi|_n;\mathbb{Z})$ can be calculated from the Serre spectral sequence. Due to dimensional constraints the only potentially non-zero differentials influencing $H_{n+1}(T\xi|_n;\mathbb{Z})$ are the transgressions $H_{n-1+j}(K(\pi_n(T\xi),n-1);\mathbb{Z}) \to H_{n+j}(T\xi;\mathbb{Z})$ for $j = 0,1,2$.

First we consider this spectral sequence when ξ is orientable. We have

$$H_{n-1+j}(K(\mathbb{Z},n-1);\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } j = 0, \\ 0 & \text{for } j = 1, \\ \mathbb{Z}_2 & \text{for } j = 2. \end{cases}$$

The generator in dimension $n-1$ is the fundamental homology class. There is a cell decomposition of $K(\mathbb{Z},n-1)$ in which there are no n-cells, hence in dimension n the homology vanishes. As a consequence, the universal coefficient theorem implies that

$$H^{n+1}(K(\mathbb{Z},n-1);\mathbb{Z}_p) \cong Hom(H_{n+1}(K(\mathbb{Z},n-1);\mathbb{Z}),\mathbb{Z}_p)$$

for all primes p. The left hand side of this equation is known to vanish for $p > 2$ and it is generated by $Sq^2 l$ for $p = 2$, where $l \in H^{n-1}(K(\mathbb{Z},n-1);\mathbb{Z}_2)$ is the modulo 2 reduction of the fundamental class. Since $Sq^1(Sq^2 l) = Sq^3 l \neq 0$, the class $Sq^2 l$ does not lift to a $\mathbb{Z}_4$ cohomology class. Therefore $H_{n+1}(K(\mathbb{Z},n-1);\mathbb{Z}) \cong \mathbb{Z}_2$, and the evaluation of $Sq^2 l$ is an isomorphism between these two groups.

In the E^∞ term of the spectral sequence, the bottom row $E^\infty_{*,0}$ is naturally identified with the image of the homomorphism $H_*(T\xi|_n) \to H_*(T\xi)$ induced by the projection $T\xi|_n \to T\xi$. Combined with the Hurewicz isomorphism $\pi_{n+1}(T\xi) \cong H_{n+1}(T\xi|_n;\mathbb{Z})$ and the Thom isomorphism $H_{n+1}(T\xi;\mathbb{Z}) \cong H_1(B\xi;\mathbb{Z})$, this identification gives the descriptions $\ker C \cong E^\infty_{0,n+1}$ (all the other terms $E^{\infty}_{i,n+1-i}$, $1 \leq i \leq n$, are 0) and $\operatorname{ran} C \cong E^{\infty}_{n+1,0}$. The transgression for $j = 0$ is an isomorphism by construction. The transgression for $j = 1$ is 0, this shows that C as a map to $H_{n+1}(T\xi;\mathbb{Z}) \cong H_1(B\xi;\mathbb{Z})$ is onto. The transgression for $j = 2$ is onto exactly if its composition with the evaluation of $Sq^2 l$ on $H_{n+1}(K(\mathbb{Z}, n-1);\mathbb{Z})$ does not vanish. Steenrod operations commute with transgressions, hence this composition is the same as the evaluation of $Sq^2 U_2 = U_2 \cup w_2(\xi)$. The kernel $\ker C \cong E^{\infty}_{0,n+1}$ is the cokernel of this evaluation, proving our claim for orientable bundles.

Now consider the case when ξ is not orientable. The homology groups $H_{n-1+j}(K(\mathbb{Z}_2, n-1);\mathbb{Z})$ are calculated in [8] to be equal to the groups

$$H_{n-1+j}(K(\mathbb{Z}, n-1);\mathbb{Z}_2) \cong H^{n-1+j}(K(\mathbb{Z}, n-1);\mathbb{Z}_2) = \begin{cases} \langle l \rangle & \text{for } j = 0, \\ 0 & \text{for } j = 1, \\ \langle Sq^2 l \rangle & \text{for } j = 2. \end{cases}$$

The transgression for $j = 0$ is an isomorphism, since the class of the fiber in $H_n(T\xi;\mathbb{Z})$ is mapped to the generator of $H_{n-1}(K(\mathbb{Z}_2, n-1);\mathbb{Z})$ by construction. The transgression for $j = 1$ maps to 0, so $E^2_{0,n+1} \cong H_{n+1}(T\xi;\mathbb{Z})$ will not change. Due to vanishing of $H_n(K(\mathbb{Z}_2, n-1);\mathbb{Z})$ we get in the same way as before that the transgression for $j = 2$ is the evaluation of an element in $Hom(H_{n+1}(K(\mathbb{Z}_2, n-1);\mathbb{Z}), \mathbb{Z}_2) \cong H^{n+1}(K(\mathbb{Z}_2, n-1);\mathbb{Z}_2) = \langle Sq^2 l \rangle$, where $l \in H^{n-1}(K(\mathbb{Z}_2, n-1), \mathbb{Z}_2)$ is the fundamental class. Hence $E^{\infty}_{0,n+1}$ is 0 or $\mathbb{Z}_2$ and vanishes exactly when the evaluation of $w_2(\xi)$ on the modulo 2 reductions of the elements of $H_{n+2}(T\xi;\mathbb{Z})$ is nonzero.

To finish the proof, we need to show that $E^{\infty}_{n+1,0} \cong H_{n+1}(T\xi;\mathbb{Z})$ is identified by the map C with the group $\ker w_1(\xi) \leq H_1(B;\mathbb{Z}_2)$. The twisted Thom isomorphism identifies $H_{n+1}(T\xi;\mathbb{Z})$ with $H_1(B;\tilde{\mathbb{Z}})$, where $\tilde{\mathbb{Z}}$ is the system of local coefficients corresponding to the representation $w_1(\xi) : \pi_1(B) \to \mathbb{Z}_2 \cong Aut(\mathbb{Z})$. The mapping C composed with the inverse of the twisted Thom isomorphism is the reduction map $H_1(B;\tilde{\mathbb{Z}}) \to H_1(B;\tilde{\mathbb{Z}}_2) = H_1(B;\mathbb{Z}_2)$ (the latter two groups are isomorphic since $Aut(\mathbb{Z}_2)$ is trivial). By the definition of homology with twisted coefficients, any twisted homology class in $H_1(B;\tilde{\mathbb{Z}})$ can be represented by a mapping of a circle into B, but not all such mappings represent a twisted homology class. They do if and only if the pullback of the local system to these circles is trivial, that is, they represent $\mathbb{Z}_2$ homology classes on which $w_1(\xi)$ vanishes. All 1-dimensional $\mathbb{Z}_2$ homology classes are reductions of integral homology classes, so all elements in $\ker w_1(\xi)$ are images under C. Finally, we claim that the reduction $H_1(B;\tilde{\mathbb{Z}}) \to H_1(B;\mathbb{Z}_2)$ is also injective. Indeed, let $c \in C_1(B;\tilde{\mathbb{Z}})$ be a cycle which lies in the kernel of the reduction: $c \bmod 2 = \partial s$ for some $s \in C_2(B;\mathbb{Z}_2)$. Choose an arbitrary lift $\tilde{s} \in C_2(B;\tilde{\mathbb{Z}})$ of s. Then $c = \partial\tilde{s}+2\delta$ for a chain $\delta \in C_1(B;\tilde{\mathbb{Z}})$, let $\delta = \sum_{\sigma\in\Sigma} \sigma$ for some 1-simplices σ with unit coefficients. Fix a homotopy class $\alpha \in \pi_1(B)$ for which $\langle w_1(\xi), \alpha\rangle \neq 0$ and for all vertices v let γ_v be a circle through v in the homotopy class α. With twisted coefficients, the curves γ_v are not cycles, we have $\partial\gamma_v = 2v$. For any 1-simplex σ with boundary $\partial\sigma = u - v$ consider the 2-chain $t \in C_2(B;\tilde{\mathbb{Z}})$ obtained by translating σ along γ_u. Then $\partial t = 2\sigma - \gamma_u + \tilde{\gamma}_v$ for some loop $\tilde{\gamma}_v$ which is homotopic to γ_u and hence γ_v as well. Therefore 2σ is homologous to $\gamma_u - \gamma_v$, and the sum $2\delta = \sum_{\sigma\in\Sigma} 2\sigma$ is homologous to

$$\sum_{\substack{\sigma\in\Sigma \\ \partial\sigma=u-v}} \gamma_u - \gamma_v = \sum_u \gamma_u \left(\sum_{\substack{\sigma\in\Sigma \\ \partial\sigma=u-v}} 1 - \sum_{\substack{\sigma\in\Sigma \\ \partial\sigma=v-u}} 1 \right),$$

where the coefficient of γ_u is the coefficient of u in $\partial\delta$ and hence it is 0. Therefore 2δ is null-homologous, and so is $c = \partial\tilde{s}+2\delta$, finishing the proof. □

In our case, we will use the following way of distinguishing the two possible conclusions of Lemma 13:

Lemma 14. *If the vector bundle ξ is associated to the universal G-bundle via a representation $\lambda : G \to O(n)$, $n > 1$, and $\lambda_*(\pi_1(G)) = \pi_1(O(n))$ (that is,* *the image of the fundamental group of G under λ contains a non-contractible loop in $O(n)$), then the mapping C from Lemma 13 is an isomorphism.*

Proof. We will check the criterion of Lemma 13 by constructing an element $[s] \in \pi_2(BG)$ such that $w_2(\xi)$ will not vanish on the modulo 2 reduction of $s_*[\mathbb{S}^2] \in H_2(BG;\mathbb{Z})$. G-bundles over $\mathbb{S}^2$ correspond in a one-to-one fashion to homotopy classes of their gluing maps, which can be identified with the elements of $\pi_1(G)$. For any $[s] \in \pi_2(BG)$, $s : \mathbb{S}^2 \to BG$, the pullback of the universal G-bundle to $\mathbb{S}^2$ by s has the gluing map $\partial[s] \in \pi_1(G)$ with ∂ being an isomorphism taken from the homotopy long exact sequence of the universal G-bundle. Since ξ is associated to the universal bundle via λ, the gluing map for the pullback of ξ will be the image of the gluing map for the universal bundle under λ and hence the degree of $s^*\xi$ (the parity of $e(\zeta)$ for a vector bundle ζ of rank 2 that is stably isomorphic to $s^*\xi$) can be regarded as $\lambda_*(\partial[s]) \in \pi_1(O(n))$. But as $[s]$ takes all values from $\pi_2(BG)$, $\partial[s]$ takes all values from $\pi_1(G)$, so we will obtain a pulled-back bundle of odd degree if and only if the whole image $\lambda_*(\pi_1(G))$ contains the generator of $\pi_1(O(n)) = \pi_1(SO(n))$.

Given such an s, the total space of $s^*\xi$ is an orientable (open) manifold since $\mathbb{S}^2$ is orientable and $s^*\xi$ is orientable as a vector bundle over a simply connected manifold. Hence the mapping $T(s^*\xi) \to T\xi$ induced by s represents an element of the integral homology group $H_{n+2}(T\xi;\mathbb{Z})$, and the class $U_2 w_2(\xi)$ evaluates on it to $\langle w_2(\xi), [s]\rangle \neq 0$, finishing the proof. □

As before, we will denote by τ the set of all multisingularities formed from regular and $\Sigma^{1,0}$ points.

Theorem 15. *a) $Cob_\tau(2k+2,k)$ is an index 2 subgroup of $\mathfrak{N}_{2k+2}$ defined by the vanishing of $\bar{w}_{k+1}^2 + \bar{w}_{k+2}\bar{w}_k$;*[2]

b$_2$) $Cob_\tau^{SO}(6,2) \cong 0$;

[2]Recall that (2.3) shows that this characteristic number is indeed not identically 0

b_*) $Cob_\tau^{SO}(2k+2,k)$ is an index 2 subgroup of Ω_{2k+2} defined by the vanishing of $\bar{w}_{k+1}^2+\bar{w}_{k+2}\bar{w}_k$ for $k\geq 4$ even;

c) $Cob_\tau^{SO}(2k+2,k)$ is a proper subgroup of Ω_{2k+2} defined by the vanishing of $\bar{p}_{(k+1)/2}$ for k odd.

Proof. Denote the set of all multisingularities formed from regular, $\Sigma^{1,0}$ and $\Sigma^{1,1,0}$ points by σ. Consider the homotopy long exact sequence of the fibration $X_\tau\to X_\sigma\to \Gamma T\tilde{\xi}_{\Sigma^{1,1,0}}$:

$$\cdots\to\pi_{3k+3}(X_\sigma)\overset{pr_1}{\to}\pi_{3k+3}(\Gamma T\tilde{\xi})\to\pi_{3k+2}(X_\tau)\to$$
$$\to\pi_{3k+2}(X_\sigma)\overset{pr_0}{\to}\pi_{3k+2}(\Gamma T\tilde{\xi})\to\ldots$$

and that of the fibration $X_\tau^{SO}\to X_\sigma^{SO}\to\Gamma T\tilde{\xi}^{SO}_{\Sigma^{1,1,0}}$:

$$\cdots\to\pi_{3k+3}(X_\sigma^{SO})\overset{pr_1}{\to}\pi_{3k+3}(\Gamma T\tilde{\xi}^{SO})\to\pi_{3k+2}(X_\tau^{SO})\to$$
$$\to\pi_{3k+2}(X_\sigma^{SO})\overset{pr_0}{\to}\pi_{3k+2}(\Gamma T\tilde{\xi}^{SO})\to\ldots$$

The singularity with the lowest codimension after $\Sigma^{1,1,0}$ is $III_{2,2}$, whose $\mathcal{A}$-orbit is dense in the Thom-Boardman class $\Sigma^{2,0}$, and its codimension in the target is $3k+4$. This means that while $\pi_{3k+2}(X_\sigma)\cong\mathfrak{N}_{2k+2}$ and $\pi_{3k+2}(X_\sigma^{SO})\cong\Omega_{2k+2}$, there is no similar a priori identification of $\pi_{3k+3}(X_\sigma)$ or $\pi_{3k+3}(X_\sigma^{SO})$, and we need to investigate pr_1 more carefully.

The representation $\tilde{\lambda}$, which defines $\tilde{\xi}$, satisfies the assumption of Lemma 14. Indeed, it maps a non-contractible loop $\gamma:\mathbb{S}^1\to SO(k)$ (here $SO(k)$ is considered as the identity component of $G_{\Sigma^{1,1,0}}$) to the non-contractible loop $t\mapsto diag(1,1,\gamma(t),\gamma(t),\gamma(t))\in SO(3k+2)$. The same is true for $\tilde{\xi}+\varepsilon^N$ as well for any N. Hence the conclusion of Lemma 14 also holds, and the elements of $\pi_{3k+3}(\Gamma T\tilde{\xi})\cong\pi_{3k+3+N}(T(\tilde{\xi}+\varepsilon^N))$ for a sufficiently large N are in one-to-one correspondence with the elements in the kernel of the class $w_1(\tilde{\xi})$. The homotopy class of $h:\mathbb{S}^{3k+3+N}\to T(\tilde{\xi}+\varepsilon^N)$ is thus identified with the class of the intersection of the image of h with the base $BG_{\Sigma^{1,1,0}}$. The homotopy group $\pi_1(BG_{\Sigma^{1,1,0}}) \cong \pi_0(G_{\Sigma^{1,1,0}})$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ in the unoriented case and $\mathbb{Z}_2$ in the oriented case, as can easily be seen from the representations (2.2): the virtual normal bundle is given by the identical representation of $O(k)$, so in the oriented case the symmetry group is $\mathbb{Z}_2 \times SO(k)$. The connected components of $G_{\Sigma^{1,1,0}}$ are in all cases separated by the locally constant functions ϵ and $\det A$. Hence $H_1(BG_{\Sigma^{1,1,0}}; \mathbb{Z}_2) \cong \pi_0(G_{\Sigma^{1,1,0}})$ via the Hurewicz map, and we can choose generators of the cohomology group $H^1(BG_{\Sigma^{1,1,0}})$ that evaluate nontrivially on loops corresponding to a nontrivial kernel bundle and to a nonorientable virtual normal bundle, respectively. By slight abuse of notation we will refer to these respective generators as ε and $\det A$.

This means that if we consider the classifying map $\kappa : \mathbb{S}^{3k+3} \to X_\sigma$ of the σ-map $f : M \to \mathbb{S}^{3k+3}$, then the image $pr_1[\kappa]$, which classifies the $\tilde{\xi}_{\Sigma^{1,1,0}}$-immersion of the cusp points of f, can be identified by the values of ε and, in the non-coorientable case, $\det A$ taken on the image of the 1-dimensional cusp manifold. We claim that these values are invariant not only under σ-cobordisms, but also under arbitrary cobordisms. Since generic mappings from $(2k+4)$-manifolds into $\mathbb{R}^{3k+4}$ only have isolated $III_{2,2}$-points apart from multisingularities from σ, we only need to show that ε and $\det A$ vanish on the boundary of a $III_{2,2}$-point. In the case of $\det A$, a neighbourhood of such a point is a ball, so the virtual normal bundle on the boundary is (stably) trivial, and, in particular, is oriented over the set of cusp points, so $\det A$ vanishes. For ε, consider the local form of a $III_{2,2}$ point of a map of codimension k [29]:

$$(x, y, u, v, w, z, \vec{s}, \vec{t}) \mapsto (xy, x^2 + ux + vy, y^2 + wx + zy, x\vec{s} + y\vec{t}, u, v, w, z, \vec{s}, \vec{t}).$$

On the boundary, the set of cusp points can be parametrized as

$$\alpha \mapsto (\sin^2\alpha\cos\alpha, \sin\alpha\cos^2\alpha, -3\sin^2\alpha\cos\alpha, -\sin^3\alpha,$$
$$-\cos^3\alpha, -3\sin\alpha\cos^2\alpha, \vec{0}, \vec{0})$$

in the source, and the kernel bundle is traced by the line in the direction

$$(\sin\alpha, -\cos\alpha, 0, 0, 0, 0, \vec{0}, \vec{0})$$

over the point at parameter value α. Since the kernel bundle is orientable, ε also vanishes as claimed.

Therefore in the exact sequence above the map pr_1 can be considered to go through $\pi_{3k+3}(X_\infty) \cong \mathfrak{N}_{2k+3}$ or $\pi_{3k+3}(X_\infty^{SO}) \cong \Omega_{2k+3}$ and is hence the evaluation of a characteristic number of the source manifold. $\det A$ is the first Stiefel-Whitney class of the virtual normal bundle, so the characteristic number corresponding to it is $w_1 \cdot Tp_{\Sigma^{1,1,0}} = w_{k+1}^2 w_1 + w_{k+2} w_k w_1$. The class corresponding to ε can be calculated analogously to [21] to be $w_{k+2}w_{k+1} + w_{k+3}w_k$.

In the unoriented case, when both ε and $\det A$ have to be considered, we will first simplify the calculation. The characteristic polynomial $(Sq^1 + w_1\cdot)(w_{k+1}^2 + w_{k+2}w_k)$ always evaluates to 0 on closed manifolds [12], and this is the sum $\varepsilon + \det A$ for k even and it is $\det A$ for k odd. Consequently the range of the entire mapping $pr_1 = (\varepsilon, \det A)$ is the same as the range of $\varepsilon = w_{k+2}w_{k+1} + w_{k+3}w_k$. As we have seen before in the proof of Theorem 9, this latter normal characteristic number evaluates to 1 on the nontrivial cobordism class in Ω_5, and multiplying a manifold by $\mathbb{R}P^2$ does not change the value of ε (for the adjusted value of k). Hence the rank of pr_1 is 1 for all $k \geq 1$. The class $w_1(\tilde{\xi})$ is not zero as $\det \tilde{\lambda}(\varepsilon, A) = \varepsilon^{k+1} \det A$ does not have constant sign, so

$$\dim_{\mathbb{Z}_2} \pi_{3k+3}(\Gamma T\tilde{\xi}) = \dim \ker w_1(\tilde{\xi}) = \dim H_1(BG_{\Sigma^{1,1,0}}) - 1 = 1 = \operatorname{rank} pr_1,$$

and $\operatorname{coker} pr_1$ vanishes.

In the oriented case, with k odd, $\tilde{\xi}$ is orientable, and the rank of pr_1 is the same as the rank of ε. As in the previous case, we have that ε evaluates nontrivially on $Y \times (\mathbb{R}P^2)^{k-1}$. It is well-known that $(\mathbb{R}P^2)^2$ is cobordant to $\mathbb{C}P^2$, so the Stiefel-Whitney characteristic numbers of $Y \times (\mathbb{R}P^2)^{k-1}$, in particular ε, are the same as the corresponding characteristic numbers of $Y \times (\mathbb{C}P^2)^{k-1/2}$, and this latter manifold is orientable. So in this case, the rank of pr_1 is also 1 for all odd $k \geq 1$, and $\operatorname{coker} pr_1$ is trivial.

In the oriented case, with k even, $\tilde{\xi}$ is not orientable, so $\pi_{3k+3}(\Gamma T\tilde{\xi}) \cong \ker w_1(\tilde{\xi}) = 0$, and $\operatorname{coker} pr_1$ is again trivial.

Since $\operatorname{coker} pr_1$ vanishes in all cases, the exact sequence we started from splits to give

$$Cob_\tau(2k+2,k) \cong \pi_{3k+2}(X_\tau) \cong \ker pr_0 : \mathfrak{N}_{2k+2} \to \mathbb{Z}_2,$$
$$Cob_\tau^{SO}(2k+2,k) \cong \pi_{3k+2}(X_\tau^{SO}) \cong \ker pr_0 : \Omega_{2k+2} \to \mathbb{Z}_2 \text{ for } k \text{ even, and}$$
$$Cob_\tau^{SO}(2k+2,k) \cong \pi_{3k+2}(X_\tau^{SO}) \cong \ker pr_0 : \Omega_{2k+2} \to \mathbb{Z} \text{ for } k \text{ odd.}$$

The mapping pr_0 has already been calculated in Theorem 9, and substituting the results for $\ker pr_0$ yields the statement of the theorem. □

To use this approach further, one would need higher-dimensional analogues of Lemma 13 as well as a way of obtaining the possible values of the corresponding τ-cobordism invariants.

Remark: An arbitrary generic mapping of the manifold $Z = Y \times (\mathbb{C}P^2)^{\frac{k-1}{2}}$ to $\mathbb{R}^{3k+3}$ from the proof of case c) has a set of cusps that cannot be eliminated by a cobordism, as evidenced by the nonvanishing of the corresponding characteristic number. At the same time, since Y can be chosen to be simply connected, Z can be assumed to be simply connected as well, so the homology class represented by the cusp points in the source lies in $H_1(Z) = 0$. Hence the Thom polynomial of the cusp vanishes both in the source and the target (which is simply connected as well), yet the cusp stratum will not be removable even by an arbitrary cobordism. Therefore the elimination problem of cusps has in this case an "invisible" obstruction (in the sense of [44]) arising from the non-trivial Postnikov invariants of $T\tilde{\xi}_\eta$. In the next section we will attempt to handle these "invisible" obstructions.

2.6 Homotopy groups of $\Gamma T\tilde{\xi}$

The groups $\pi_m(\Gamma T\tilde{\xi})$ play a crucial role in the method we employed above, namely they contain a necessary and sufficient obstruction to the elimination of the top monosingularity η from a τ-map by τ-cobordism. In this section, we will try to find a geometric interpretation of these groups with the goal of making the obstruction more transparent.

Since $T\tilde{\xi}$ is $(\operatorname{rank}\tilde{\xi}-1)$-connected, Freudenthal's theorem shows us that for $m \le 2\operatorname{rank}\tilde{\xi}-2$ the following holds:

$$\pi_m(\Gamma T\tilde{\xi}) \cong \pi_{N+m}(S^N T\tilde{\xi}) \cong \pi_m(T\tilde{\xi}).$$

If $\tilde{\xi}$ was the tautological bundle γ_N over $BO(N)$ (or γ_N^{SO} over $BSO(N)$) for a large enough N, then these homotopy groups would be just the unoriented (respectively, oriented) cobordism groups of manifolds. But $\tilde{\xi}$ is associated to the universal $G = G_\eta$-bundle by the representation $\tilde{\lambda}$, which is not the identical representation of $O(N)$ or $SO(N)$ in general. In the case of Morin singularities, however, we can approximate the space $T\tilde{\xi}$ by $T\gamma_k$ or $T\gamma_k^{(SO)}$ modulo some primes.

Given a natural number r, denote by $\mathcal{C}$ the class of finite abelian groups of odd order divisible only by primes that divide $r+1$ as well, and let $\mathcal{C}^+$ be the class of finite abelian groups of order divisible only by primes that divide $2(r+1)$. Isomorphism modulo groups in $\mathcal{C}$ and $\mathcal{C}^+$ will be denoted by $\cong_{\mathcal{C}}$ and $\cong_{\mathcal{C}^+}$ respectively. With this notation, we have the following identifications:

Theorem 16. *Let $\tilde{\xi}$ be the universal target bundle of the Morin singularity $\Sigma^{1_r,0}$ of codimension $k > 0$ maps in either the unoriented or the oriented case (recall that $\operatorname{rank}\tilde{\xi} = rk + r + k$). For a given mapping $\kappa : \mathbb{S}^{n+k} \to T\tilde{\xi}$, set $M = M(\kappa) = \kappa^{-1}(0_{\tilde{\xi}})$ the transverse preimage of the zero section of $\tilde{\xi}$ (with the induced orientation when $\tilde{\xi}$ is orientable). Then for all m such that $0 \le m < k$ the following statements hold.*

- *If $\tilde{\xi}$ is not orientable, then*

$$\pi_{m+rk+k+r}(\Gamma T\tilde{\xi}) \in \mathcal{C}^+.$$

- *If $\tilde{\xi}$ is orientable, then*

$$\pi_{m+rk+k+r}(\Gamma T\tilde{\xi}) \cong_{\mathcal{C}^+} \Omega_m.$$

- *If r is even and $\tilde{\xi}$ is not orientable, then*

$$\pi_{m+rk+k+r}(\Gamma T\tilde{\xi}) \cong_{\mathcal{C}} \pi_m(T\gamma_{rk+k+r} \wedge (\mathbb{R}P^\infty \cup \{point\})) \cong$$
$$\cong \mathfrak{N}_m(\mathbb{R}P^\infty) \cong \bigoplus_{j=0}^{m} \mathfrak{N}_j.$$

- *If r is even and $\tilde{\xi}$ is orientable, then*

$$\pi_{m+rk+k+r}(\Gamma T\tilde{\xi}) \cong_{\mathcal{C}} \pi_m(T\gamma^{SO}_{rk+k+r} \wedge (\mathbb{R}P^\infty \cup \{point\})) \cong \Omega_m(\mathbb{R}P^\infty).$$

All these modulo $\mathcal{C}$ or $\mathcal{C}^+$ isomorphisms are given by associating to a homotopy class $[\kappa] \in \pi_{n+k}(T\tilde{\xi})$ the cobordism class of M decorated with the kernel bundle of the mapping f classified by κ.

Proof. The symmetry group of Σ^{1_r} is $G \cong \mathbb{Z}_2 \times O(k)$ (the coorientation-preserving part is $G^{SO} \cong O(k)$ for r odd and $G^{SO} \cong \mathbb{Z}_2 \times SO(k)$ for r even), and the target representation $\tilde{\lambda}$ has the form [30]

$$\tilde{\lambda}(\varepsilon, A) = diag(1, \varepsilon, \dots, \varepsilon^{r-1}, A, \varepsilon A, \dots, \varepsilon^r A), \tag{2.5}$$

where $\varepsilon \in \mathbb{Z}_2$ and $A \in O(k)$. We will denote by l the canonical line bundle over $B\mathbb{Z}_2$, and will not indicate the obvious pullbacks when talking about bundles over $BG = \mathbb{R}P^\infty \times BO(k)$. Let $i : BG \to BO(rk + k + r)$ be the mapping inducing $\tilde{\xi}$ from the universal vector bundle, and denote by

$I : T\tilde{\xi} \to T\gamma_{rk+k+r}$ the corresponding bundle map. For some combinations of r and k the bundle $\tilde{\xi}$ is oriented:

$$\det \tilde{\lambda}(\varepsilon, A) = (\det A)^{r+1} \varepsilon^{\frac{r(r-1)}{2} + k\frac{r(r+1)}{2}} = 1 \Leftrightarrow r \equiv 1 \bmod 4 \text{ and } k \text{ is even}$$

in the unoriented case,

$$\det \tilde{\lambda}(A) = (\det A)^{r+1} (\det A)^{\frac{r(r-1)}{2} + k\frac{r(r+1)}{2}} = 1 \Leftrightarrow r \equiv 1 \bmod 4 \text{ and } k \text{ is even}$$

in the oriented case with r odd, and

$$\det \tilde{\lambda}(\varepsilon, A) = \varepsilon^{\frac{r(r-1)}{2} + k\frac{r(r+1)}{2}} = 1 \Leftrightarrow k \text{ is odd or } r \equiv 0 \bmod 4$$

in the oriented case with r even. In these cases, we can pull back $\tilde{\xi}$ from γ^{SO}_{rk+k+r}; denote by $i^{or} : BG \to BSO(rk+k+r)$ and $I^{or} : T\tilde{\xi} \to T\gamma^{SO}_{rk+k+r}$ the resulting maps analogous to i and I. Note that given a $\tilde{\xi}$-embedding $f : M^n \to \mathbb{R}^{n+k}$ with the classifying map $\kappa : \mathbb{S}^{n+k} \to T\tilde{\xi}$, the composition $I \circ \kappa$ classifies the same embedding f without the extra normal structure, so its homotopy class is determined by the cobordism class of the source manifold M under the additional assumption that $n + k < 2(rk + k + r)$. The analogous statement is true for oriented $\tilde{\xi}$ and the oriented cobordism class of M.

We will calculate the induced map in cohomology, with coefficient rings $\mathbb{Z}_2$, $\mathbb{Z}_q$ with odd prime power q and $\mathbb{Q}$. In each case, $H^*(BO(rk + k + r))$ is freely generated by the appropriate characteristic classes, and i^* maps the characteristic class α to its evaluation on $\tilde{\xi}$, $i^*\alpha = \alpha(\tilde{\xi})$. Therefore, we compute the characteristic classes of the component bundles of $\tilde{\xi}$ corresponding to the splitting (2.5):

$$\tilde{\xi} = \left\lfloor \frac{r}{2} \right\rfloor l \bigoplus \left\lceil \frac{r+1}{2} \right\rceil \gamma_k^{(SO)} \bigoplus \left\lfloor \frac{r+1}{2} \right\rfloor l \otimes \gamma_k^{(SO)}.$$

Denote by c the generator of $H^1(\mathbb{R}P^\infty)$, let v_i be the generator $w_i(\gamma_k) \in H^i(BO(k))$ and $w_i(\gamma_k^{SO}) \in H^i(BSO(k))$, and set q_i to be the generator $p_i(\gamma_k^{SO}) \in H^{4i}(BSO(k))$ as well as the corresponding Pontryagin class in $H^{4i}(BO(k))$. Again, the pullbacks to BG will be assumed implicitely.

$$w(l) = 1 + c,$$
$$w_j(\gamma_k) = \begin{cases} v_j, & \text{for } j \leq k, \\ 0, & \text{otherwise,} \end{cases}$$
$$w_j(\gamma_k^{SO}) = \begin{cases} v_j, & \text{for } 1 \neq j \leq k, \\ 0, & \text{otherwise.} \end{cases}$$

The Stiefel-Whitney characteristic classes of the tensor products $l \otimes \gamma_k$ are easily calculated using the splitting lemma:

$$w_j(l \otimes \gamma_k) = \sum_{t=0}^{j} \binom{k-t}{j-t} v_t c^{j-t},$$
$$w_j(l \otimes \gamma_k^{SO}) = \binom{k}{j} c^j + \sum_{t=2}^{j} \binom{k-t}{j-t} v_t c^{j-t}.$$

Next we compute the Pontryagin characteristic classes. We will use only odd and rational coefficients, so we can omit elements of order 2 to obtain

$$p(l \oplus l) = 1,$$
$$p_j(\gamma_k \oplus \det \gamma_k) = p_j(\gamma_k^{SO}) = \begin{cases} q_j, & \text{for } j \leq k/2, \\ 0, & \text{otherwise,} \end{cases}$$

and the last formula also holds for $\gamma_k \otimes \det \gamma_k$ if k is odd and for $\det \gamma_k \oplus (\gamma_k \otimes \det \gamma_k)$ as well as for $l \otimes \gamma_k^{SO}$ if k is even.

$\mathbb{Z}_2$ *coefficients.* All vector bundles are oriented in this cohomology theory, so I^* is related to i^* by the Thom isomorphism. $H^*(BO(rk+k+r))$ is freely generated by the Stiefel-Whitney classes $w_1, \dots, w_{rk+k+r}$, and we have

$$w(\tilde{\xi}) = w(l)^{\lfloor r/2 \rfloor} w(\gamma_k)^{\lfloor r/2 \rfloor + 1} w(l \otimes \gamma_k)^{\lceil r/2 \rceil}.$$

Notice that when r is even, $w_j(\tilde{\xi})$ with $j \le k$ is the sum of $(r+1)v_j = v_j$ and some products containing only v_t with $t < j$, so i^* is injective and its image is complementary to the span $\langle c^j \rangle_{j \ge 0}$ in the range of dimensions up to k. In the oriented setting, when r is even, the symmetry group is $\mathbb{Z}_2 \times SO(k)$ and the same statement remains true.

$\mathbb{Q}$ *coefficients.* Now $\tilde{\xi}$ is not necessarily orientable, so we need to first investigate it oriented cover $\hat{\xi}$, which corresponds to the class

$$w_1(\tilde{\xi}) = (r+1)v_1 + \left(\frac{r(r-1)}{2} + k\frac{r(r+1)}{2} \right) c.$$

As mentioned above, this class vanishes for $r \equiv 1 \bmod 4$ and k even in the unoriented case, while in the oriented case it vanishes when $r \equiv 0 \bmod 4$, when r is even and k is odd and when $r \equiv 1 \bmod 4$ and k is even. In all cases, the Thom class of $\hat{\xi}$ is the pullback of the Thom class of γ^{SO}_{rk+k+r} by the inducing map $\hat{i}$, so it is enough to check the mapping between the bases. By the computations above, the class $\hat{i}^* p_j$ is the sum of $(r+1)q_j$ and products of q_t with all $t < j$ for $j < k/2$; the Euler class lies in $H^k(BSO(k))$ and $H^{rk+k+r}(BSO(rk+k+r))$, so $\hat{i}^*$ is an isomorphism for dimensions up to $k-1$. The covering $B\hat{\xi} \to B\tilde{\xi}$ induces isomorphism in cohomology for dimensions up to $k-1$, so $\hat{i}^*$ is bijective in dimensions up to $k-1$. The Thom isomorphism for the oriented covers shows that $\hat{I}^*$ is an isomorphism as well. When $\tilde{\xi}$ is orientable, I^* is the same as $\hat{I}^*$. When $\tilde{\xi}$ is not orientable, the deck transformation of the projection $T\hat{\xi} \to T\tilde{\xi}$ sends the Thom class U to $-U$ while keeping the cohomology classes of the base having degree less than k invariant (they are all Pontryagin classes and do not depend on the choice of orientation), so there are no invariant classes in these degrees. In this case we have $H^*(T\tilde{\xi}) \cong H^*(\mathit{point}) \cong H^*(T\gamma_{rk+k+r})$ and I^* is still an isomorphism in dimensions not exceeding $k - 1 + \operatorname{rank}\tilde{\xi}$.

$\mathbb{Z}_q$ *coefficients,* $q = p^\alpha$ *an odd prime power.* There is no odd torsion in the cohomology of $\mathbb{R}P^\infty$, $BO(k)$, $BSO(k)$, $BO(rk+k+r)$ or $BSO(rk+k+r)$. Hence the situation is very much analogous to that of the rational coefficients, except the multiplication by $r+1$ is not an isomorphism if and only if p divides $r+1$, so I^* is an isomorphism in the discussed dimension range for all q except those with p dividing $r+1$.

We see that I^* and I^{or*} are isomorphisms in dimensions not exceeding $k-1+\operatorname{rank}\tilde{\xi}$ for all coefficients $\mathbb{Z}_p$ with p an odd prime not dividing $r+1$ as well as rational coefficients, so by the modulo $\mathscr{C}$ Whitehead theorem [36] the mapping induced by I in homotopy is an isomorphism modulo the class $\mathcal{C}^+$ in dimensions up to $k-1+\operatorname{rank}\tilde{\xi}$. This proves the statement of the theorem up to $\mathcal{C}^+$-isomorphism.

In addition, for r even we can approximate the 2-primary component as well. Consider the map $\ker : BG \to \mathbb{R}P^\infty$ induced by the representation $(\varepsilon, A) \mapsto \varepsilon$. By definition, the map $\ker^*$ in $\mathbb{Z}_2$ cohomology is a bijection onto the span $\langle c^j \rangle_{j\geq 0}$, so the product maps $i \times \ker : BG \to BO(rk+k+r) \times \mathbb{R}P^\infty$ and $i^{or} \times \ker : BG^{SO} \to BSO(rk+k+r) \times \mathbb{R}P^\infty$ induce isomorphisms in $\mathbb{Z}_2$ cohomology up to dimension k. Let $pr : BO(rk+k+r) \times \mathbb{R}P^\infty \to BO(rk+k+r)$ and $pr^{SO} : BSO(rk+k+r) \times \mathbb{R}P^\infty \to BSO(rk+k+r)$ be the projection maps. The vector bundle $\tilde{\xi}$ can be considered to be pulled back from $pr^*\gamma_k$ (respectively, $pr^{SO*}\gamma_k^{SO}$), so there are bundle maps $I \wedge \ker : T\tilde{\xi} \to T(pr^*\gamma_k) \cong T\gamma_{rk+k+r} \wedge (\mathbb{R}P^\infty \cup \{point\})$ and $I^{or} \wedge \ker : T\tilde{\xi} \to T(pr^{SO*}\gamma_k^{SO}) \cong T\gamma_{rk+k+r}^{SO} \wedge (\mathbb{R}P^\infty \cup \{point\})$. By Thom isomorphism, these maps induce isomorphisms in $\mathbb{Z}_2$ cohomology up to dimension $k+\operatorname{rank}\tilde{\xi}$, while the induced mappings in cohomology with odd or rational coefficients are the same as I^* (respectively, I^{or*}) since $H^*(\mathbb{R}P^\infty) \cong H^*(point)$ in these cohomology theories. These maps can be identified with the forgetful maps sending $\tilde{\xi}$-embeddings to their underlying maps equipped solely with the kernel bundle. Also, it is known that the integral cohomology of the spaces $BO(m)$ does not have 4-torsion elements, even in the case of twisted coefficients [7], so i^* and i^{or*} are isomorphisms for all $\mathbb{Z}_{2^m}$, possibly twisted coefficient rings in the investigated dimension range. Hence I^*, identified with i^* by the Thom isomorphism (with twisted coefficients when $\tilde{\xi}$ is not orientable), is also an isomorphism in dimensions not exceeding $k + \operatorname{rank} \tilde{\xi}$ with $\mathbb{Z}_{2^m}$ coefficients. This finishes the proof of the remaining two cases of $\mathcal{C}$-isomorphisms. □

2.7 Negative codimensional application

Finally, we give an application of Theorem 6 for the cases $k = -1$ and $k = -2$. Considering mappings of 4-manifolds into 3-manifolds, the generic maps have only the following germs [55, Proposition 3.1]:

- regular point: $(x, y, z, w) \mapsto (y, z, w)$
- definite fold: $(x, y, z, w) \mapsto (x^2 + y^2, z, w)$
- indefinite fold: $(x, y, z, w) \mapsto (x^2 - y^2, z, w)$
- cusp: $(x, y, z, w) \mapsto (x^3 + zx + y^2, z, w)$
- definite swallowtail: $(x, y, z, w) \mapsto (x^4 + zx^2 + wx + y^2, z, w)$
- indefinite swallowtail: $(x, y, z, w) \mapsto (x^4 + zx^2 + wx - y^2, z, w)$

Theorem 17. *If M is a closed 4-manifold and P is a closed 3-manifold, then any smooth generic mapping $f : M \to P$ is cobordant to a generic mapping without definite swallowtails. Should M and P be given an orientation, the source manifold of the cobordism can be chosen to be oriented as well.*

Proof. Let τ' consist of all combinations of regular points, folds, cusps and indefinite swallowtails, set η to be the definite swallowtail and denote by τ the set of all multisingularities composed of η and elements of τ'. Since f

is generic, f is a τ-map, and we will show that it is τ-cobordant to a τ'-map. According to Theorem 6, it is enough to show that the image of the 0-dimensional manifold formed by the definite swallowtail points of f in P with the normal structure arising from f is null-cobordant. The maximal compact subgroup G of the symmetry group of the definite swallowtail is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and has source and target representations

$$\lambda = \begin{pmatrix} \alpha & & & \\ & \beta & & \\ & & 1 & \\ & & & \alpha \end{pmatrix} \text{ and } \tilde{\lambda} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & \alpha \end{pmatrix}$$

respectively, with $\alpha, \beta \in \{-1, 1\}$. Fixing a coorientation is equivalent to restricting G to the subgroup $\ker \alpha\beta$. Since the image of $\tilde{\lambda}(\ker \alpha\beta)$ is not in $SO(3)$, the bundle $\tilde{\xi}$ is not orientable regardless of whether the virtual normal bundle is required to be oriented or not. Hence the group $\pi_3(\Gamma T\tilde{\xi}) \cong \pi_{3+N}(T(\tilde{\xi} + \varepsilon^N))$ is isomorphic to $\mathbb{Z}_2$, with the isomorphism given by the number of (transverse) intersection points with the base modulo 2 (Lemma 8). Consequently, the $\tilde{\xi}$-structured immersion cobordism class of the definite swallowtail points can be identified with their number modulo 2. It is easy to check that the definite swallowtails are exactly the boundary points of the self-intersection 1-manifold of the definite fold surface, therefore the total number of definite swallowtails is even, and thus their (structured) immersion into P is null-cobordant as claimed. $\square$

One may notice that the indefinite swallowtails share all the properties mentioned in the proof with the definite swallowtails: their symmetry groups and their representations are the same, and the self-intersection lines of the indefinite fold surface connect the indefinite swallowtails. However, the germ quoted above does not define a monosingularity in our sense, because the preimage of the singular point $(0, 0, 0)$ is not compact: it consists of the two parabolas $(t, \pm t^2, 0, 0)$. In a mapping of a closed 4-manifold into a closed

3-manifold, the indefinite swallowtail is an isolated singularity on the fiber containing it, this fiber is topologically an $S^1 \vee S^1$. The complement of the swallowtail point in its (global) fiber consists of two arcs. The arc containing the branch $(t, t^2, 0, 0)$, $t > 0$, lying in the first quadrant in the local form given above, connects it to exactly one of the other three branches of the local form $(t, \pm t^2, 0, 0)$; let us denote this arc by C. There are three different monosingularities containing an indefinite swallowtail, depending on whether C contains also the branch in the second, fourth or third quadrant. They will be denoted by III^e, III^f and III^g respectively.

Luckily, the singular fiber type of the 1-dimensional self-intersection set of the indefinite fold surface that ends in an indefinite swallowtail point determines the singular fiber type of that swallowtail point. Indeed, in the local form, the indefinite fold points correspond to the saddle-points of the mapping $(x, y) \mapsto (x^4 + zx^2 + wx - y^2)$, and the self-intersection of the indefinite fold surface corresponds to parameter values where the two minima of the function $x \mapsto x^4 + zx^2 + wx$ are equal. This happens exactly when $w = 0$ and $z = -2x^2$ (with $y = 0$ implied). Hence the singular fibers at these points are the preimages of the points on the curve $t \mapsto (-t^4, -2t^2, 0)$, and have the form $(x, \pm(x^2 - t^2), -2t^2, 0)$. So when the singular fiber through the swallowtail point is of type III^e, the type of the singular line ending there is II^4 in the notation of [55] (the singular fiber is two curves intersecting in two points); when the swallowtail is on a fiber of type III^f, the singular line is of type II^3 (the singular fiber is a curve visiting its self-intersections in order *aabb*), and when the swallowtail is on a fiber of type III^g, the singular line is of type II^5 (the singular fiber is a curve visiting its self-intersections in order *abab*).

Using the description of singular fibers [55], it is a mechanical computation to verify that the lines of fibers of types II^3, II^4 and II^5 can indeed end only at swallowtails located on fibers of type III^f, III^e and III^g, respectively, and the normal structures of the images of the swallowtail points are trivial in each case, independently of whether coorientation is required or not. Therefore we can repeat the proof of Theorem 17 word by word, obtaining the following corollary:

Theorem 18. *If M is a closed 4-manifold and P is a closed 3-manifold, then any smooth generic mapping $f : M \to P$ is τ-cobordant to a mapping without any swallowtails, where τ consists of all combinations of regular points, definite and indefinite folds, cusps and swallowtails. Should M and P be given an orientation, the source manifold of the cobordism can be chosen to be oriented as well.*

Remark: These results can be considered to be cobordism-theoretic analogues of those of Saeki [35] and Ohmoto et al. [28], which give necessary conditions for the existence of a fold map in a fixed homotopy class and give a necessary and sufficient condition for the case when the target is $\mathbb{R}^3$.

Chapter 3

Algebraic line of attack

3.1 Cohomologies of the Kazarian space

We will consider the Kazarian construction for fold maps in order to derive calculable homological constraints. In the following, all cohomologies will be taken with $\mathbb{Z}_2$ coefficients. The Kazarian spaces for immersions, for fold maps and for all maps will be denoted by K_0, $K_{1,0}$ and K_∞ respectively. Recall that up to a homotopy equivalence, $K_0 \sim BO(k)$ and $K_\infty \sim BO$. There are natural embeddings $K_0 \xrightarrow{u} K_{1,0} \xrightarrow{g} K_\infty$, we denote the composition $g \circ u$ by $\bar{u}$. It is clear from the construction that $\bar{u}$ is (homotopic to) the standard embedding $BO(k) \to BO$.

Near the fold points the mappings of codimension k have the normal form

$$(x, y_1, \ldots, y_k) \mapsto (x^2, y_1, \ldots, y_k, xy_1, \ldots, xy_k). \tag{3.1}$$

The maximal compact subgroup of the symmetry group is $G_{\Sigma^{1,0}} \cong \mathbb{Z}_2 \times O(k)$. The source and target representations of $G_{\Sigma^{1,0}}$ are

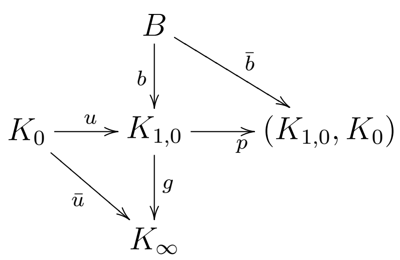
$$\lambda(\varepsilon, A) = \begin{pmatrix} \varepsilon & \\ & A \end{pmatrix} \text{ and } \tilde{\lambda}(\varepsilon, A) = \begin{pmatrix} 1 & & \\ & A & \\ & & \varepsilon A \end{pmatrix}. \tag{3.2}$$

Using the construction of the Kazarian space from the neighbourhoods of the strata $B\eta$, we can obtain $K_{1,0}$ as the total space of the (universal source) vector bundle ξ over the base $B = BG_{\Sigma^{1,0}}$ (which is homotopically equivalent to $\mathbb{R}P^\infty \times BO(k)$), glued to K_0. The universal source bundle $\xi = \xi_{\Sigma^{1,0}}$ (defined in Section 1.3) has the form

$$\xi = l \oplus \gamma$$

with l and γ being the pullbacks of the tautological bundles over $\mathbb{R}P^\infty$ and $BO(k)$ respectively. This gives us an embedding $b : B \to K_{1,0}$, and we denote its composition with the map $p : K_{1,0} \to (K_{1,0}, K_0)$ by $\bar{b} : B \to (K_{1,0}, K_0)$. By excision, for all cohomological purposes $\bar{b}$ is the embedding of B into the pair $(D\xi, S\xi)$ of the ball and sphere bundles of ξ. For the calculations, we will consider the restrictions of the elements of $H^*(K_\infty)$ to B, which are the characteristic classes of the virtual normal bundle ν over K_∞ restricted to B. It can be seen from the forms of the representations λ and $\tilde{\lambda}$ that the virtual bundle $\nu|_B$ is $l \otimes \gamma - l$.

The mappings defined above commute, implying the commutativity of the corresponding diagram of cohomology groups. The elements of these groups will be expressed in the terms of the usual generators $w_I \in H^*(BO)$ in case of K_0 and K_∞, while the elements of $H^*(B)$ and $H^*(D\xi, S\xi)$ will be expressed in terms of the generators $c \in H^1(\mathbb{R}P^\infty)$, $v_I = w_I(\gamma) \in H^*(BO(k))$ and the Thom class $U \in H^{k+1}(D\xi, S\xi)$.



$$\begin{array}{ccccc} & & \langle v_I, c| \max I \leq k\rangle & & \\ & & \uparrow b^* & \nwarrow^{Ua\mapsto aw_{k+1}(\xi)} & \\ \langle w_I| \max I \leq k\rangle & \xleftarrow{u^*} & H^*(K_{1,0}) & \xleftarrow{p^*} & U\langle v_I, c| \max I\leq k\rangle \\ & \nwarrow^{w_{>k}\mapsto 0} & \uparrow g^* & & \\ & & \langle w_I\rangle & & \end{array}$$

The Stiefel-Whitney characteristic classes of the tensor product $l \otimes \gamma$ can be easily calculated using the splitting lemma:

$$w_i(l \otimes \gamma) = \sum_{j=0}^{i} \binom{k-j}{i-j} v_j c^{i-j}.$$

Inverting the total Stiefel-Whitney class of l we have

$$w(-l) = w(l)^{-1} = 1 + c + c^2 + ...;\ w_i(-l) = c^i,$$

so the characteristic classes of the sum are

$$w_i(\nu|_B) = \sum_{s=0}^{i} w_s(l \otimes \gamma) w_{i-s}(-l) = \sum_{s=0}^{i} \sum_{j=0}^{s} \binom{k-j}{s-j} v_j c^{s-j} c^{i-s} =$$
$$= \sum_{j=0}^{i} v_j c^{i-j} \sum_{s=j}^{i} \binom{k-j}{s-j}.$$

If $i \geq k \geq j$, then the inner sum takes the form

$$\sum_{s=j}^{i} \binom{k-j}{s-j} = \binom{k-j}{0} + \cdots + \binom{k-j}{k-j} + 0 + \cdots + 0 = 2^{k-j},$$

and the sum is 0 if $i \geq k < j$. So for $i \geq k$ we have

$$w_i(\nu|_B) = \sum_{j=0}^{k} 2^{k-j} v_j c^{i-j} = v_k c^{i-k}.$$

Consider now the mapping $b^* \circ g^* : w_I \mapsto w_I(\nu|_B)$ on monomials $w_I = w_{i_1} \dots w_{i_l}, i_1 \geq \dots \geq i_l \geq 0$, with $\max I > k$. By the formula derived above, the image will be divisible by $w_{k+1}(\xi) = v_k c$. Let $a_I \in H^*(B)$ be such that $b^* g^* w_I = v_k c a_I$ (if $I = I^+ \cup I^-$ with $\max I^- \leq k$, $\min I^+ \geq k+1$ and $\sum_{i \in I^+}(i-k) = S$, then $a_I = w_{I^-}(\nu) v_k^{|I^+|-1} c^{S-1}$). The element $g^* w_I$ is sent to 0 by u^* since $u^* g^* = \bar{u}^*$ annihilates all w_i with $i > k$, so by exactness of the horizontal row of our diagram (it is a fragment of the cohomological long exact sequence of the pair $(K_{1,0}, K_0)$) there is a class $Ub_I \in H^*(D\xi, S\xi)$ such that

$$g^* w_I = p^*(Ub_I).$$

Applying b^* to both sides of this equation, we get that $v_k c a_I = b^* g^* w_I = b^* p^*(Ub_I) = \bar{b}^*(Ub_I) = v_k c b_I$. Since $H^*(B)$ has no zero divisors, this implies $a_I = b_I$ and hence

$$g^* w_I = p^*(Ua_I). \tag{3.3}$$

The mapping p^* is injective since even $\bar{p}^* = b^* \circ p^*$ is injective.

Assume that a class $\sum_{I \in \mathcal{I}} w_I \in H^*(K_\infty)$ lies in the kernel of g^*. Then whenever $\max I \leq k$, we have $u^* g^* w_I = \bar{u}^* w_I = w_I$, so all index sets $I \in \mathcal{I}$ have to satisfy $\max I > k$. Hence a_I is defined for them and we have

$$0 = g^* \sum_{I \in \mathcal{I}} w_I = p^* \sum_{I \in \mathcal{I}} Ua_I \Leftrightarrow \sum_{I \in \mathcal{I}} a_I = 0.$$

We can now calculate the avoiding ideal $\mathcal{A}_k = \ker g^* \subset H^*(K_\infty)$. This will give us cohomological obstructions to the existence of fold maps in Section 3.2.

Theorem 19. *The avoiding ideal $\mathcal{A}_k$ for the singularity $\Sigma^{1,1}$ of mappings of codimension k is generated as an $H^*(K_\infty)$ ideal by the set*

$$\{w_{k+l} w_{k+m} + w_{k+q} w_{k+r} | l, m, q, r \geq 0 \text{ and } l + m = q + r \geq 2\}.$$

Proof. Fix k and denote by

$$\mathcal{B} = (w_{k+l}w_{k+m} + w_{k+q}w_{k+r} | l, m, q, r \geq 0 \text{ and } l + m = q + r \geq 2)_{H^*(K_\infty)}$$

the ideal generated by the elements given in the statement of the theorem. It is easy to see that $\mathcal{B} \subset \ker \bar{u}^*$ and

$$a_{\{k+l,k+m\}} = v_k^2 c^{l+m} = v_k^2 c^{q+r} = a_{\{k+q,k+r\}}$$

holds for all the involved quartuples (l, m, q, r), so by equality (3.3)

$$\mathcal{B} \subseteq \mathcal{A}_k.$$

To finish the proof, it is sufficient to verify that

$$\dim \mathcal{A}_k^n \leq \dim \mathcal{B}^n \text{ for all } n. \tag{3.4}$$

In order to compute $\dim \mathcal{A}_k^n$ we show that $\ker g^* = \ker b^* g^*$. Indeed, if $b^* g^* \alpha = 0$ for some $\alpha \in H^*(BO)$, then set

$$\alpha^- = \bar{u}^* \alpha \in H^*(BO(k)) \subset H^*(BO)$$

and $\alpha^+ = \alpha - \alpha^-$. We have $b^* g^* \alpha = \nu^* \alpha = \nu^* \alpha^- + \nu^* \alpha^+$. Observe that the mapping $H^*(BO(k)) \ni \alpha^- \mapsto \nu^* \alpha^- \in H^*(B)$ is the sum of coordinate maps $w_I \mapsto w_I(l \otimes \gamma - l) = w_I(\gamma) + c \cdot (\dots)$, so $\nu^* \alpha^-$ written in the basis we use will contain every v_I for which α^- contains w_I. On the other hand, all of the monomials of $\nu^* \alpha^+$ contain c (since all $w_{k+1+a}(\nu) = cc^a v_k$ do), so if $b^* g^* \alpha = 0$, then $\alpha^- = 0$. By (3.3) we then have $g^* \alpha = g^* \alpha^+ = p^*(U \nu^* \alpha / v_k c) = 0$ and $\alpha \in \ker g^*$.

To calculate the image of $b^* g^*$, we know that $b^* g^* w_I = w_I(\nu)$, in particular, $b^* g^* w_{k+a} w_I = v_k c^a w_I(\nu)$. If we choose any I with $\max I \leq k$, then $w_I(\nu) = v_I + c \cdot (\dots)$ shows that $b^* g^*$ is onto the factor ring $H^*(BO(k)) =$

$H^*(B)/(c)$, and $w_{k+a}w_I(\nu) = v_k v_I c^a + c^{a+1}\cdot(\dots)$ shows that the image of b^*g^* in the slice $c^a H^*(BO(k)) = c^a H^*(B)/(c^{a+1})$ contains exactly the elements divisible by v_k. Thus the image of b^*g^* is spanned by v_I with $\max I \leq k$ and $c^a v_I$ with $\max I = k$, $a > 0$.

Therefore

$$\begin{aligned}
\dim \mathcal{A}_k^n &= \dim\ker g^n = \dim\ker b^n g^n = \dim H^n(K_\infty) - \dim\operatorname{im} b^n \circ g^n = \\
&= |\{a_0 \geq \cdots \geq a_m \geq 0 | n = a_0 + \cdots + a_m\}| \\
&\qquad -|\{k \geq a_0 \geq \cdots \geq a_m \geq 0 | n = a_0 + \cdots + a_m\}| \\
&\qquad\qquad -|\{k = a_0 \geq \cdots \geq a_m \geq 0 | n \geq a_0 + \cdots + a_m\}| \\
&= |\{a_0 \geq \cdots \geq a_m \geq 0 | a_0 > k \text{ and } n = a_0 + \cdots + a_m\}| \\
&\qquad -|\{a'_0 > k \geq a_1 \geq \cdots \geq a_m \geq 0 | n = a'_0 + a_1 + \cdots + a_m\}| \\
&= |\{a_0 \geq \cdots \geq a_m \geq 0 | a_1 > k \text{ and } n = a_0 + \cdots + a_m\}|.
\end{aligned}$$

$\dim \mathcal{B}^n$ can be estimated similarly, once we observe that the elements of $H^n(BO)/\mathcal{B}^n$ can be represented as sums of monomials w_I or $w_{k+a}w_I$ with $\max I \leq k$: indeed, if a monomial is of the form $w_{k+a}w_{k+b}\hat{w}$, we can change it by $(w_{k+a}w_{k+b} + w_{k+a+b}w_k)\hat{w} \in \mathcal{B}$ to get an equivalent representation $w_{k+a+b}w_k\hat{w}$ with less indices larger than k. Thus we have $\dim H^n(K_\infty)/\mathcal{B}^n \leq \dim\operatorname{im} b^n \circ g^n$ (the number of words $c^a w_k w_I$ with $\max I \leq k$ is the same as the number of words $w_{k+a}w_I$ with $\max I \leq k$), implying

$$\begin{aligned}
\dim \mathcal{B}^n &= \dim H^n(BO) - \dim H^n(BO)/\mathcal{B}^n \geq \\
&\geq \dim H^n(K_\infty) - \dim b^n \circ g^n = \dim \mathcal{A}_k^n.
\end{aligned}$$

Hence $\dim \mathcal{A}_k^n = \dim \mathcal{B}^n$, and that completes the proof. □

As an immediate consequence, we obtain the following corollary, which allows us to decide efficiently whether a particular characteristic number lies in the avoiding ideal or not:

Theorem 20. *The avoiding ideal for the singularity* $\Sigma^{1,1}$ *consists of elements*

$$\sum_{I\in\mathcal{I}} w_I \text{ such that } \sum_{I\in\mathcal{I}} c^S w_k^{|I^+|} w_{I\setminus I^+} = 0,$$

where $\mathcal{I}$ *contains only index sets* I *with* $\max I > k$*,* I^+ *denotes* $\cup\{J \subseteq I |\min J > k\}$ *and* $S = \sum_{i\in I^+}(i-k)$*.*

The proof of Theorem 19 also shows that g^* is surjective. Indeed, elements of the form a_I are a basis of $H^*(K_{1,0}, K_0)$ since we proved in the course of the proof that $\operatorname{ran} b^*g^*$ contains all the classes $v_k c v_I$. Hence by (3.3) we have $\operatorname{ran} g^* \supseteq \operatorname{ran} p^* = \ker u^*$, and the surjectivity of the composition $\bar{u}^* = u^*g^*$ implies the surjectivity of g^*. Consequently, $H^*(K_{1,0})$ is the factor ring $H^*(BO)/\mathcal{A}_k$.

3.2 Fold maps of $\mathbb{R}P^n$ into $\mathbb{R}^{n+k}$

As an application of Theorem 19, we will consider maps of projective spaces into Euclidean space. If we have a mapping $f : M^n \to \mathbb{R}^{n+k}$ with only regular points and folds (with any multiplicity), then the classifying map of its virtual normal bundle $\nu_f : M \to BO$ is homotopic to the composition of a suitable $\tilde{\nu}_f : M \to K_{1,0}$ and the canonical embedding $g : K_{1,0} \to K_\infty$. Hence the induced mapping in cohomology $\nu_f^* : H^*(BO) \to H^*(M)$ decomposes as $\nu_f^* = (\tilde{\nu}_f)^* \circ g^*$ and consequently $\ker \nu_f^* \supseteq \ker g^*$. In particular, all elements

$$\alpha(l, m, q, r) = w_l w_m + w_q w_r$$

with $l + m = q + r \geq 2k + 2$, $l, m, q, r \geq k$, must evaluate to 0 on ν_f. If $M = \mathbb{R}P^n$, then this evaluation is particularly easy to compute. Indeed, denote by a the generator of $H^1(\mathbb{R}P^n)$, then $a^{n+1} = 0$. Define the nonnegative integers s and t by the property that

$$n = 2^s + t,\ t < 2^s.$$

Then

$$\begin{aligned} w(\nu_f) = w(-\tau_{\mathbb{R}P^n}) = (1+a)^{-n-1} = (1+a^{2^{s+1}})(1+a)^{-n-1} = \\ = (1+a)^{2^{s+1}}(1+a)^{-n-1} = \sum_{j=0}^{n} \binom{2^{s+1}-n-1}{j} a^j = \\ = \sum_{j=0}^{n} \binom{2^s-t-1}{j} a^j. \end{aligned} \tag{3.5}$$

Therefore $\nu_f^* \alpha(l,m,q,r) = \left(\binom{2^s-t-1}{k+l}\binom{2^s-t-1}{k+m} + \binom{2^s-t-1}{k+q}\binom{2^s-t-1}{k+r}\right) a^{2k+l+m}$ is null if and only if $\binom{2^s-t-1}{k+l}\binom{2^s-t-1}{k+m} + \binom{2^s-t-1}{k+q}\binom{2^s-t-1}{k+r}$ is even or $2k+l+m > n$. If we produce an $\alpha(l,m,q,r)$ such that $\nu_f^* \alpha(l,m,q,r) \neq 0$, then the smallest k for which this element will be in $\mathcal{A}_k$ is the minimum of $\{l,m,q,r\}$, so we try to maximize this quantity in order to optimize our estimate on k.

If $t > \frac{2^s}{3}$, then the maximal j in the sum (3.5) for which the corresponding term is nonzero equals $2^s - t - 1$, and this is less than $\frac{n}{2}$, so the best α which does not evaluate to 0 is $\alpha(2^s-t-2, 2^s-t, 2^s-t-1, 2^s-t-1) = \left(0 + \binom{2^s-t-1}{2^s-t-1}\right) a^{2^{s+1}-2t} = a^{2^{s+1}-2t} \neq 0$, and we have to consider this element if $k \leq 2^s - t - 2$. Hence in this case, the existence of a fold map from $\mathbb{R}P^n$ to $\mathbb{R}^{n+k}$ implies $k \geq 2^s - t - 1 = 2^{s+1} - n - 1$.

If $t < \frac{2^s}{3}$, the calculation is less obvious. All $\alpha(l,m,q,r)$ with $l+m > n$ evaluate to 0 since $H^{l+m}(\mathbb{R}P^n) = 0$, so we can assume that $l + m \leq n$. Start listing the values

$$\binom{2^s-t-1}{\lfloor \frac{n}{2} \rfloor}, \binom{2^s-t-1}{\lfloor \frac{n}{2} \rfloor - 1}, \binom{2^s-t-1}{\lfloor \frac{n}{2} \rfloor - 2}, \ldots, \binom{2^s-t-1}{\lfloor \frac{n}{2} \rfloor - h}$$

and assume that the first h elements of this sequence have the same parity while the next one has the opposite parity. If the sequence starts with even elements, then it is clear that any term $w_b w_c$ which does not evaluate to zero has $\min\{b,c\} \le \lfloor \frac{n}{2} \rfloor - h$, and an optimal α is either $\alpha(j, j+i, j+1, j+i-1)$ with an i such that $h < i \le n-j$ and $\binom{2^s-t-1}{j+i}$ is odd, or $\alpha(j-1, j+1, j, j)$ with $j = \lfloor \frac{n}{2} \rfloor - h$ if there is no such i. If the sequence starts with odd elements, then in the same way any term $w_b w_c$ which does not evaluate to a^{b+c} has $\min\{b,c\} \le \lfloor \frac{n}{2} \rfloor - h$, and we get that an optimal α is either $\alpha(j, j+i, j+1, j+i-1)$ with an i such that $h < i \le n-j$ and $\binom{2^s-t-1}{j+i}$ is even, or $\alpha(j-1, j+1, j, j)$ with $j = \lfloor \frac{n}{2} \rfloor - h$ if there is no such i.

Therefore, we have to investigate the parity of $F(j) = \binom{2^s-t-1}{j}$ for values of j close to $n/2$, and for that, we need to look at the binary expansions of $2^s - t - 1$ and j: a binomial coefficient $\binom{b}{c}$ is odd precisely when the binary expansion of b has digits 1 at all the places where the binary expansion of c has digits 1 [17]. Given the binary expansion of $n = 2^s + t$ we can obtain the binary expansion of $2^{s+1} - n - 1 = \underbrace{1...1}_{s+1}{}_2 - n$ by bitwise negation, and the binary expansion of $\lfloor \frac{n}{2} \rfloor$ is obtained by shifting to the right by one position. This implies that $F(\lfloor n/2 \rfloor)$ is odd precisely when the binary expansion of n does not contain the substring ...11..., and according to this we have two cases.

n contains ...11..., first at position u: $n = 2^u(8a+3) + b$ with $u \ge 0$ maximal and $0 \le b < 2^u$. Then decreasing j starting from $\lfloor n/2 \rfloor$ we will get even values of $F(j)$ until the decrease does not affect the u^{th} digit since this is the highest 1 at a place where $2^s - t - 1$ has a 0; once that location is reached, the highest value of j for which $F(j)$ is odd has to copy the rest of the string from $2^s - t - 1$, that is, $j = 2^{u+2}a + 2^u - 1 - b > \frac{n}{2} - 2^{u+1}$. Increasing j, on the other hand, does not change the parity of $F(j)$ as long as $j < 2^{u+2}(a+1)$ due to either the u^{th} or the $u+1^{st}$ digit, which is more than 2^{u+1} steps so we don't get a better estimate on k than $k \ge j - 1 = 2^{u+2}a + 2^u - b - 2$.

n does not contain ...11.... In this case we first deal with the case of n odd; $2^s - t - 1$ is even and both $\lfloor n/2 \rfloor - 1$ and $\lfloor n/2 \rfloor + 1$ are odd, so the

sharpest possible estimate holds, $k \geq \frac{n-3}{2}$. And if $n \in 2^{p+1}\mathbb{Z} + 2^p$, $p > 0$, it is easy to see that increasing j first produces a parity change after 2^{p-1} steps and decreasing j does the same after $2^{p-1}+1$ steps, so the estimate is $k \leq \frac{n}{2} - 2^{p-1} - 1$.

We have thus proved the following result:

Theorem 21. *Let $n = 2^s + t$ be such that s and $t < 2^s$ are nonnegative integers. If there exists a fold map $\mathbb{R}P^n \to \mathbb{R}^{n+k}$, then*

$$k \geq \begin{cases} 2^{s+1} - n - 2 & \text{if } \frac{4}{3}2^s < n < 2^{s+1} \\ \lfloor \frac{n}{2} \rfloor - 1 & \text{if } 2^s < n < \frac{4}{3}2^s \text{ is odd and } \forall u \quad \lfloor \frac{n}{2^u} \rfloor \not\equiv 3 \ mod\ 4 \\ \frac{n}{2} - 2^{p-1} & \text{if } 2^s < n = 2^p m < 2^s \frac{4}{3} \text{ with } p > 0, \text{ odd } m \\ & \qquad \text{and } \forall u \quad \lfloor \frac{n}{2^u} \rfloor \not\equiv 3 \ mod\ 4 \\ 2^{u+2}a + 2^u - b - 2 & \text{if } 2^s < n = 2^u(8a+3) + b < 2^s \frac{4}{3} \\ & \qquad \text{with } 0 \leq b < 2^u \text{ and } u \text{ maximal.} \end{cases}$$

Remark: When $t > 2^s/3$, our estimate on the codimension is one less than the largest k' for which $\bar{w}_{k'}(\mathbb{R}P^n)$ does not vanish, so allowing fold singularities does not decrease the necessary codimension for a mapping to $\mathbb{R}^{n+k}$ to exist by more than 1 compared to the analogous estimate for immersions (known not to be sharp). In the case $0 < t < 2^s/3$ this is no longer the case, but our estimate stays close to the sharpest possible value $k = \lfloor n/2 \rfloor$ (for which any generic mapping can only have fold singularities anyway): the restriction on t implies $p \leq s-2 \Rightarrow k > 3n/8$ in the second and third cases as well as $a \geq 2 \Rightarrow 2^{u+2}a + 2^u - b - 2 = \frac{n}{2} - \left(2^{u-1} + \frac{3b}{2} + 2\right) \geq \frac{n}{2} - \left(2^{u+1} + \frac{1}{2}\right) > \frac{3n}{8}$ in the last case with the sole exception of $n = 19$ (alternatively, $k \geq 7n/19$ for all n).

3.3 Fold bordism groups

We will combine several previously known results to obtain the left-right bordism groups of fold maps without restrictions on multiplicities of points. Denote by $fold$ the set of all multisingularities $r\Sigma^0 + s\Sigma^{1,0}$ for mappings of a fixed codimension $k > 0$. We have the usual isomorphism

$$Bord_{fold}(n) \cong \mathfrak{N}_{n+k}(X_{fold}).$$

By [10, Theorem 1.9],

$$\mathfrak{N}_{n+k}(X_{fold}) \cong (H_*(X_{fold}; \mathbb{Z}_2) \otimes \mathfrak{N}_*)_{n+k}.$$

[45, Corollary 72] expresses X_{fold} as $\Gamma T\nu^k$ with ν^k a virtual bundle over $K_{1,0}$, so we can apply the results of Araki and Kudo [2], Dyer and Lashof [13] to calculate the ranks of $H_*(X_{fold}; \mathbb{Z}_2)$ from an additive basis of $\overline{H}_*(T\nu^k) \cong H_{*-k}(K_{1,0})$ (and the remark at the end of Section 3.1 together with Theorem 19 provides us with one).

To formulate the result in a closed form we introduce some notation. Let q_n denote the number of partitions of the natural number n into natural numbers not exceeding k, and let d_n be the number of its non-dyadic partitions (that is, partitions with no entry having the form $2^r - 1$ for any r). A pair (J, t) with $J = (j_1, \ldots, j_l)$, t and j_u positive integers for $u = 1, \ldots, l$, will be called an *admissible pair* of height $h(J) = 2^l$ and weight $w(J, t) = t + j_1 + \cdots + j_l$, if and only if either $J = \emptyset$ or $j_u \leq 2j_{u+1}$ for all $u < l$ and $j_1 - j_2 - \cdots - j_l > t$. Denote by $\mathcal{J}$ the set of all admissible pairs, and let $\mathcal{J}^{[m]}$ be the set of all m-element subsets of $\mathcal{J}$. For given nonnegative integers i, h and w, we define $\mathcal{B}(i, h, w)$ as the set of all sets $(\vec{J}, \vec{t}, \vec{\lambda}) = ((J_1, \ldots, J_i), (t_1, \ldots, t_i), (\lambda_1, \ldots, \lambda_i))$ for which

1. $\lambda_1 \geq \cdots \geq \lambda_i \geq 1$;

2. $\{(J_u, t_u) | 1 \leq u \leq i\} \in \mathcal{J}^{[i]}$;

3. $\lambda_1 h(J_1, t_1) + \cdots + \lambda_i h(J_i, t_i) = h$; and

4. $\lambda_1 w(J_1, t_1) + \cdots + \lambda_i w(J_i, t_i) = w$.

Theorem 22. *$Bord_{fold}(n) \cong \mathbb{Z}_2{}^{r_{n,k}}$, where*

$$r_{n,k} = \sum_{h=0}^{n+k} d_{n+k-h} \sum_{\substack{i\geq 0\\ s\geq 0}} \sum_{(\vec{J},\vec{t},\vec{\lambda})\in\mathcal{B}(i,s,h)} \prod_{u=1}^{i} \binom{\lambda_u - 1 + q_{t_u-k} + \sum_{j=0}^{t_u-2k-1} q_j}{\lambda_u}$$

Proof. We have

$$Bord_{fold}(n) \cong \mathfrak{N}_{n+k}(X_{fold}) \cong \bigoplus_{h=0}^{n+k} \mathfrak{N}_{n+k-h} \otimes H_h(\Gamma T\nu).$$

By Thom's theorem [48] $\mathfrak{N}_d$ is a vector space over $\mathbb{Z}_2$ with a basis enumerated by partitions of d into natural numbers not of form $2^v - 1$, so $Bord_{fold}(n)$ is a vector space over $\mathbb{Z}_2$ as well and we have

$$\dim Bord_{fold}(n) = \sum_{h=0}^{n+k} d_{n+k-h} \dim H_h(\Gamma T\nu). \tag{3.6}$$

$H_*(\Gamma T\nu)$ has an additive basis consisting of (possibly empty) products of admissible elements of form $Q^J(\Phi a)$ with Q^J an admissible Kudo-Araki operation [2], Φ the Thom isomorphism of ν and a chosen from a homogeneous additive basis of $H_*(K_{1,0};\mathbb{Z}_2) \cong H^*(K_{1,0};\mathbb{Z}_2)$. In this case, let the basis be the elements of form $g^* w_I$ with $\max(I \setminus \{\max I\}) \leq k$ as in the proof of Theorem 19, so we get $\dim H_m(K_{1,0}) = q_m + q_{m-k-1} + q_{m-k-2} + \cdots + q_0$ generators in dimension m. Choosing λ elements without order out of this set can be done in $\binom{\dim H_m(K_{1,0})+\lambda-1}{\lambda}$ different ways, and substituting this value into (3.6) gives us the claimed formula. □

3.4 Fold bordism groups with restriction on the multiplicity of self-intersections

Fix a codimension $k > 0$ and consider a finite set of multisingularities of the form $I = I_r = \{\Sigma^{1,0}\} \cup \{j\Sigma^0 | j \leq r\}$ with some $r \geq 2$. We cannot calculate the left-right bordism groups of I-maps $Bord_I(n)$ in the same way as we proceeded with $Bord_{fold}(n)$ because of the presence of global restrictions. Note that since there are global restrictions, we cannot apply Theorem 3 either. As usual, we have

$$\mathfrak{N}_{n+k}(X_I) = Bord_I(n),$$

and since $\mathfrak{N}_{n+k}(X)$ is a vector space over $\mathbb{Z}_2$, so is $Bord_I(n)$, hence its dimension describes its structure completely. In this section, homology is always taken with $\mathbb{Z}_2$ coefficients, which are suppressed for brevity.

We will show an algorithmic way of calculating the dimensions (over $\mathbb{Z}_2$) of $Bord_I(n)$. The simplest I that we consider here is $I = I_2$, that is, when an I-map may have the following multisingularities: $\Sigma^0, 2\Sigma^0, \Sigma^{1,0}$. The dimension over $\mathbb{Z}_2$ of the left-right I_2-bordism group is given by the following formula (with q_m and d_m defined as on page 76):

Theorem 23.

$$\dim Bord_{I_2}(n) = \sum_{s=0}^{n+k} \sum_{j=0}^{\left\lfloor \frac{n-k-s-1}{2} \right\rfloor} q_j q_{n-k-s-j} d_s + \sum_{s=0}^{n+k} \sum_{r=\left\lceil \frac{n-k-s}{2} \right\rceil}^{n-k-s-1} q_r d_s + \\ + \sum_{s=0}^{n+k} q_{n-s} d_s + d_{n+k}.$$

In order to give the result on $Bord_I(n)$ in general we introduce some more notation. When $I = I_r$ and $I' = I_{r-1}$, we obtained X_I from $X_{I'}$ by gluing along its boundary a disc bundle $\tilde{D}_r$; the underlying vector bundle will be denoted by $\tilde{\xi}_r$. The maximal compact subgroup of the symmetry group of the singularity $r\Sigma^0$ will be denoted by G_r; it is isomorphic to the wreath product $S_r \wr O(k)$. The representation of G_r defining $\tilde{D}_r$ is the natural representation of the wreath product associated to the identical representation of $O(k)$.

Theorem 24. *For $I = I_r$ and $I' = I_{r-1}$, the dimensions of the singular bordism groups satisfy*

$$\dim Bord_I(n) = \dim Bord_{I'}(n) + \sum_{i=0}^{n+k} d_{n+k-i} \dim H_{i-rk}(BG_r),$$

where

$$\dim H_m(BG_r) = \sum_{i\geq 1} \sum_{(\vec{J},\vec{t},\vec{\lambda})\in\mathcal{B}(i,r,m)} \prod_{u=1}^{i} \binom{q_{t_u}+\lambda_u-1}{\lambda_u}.$$

Let Γ denote the classifying space of immersions without triple points and set $X_f = X_{I_2}$. The space X_f (first constructed in [40]) was obtained from Γ by gluing a disc bundle $\tilde{D}_f$ to it along its boundary. The underlying vector bundle of the disc bundle $\tilde{D}_f$ is the universal target bundle for the fold singularity and will be denoted by $\tilde{\xi}_f$. The maximal compact subgroup of the symmetry group of the fold will be denoted by G_f; it is isomoprhic to the product $\mathbb{Z}_2 \times O(k)$. The target representation of G_f defining $\tilde{D}_f$ is

$$\tilde{\lambda}(\varepsilon, A) = \begin{pmatrix} 1 & & \\ & A & \\ & & \varepsilon A \end{pmatrix}.$$

The following two lemmas will be used to prove Theorem 23:

Lemma 25. *In the long exact sequence of the triple $(X_f, \Gamma, MO(k))$ the* *boundary map* $\partial_n : H_n(X_f, \Gamma) \to H_{n-1}(\Gamma, MO(k))$ *satisfies*

$$\dim \operatorname{im} \partial_n = \sum_{r=0}^{\lfloor \frac{n-2k-1}{2} \rfloor} q_r$$

and

$$\dim \ker \partial_n = \sum_{r=\lfloor \frac{n-2k-1}{2} \rfloor + 1}^{n-2k-1} q_r.$$

Proof. Consider the long exact sequence of the triple $(X_f, \Gamma, MO(k))$ in homology and the following identifications on the involved groups:

$$\begin{array}{ccccccc}
\cdots \longrightarrow & H_n(X_f, \Gamma) & \xrightarrow{\partial_n} & H_{n-1}(\Gamma, MO(k)) & \longrightarrow \cdots \\
 & \downarrow \cong & & \downarrow \cong & \\
\cdots \longrightarrow & \tilde{H}_n(X_f/\Gamma) & \longrightarrow & \tilde{H}_{n-1}(\Gamma/MO(k)) & \longrightarrow \cdots \\
 & \downarrow \cong & & \downarrow \cong & \\
\cdots \longrightarrow & H_{n-2k-1}(BG_f) & \xrightarrow{\hat{\partial}_{n-2k-1}} & H_{n-2k-1}(BG_2) & \longrightarrow \cdots
\end{array}$$

Here the vertical arrows between the bottom rows are the inverse Thom isomorphisms of the bundles $\tilde{\xi}_f$ and $\tilde{\xi}_2$. The local form of a fold point is (3.1),

$$(x, y_1, \ldots, y_k) \mapsto (x^2, y_1, \ldots, y_k, xy_1, \ldots, xy_k).$$

This gives a representation $\rho : G_f \to G_2$ as detailed below. In the target, considered with local coordinates in which the fold point has the form (3.1), there is a unique double point on the boundary of a small ball around the fold point. Any symmetry $\alpha \in G_f$ must leave this double point fixed and hence is a symmetry of its normal structure, that is, defines an element $\rho(\alpha) \in G_2$. In particular, whenever there is a fold manifold with its normal bundle having transition maps $g_{\alpha\beta} \in G_f$, the G_2-bundle of the regular double points in the boundary of a small neighbourhood can be taken to have transition maps

$\rho \circ g_{\alpha\beta}$. Therefore the boundary mapping $\hat{\partial}_m : H_m(BG_f) \to H_m(BG_2)$ is induced by ρ, or in other words, we obtain $\hat{\partial}_m$ by applying the functors B and H_m to the map ρ.

If in the target instead of the given basis $e, u_1, \ldots, u_k, v_1, \ldots, v_k$ we consider the basis $e, \{z_j^+, z_j^-\}_{1 \le j \le k}$ with $z_j^+ = u_j + v_j$ and $z_j^- = u_j - v_j$, then the double point on the boundary of the model fold mapping will have its branches spanned by $(z_1^+, \ldots, z_k^+)$ and $(z_1^-, \ldots, z_k^-)$ respectively. Hence the action of the fold symmetry $(x, \vec{y}) \mapsto (-x, \vec{y})$ is mapped to the matrix $\begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}$, while the symmetry $(x, \vec{y}) \mapsto (x, A\vec{y})$ is mapped to the matrix $\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$. Therefore the underlying mapping ρ can be identified with the usual embedding $\mathbb{Z}_2 \times O(k) \to \mathbb{Z}_2 \wr O(k) \cong G_2$, and [22, pp.55–56] gives a description of the induced mapping in cohomology. The cohomology ring $H^*(BG_2)$ is generated by the elements of the form $e^j \otimes x \otimes x$, where $H^j(B\mathbb{Z}_2) = \langle e^j \rangle$, $j \ge 0$, and $x \in H^*(BO(k))$, and the elements of the form $1 \otimes x \otimes y + 1 \otimes y \otimes x$ with $x, y \in H^*(BO(k))$. The relations between these elements are exactly those of the tensor product, so if $\{x_i\}_{i \ge 1}$ is a graded vector space basis consisting of homogeneous elements in $H^*(BO(k))$, then the elements

$$\{1 \otimes x_i \otimes x_j + 1 \otimes x_j \otimes x_i : 1 \le i < j\} \text{ and } \left\{e^j \otimes x_i \otimes x_i : j \ge 0, i \ge 1\right\} \tag{3.7}$$

form a homogeneous basis of $H^*(BG_2)$.

The mapping $\delta : H^*(BG_2) \to H^*(BG_f)$, which is dual to $\hat{\partial}$, acts as follows:

$$e^j \otimes x \otimes x \mapsto \sum_{r=0}^{\deg x} e^{j+r} \otimes Sq^{\deg x - r} x,$$

$$1 \otimes x \otimes y + 1 \otimes y \otimes x \mapsto 1 \otimes (x \cup y + y \cup x) = 0.$$

Thus, we can calculate the rank of the mapping $\hat{\partial}$. The kernel $\ker\delta$ contains all the classes of the form $1\otimes x\otimes y+1\otimes y\otimes x$, and we claim that no other class is mapped to 0 by δ. Up to a sum of classes of this form, every cohomology class z in $H^*(BG_2)$ is a finite sum of classes of form $e^j \otimes x \otimes x$. We can merge the classes with the same j exchanging $e^j \otimes x \otimes x + e^j \otimes y \otimes y$ with $e^j \otimes (x+y) \otimes (x+y)$; this does not change the sum for $j > 0$ and adds $1 \otimes x \otimes y + 1 \otimes y \otimes x \in \ker\delta$ to it for $j = 0$. So z can be assumed to be a finite sum of the form

$$z = \sum_{j\geq i} e^j \otimes x_j \otimes x_j,$$

where i is chosen to have the property $x_i \neq 0$; unless $z = 0$, this is possible. Applying the mapping δ, we arrive at

$$\delta(z) = \sum_{j\geq i}\sum_{r\geq 0} e^{j+r} \otimes Sq^{\deg x_j - r} x_j = e^i \otimes Sq^{\deg x_i} x_i + \cdots = e^i \otimes x_i^2 + \ldots,$$

with the omitted summands all containing e^{j+r} with $j + r > i$. If $z \in \ker\delta$, then $e^i \otimes x_i^2$ also has to vanish, and that contradicts the assumption that $x_i \neq 0$ since $H^*(BO(k))$ is an integral domain.

Therefore we have

$$\ker\delta = \langle 1 \otimes x \otimes y + 1 \otimes y \otimes x \rangle_{x,y\in H^*(BO(k))}.$$

According to the description (3.7), we have

$$\dim \operatorname{im} \hat{\partial}_m = \dim H^m(BG_2)/\ker\delta_m = \dim(\langle e^j \otimes x \otimes x\rangle)_m = \sum_{r=0}^{\lfloor m/2\rfloor} q_r,$$

therefore

$$\begin{aligned}\dim\ker\hat{\partial}_m &= \dim H_m(BG_f) - \dim\operatorname{im}\delta_m = \\ &= \dim H_m(B\mathbb{Z}_2 \times BO(k)) - \dim\operatorname{im}\hat{\partial}_m = \\ &= \sum_{r=0}^{m} q_r - \sum_{r=0}^{\lfloor m/2\rfloor} q_r = \sum_{r=\lfloor m/2\rfloor+1}^{m} q_r.\end{aligned}$$

Substituting $m = n - 2k - 1$ yields the statement of the lemma. □

Lemma 26. *The long exact sequence of the pair $(X_f, MO(k))$ splits at the boundary mapping.*

Proof. The statement of the lemma is equivalent to claiming that the embedding $\bar{i} : MO(k) \to X_{I_2}$ induces monomorphisms $\bar{i}_{*,n} : H_n(MO(k)) \to H_n(X_{I_2})$ for all $n \in \mathbb{Z}$. We recall the proof of a stronger statement due to Aguilar and Pastor [1], that even the natural map $j : MO(k) \to \Omega^\infty MO(k+\infty)$ of the classifying space of embeddings into the classifying space of all maps induces a monomorphism in homology. Here we present a more detailed proof than that of [1], but the argument is essentially the same. Our claim follows immediately as j factors through $\bar{i}$ and hence j_* factors through $\bar{i}_*$.

Consider the following diagram:

$$\begin{array}{ccc} \tilde{H}_n(MO(k)) & \xrightarrow{\bar{i}_{*,n}} & \tilde{H}_n(\Omega^N MO(k+N)) \\ {\scriptstyle S^N}\downarrow\cong & & \downarrow{\scriptstyle eval} \\ \tilde{H}_{n+N}(S^N MO(k)) & \xrightarrow{\bar{i}'_{*,n+N}} & \tilde{H}_{n+N}(MO(k+N)) \end{array}$$

Here S^N on the left-hand side of the diagram denotes the N-fold suspension isomorphism and the evaluation map *eval* on the right-hand side of the diagram sends the singular chain $Z \ni z \mapsto f(z) \in \Omega^N MO(k+N) = Map(\mathbb{S}^N, MO(k+N))$ to the chain $Z \times \mathbb{S}^N \ni (z,p) \mapsto f(z)(p) \in MO(k+N)$.

It is easy to see that this map commutes with taking the chain boundary, hence it induces a map in homology. $\bar{i}'_*$ is the map induced by the classifying map $S^N MO(k) = T(\gamma_k + \varepsilon^N) \to T\gamma_{N+k} = MO(k+N)$, and it corresponds to $\bar{i}_*$ under the natural isomorphism $[-, \Omega^N MO(k+N)] \cong [S^N(-), MO(k+N)]$.

We claim that this diagram commutes. Consider a singular chain $f : Z \to MO(k)$, representing an element of $\tilde{H}_n(MO(k))$. On the southwestern path of the diagram $\bar{i}'_{*,n+N} \circ (S^N)$, it is first sent to the chain $Z \times \mathbb{S}^N \ni (z,p) \mapsto (f(z), p)$, which is then sent further by the map $S^N MO(k) \to MO(k+N)$ identifying the suspension of the fiber of $MO(k)$ over $b \in BO(k)$ with the fiber of $MO(k+N)$ over $e(b)$ for some (fixed) embedding $e : BO(k) \to BO(k+N)$, $e^*\gamma^{k+N} \cong \gamma^k \oplus \varepsilon^N$. On the northeastern path $eval \circ \bar{i}_{*,n}$ a singular chain $f : Z \ni z \mapsto f(z)$ with $f(z)$ lying in the fiber over $b = b(z)$ is first mapped to a parametrization of the ε^N part of the fiber over $e(b)$, and then the resulting chain is transformed by *eval* by sending $(z,p) \in Z \times \mathbb{S}^N$ to the point $f(z) + p \in T(\gamma \oplus \varepsilon^N)$ (in the fiber over $e(b)$). This is the same mapping as we obtained from the southwestern path.

Finally, observe that the bottom map $\bar{i}'_{*,n+N}$ is the dual of the surjective mapping

$$\begin{aligned}\tilde{H}^{n+N}(MO(k+N)) \cong H^{n-k}(BO(k+N)) \to \\ \to H^{n-k}(BO(k)) \cong \tilde{H}^{n+N}(S^N MO(k)),\end{aligned}$$

defined by

$$H^{n-k}(BO(k+N)) \ni w_I \mapsto w_I(\gamma^k) = \begin{cases} 0 \text{ if } \max_{i \in I} i > k, \\ w_I \text{ otherwise.} \end{cases}$$

Hence the map $\bar{i}_{*,n}$ is injective: $\ker \bar{i}_{*,n} \subseteq (S^N)^{-1} \ker \bar{i}'_{*,n+N} = 0$, and consequently j_* has to be injective as well. □

Proof of Theorem 23. The application of [10, Theorem 1.9] shows that:

$$Bord_{I_2}(n) \cong \mathfrak{N}_{n+k}(X_f) \cong \bigoplus_{s=0}^{n+k} \mathfrak{N}_s \otimes H_{n+k-s}(X_f).$$

Since all elements of $\mathfrak{N}_*$ have order at most 2, the same is true for the elements of $\mathfrak{N}_*(X_f)$, so $\mathfrak{N}_*(X_f)$ is a vector space over $\mathbb{Z}_2$. The dimensions $\dim \mathfrak{N}_m = d_m$ [48] are explicitly calculable, therefore we only need to obtain the dimensions of the homology groups $H_*(X_f)$.

Since the coefficient ring $\mathbb{Z}_2$ is a field, Lemma 26 implies that

$$H_n(X_f) \cong H_n(X_f, MO(k)) \oplus H_n(MO(k)).$$

The dimension of $H_n(X_f, M)$ can be obtained from Lemma 25:

$$\begin{aligned}\dim H_n(X_f, MO(k)) &= \dim \operatorname{coker} \partial_{n+1} + \dim \ker \partial_n = \\ &= \dim H_{n-2k}(BG_2) - \dim \operatorname{im} \partial_{n+1} + \dim \ker \partial_n.\end{aligned}$$

Consequently

$$\begin{aligned}&\dim \tilde{H}_n(X_f) = \\ &\quad = \dim H_{n-2k}(BG_2) - \dim \operatorname{im} \partial_{n+1} + \dim \ker \partial_n + \dim \tilde{H}_n(MO(k)) = \\ &\quad = \sum_{j=0}^{\lfloor \frac{n-2k-1}{2} \rfloor} q_j q_{n-2k-j} + \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor - k} q_r - \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor - k} q_r + \sum_{r=\lfloor \frac{n+1}{2} \rfloor - k}^{n-2k-1} q_r + q_{n-k} = \\ &\quad = \sum_{j=0}^{\lfloor \frac{n-2k-1}{2} \rfloor} q_j q_{n-2k-j} + \sum_{r=\lceil \frac{n}{2} \rceil - k}^{n-2k-1} q_r + q_{n-k}.\end{aligned}$$

The resulting value for the dimension of the left-right bordism group is

$$\dim Bord_{I_2}(n) = \dim \mathfrak{N}_{n+k}(X_f) = \sum_{s=0}^{n+k} \dim H_{n+k-s}(X_f) \dim \mathfrak{N}_s =$$
$$= \sum_{s=0}^{n+k} \sum_{j=0}^{\left\lfloor \frac{n-k-s-1}{2} \right\rfloor} q_j q_{n-k-s-j} d_s + \sum_{s=0}^{n+k} \sum_{r=\left\lceil \frac{n-k-s}{2} \right\rceil}^{n-k-s-1} q_r d_s +$$
$$+ \sum_{s=0}^{n+k} q_{n-s} d_s + d_{n+k}$$

as claimed. □

Theorem 24 is the corollary of the following two lemmas:

Lemma 27. *For $I = I_r$ and $I' = I_{r-1}$, the long exact sequence of the pair $(X_I, X_{I'})$ in unoriented bordism splits at the boundary mapping. Consequently, $\mathfrak{N}_*(X_I) \cong \mathfrak{N}_*(X_{I'} \vee T\tilde{\xi}_r) \cong \mathfrak{N}_*(X_{I'}) \oplus \tilde{\mathfrak{N}}_*(T\tilde{\xi}_r)$.*

Proof. The boundary mapping of the long exact sequence of $(X_I, X_{I'})$ in unoriented bordism sends a singular mapping $f : M \to X_I$ with $[f] \in \mathfrak{N}_*(X_I, X_{I'})$ to $\partial f : \partial M \to X_{I'}$. To prove that this latter mapping is null-bordant in $X_{I'}$, it is enough to show that the I'-mapping classified by it is null-I'-bordant, that is, it bounds an I'-mapping. By a homotopy inside $X_I \setminus X_{I'}$ we can make f transverse to the zero section BG_r of the attached bundle, and a further homotopy makes f intersect the zero section in the direction of the fibers. Hence without loss of generality we can assume that f restricted to a tubular neighbourhood of the r-tuple point manifold is a bundle map $D\psi \to \tilde{D}_r$ for some disc bundle $D\psi \to B\psi$ of rank rk, and the classified mapping is the tubular neighbourhood of the r-tuple point manifold. Since $\partial f : \partial N \to X_{I'}$ is I'-bordant to the restriction of f to the sphere bundle $\partial\psi$, it is enough to prove the statement in the case when N is equal to this disc bundle.

We will now construct a manifold P^{rk} with spherical boundary $\partial P \cong \mathbb{S}^{rk-1}$, an I'-map $g : (N, \partial N) \to (P, \partial P)$ such that ∂g is the boundary of a regular r-tuple point, and an action of G_r under which g will be invariant. Choose a point $s \in \mathbb{S}^k$ and a small ball neighbourhood U_s thereof, and let P be the product manifold $(\mathbb{S}^k)^r$ with the product $U_s \times \cdots \times U_s$ removed. Set the source manifold N to be the disjoint union of r copies of k-balls $\mathbb{S}^k \setminus U_s$, and let the mapping g be the restriction to N of the union of coordinate mappings $\mathbb{S}^k \to (\mathbb{S}^k)^r$, $x \mapsto (s, \ldots, s, x, s, \ldots, s)$ with the m^{th} member of the disjoint union corresponding to the m^{th} coordinate. Clearly, this mapping has no r-tuple points. The natural action of S_r and $O(k)$ extends from $\partial(U_s \times \cdots \times U_s)$ to the entire P, and g is left invariant by these actions. Consequently, the same is true for the action of $G_r \cong S_r \wr O(k)$.

Given such a map g, we can construct the associated bundle $D\psi[P]$ over $B\psi$ with the fiber P; it will be the pullback of the bundle $\tilde{\xi}_r[P]$ associated to $\tilde{\xi}_r$ by the action of G_r on P. The invariance of g under the action of G_r means that we can define a mapping $\bar{g} : \tilde{\xi}_r[N] \to \tilde{\xi}_r[P]$ that restricts to g on each fiber. In particular, $\bar{g}$ and the universal immersion $\xi_r \to \tilde{\xi}_r$ agree on the boundary $\partial\tilde{\xi}_r \cong \partial\tilde{\xi}_r[P]$, hence the pullback of g to $D\psi[P]$ agrees with the map classified by f on the boundary $\partial D\psi \cong \partial D\psi[P]$ and it is an I'-map (it is an immersion without r-tuple points on each fiber). This proves that ∂f classifies a null-I'-bordant map, consequently ∂f is null-bordant in $X_{I'}$, as claimed. $\square$

Remark: The gluing map $\partial\tilde{D}_r \to X_{I'}$ is in fact stably nullhomotopic [14] [3].

Lemma 28.

$$\dim H_m(BG_r) = \sum_{i \geq 1} \sum_{(\vec{J}, \vec{t}, \vec{\lambda}) \in \mathcal{B}(i,r,m)} \prod_{u=1}^{i} \binom{q_{t_u} + \lambda_u - 1}{\lambda_u},$$

Proof. Consider the space $\Gamma BO(k)$. In the case of a connected space Z,

$H_*(\Gamma Z)$ is a polynomial ring with generators of the form

$$Q^J(x_\alpha) = Q^{j_1} \dots Q^{j_s} x_\alpha,$$

where $\{x_\alpha\}$ is a homogeneous basis of the reduced homology group $\tilde{H}_*(Z)$ (as a graded vector space over $\mathbb{Z}_2$) and $(J, \dim x_\alpha)$ is an admissible pair as previously described [14] [27]. Define a height function h by

1. $h(x) = 1$ for $x \in H_*(Z)$,
2. $h(u \cdot v) = h(u) + h(v)$,
3. $h(Q^j y) = 2h(y)$.

Then the grading of $H_*(\Gamma Z)$ by height corresponds to the filtration of ΓZ by subspaces $\Gamma_r Z$, which are stably homotopically equivalent to the bouquets $\bigvee_{1 \le i \le r} ES_i \underset{S_i}{\times} Z^i$ with S_i acting on Z^i by permuting the coordinates. Hence in the case of $Z = BO(k)$ we have

$$\begin{aligned}
\dim H_m(B(S_r \wr O(k))) &= \dim H_m(ES_r \underset{S_r}{\times} BO(k)^r) = \\
&= \dim H_m(\Gamma_r BO(k)) - \dim H_m(\Gamma_{r-1} BO(k)) = \\
&= \left| \left\{ x = \prod_{i=1}^{t} Q^{J_i} x_{\alpha_i} \,\middle|\, h(x) = r, \dim x = m, t \ge 1, J_i \in \mathcal{J} \right\} \right| = \\
&= \sum_{t \ge 1} \big| \{ [((J_1, x_{\alpha_1}), \dots, (J_t, x_{\alpha_t}))] \in \mathcal{J}^t / S_t \big| \\
&\qquad h(J_1) + \dots + h(J_t) = r \text{ and} \\
&\qquad w(J_1, \dim x_{\alpha_1}) + \dots + w(J_t, \dim x_{\alpha_t}) = m \} \big| .
\end{aligned}$$

Collecting the same pairs $(J_i, \dim x_{\alpha_i})$, this expression can be rewritten as

$$\sum_{i\geq 1} \Big| \big\{ \{ (J_1, x_{\alpha_{1,1}}, \dots, x_{\alpha_{1,\lambda_1}}), \dots, (J_i, x_{\alpha_{i,1}}, \dots, x_{\alpha_{i,\lambda_i}}), (t_1, \dots, t_i) \} \Big|$$

$$\{(J_1, t_1) \dots, (J_i, t_i)\} \in \mathcal{J}^{[i]}, \dim x_{\alpha_{u,v}} = t_u \text{ for all } u \leq i, v \leq \lambda_u,$$

$$\lambda_1 h(J_1) + \dots + \lambda_i h(J_i) = r,$$

$$\lambda_1 w(J_1, t_1) + \dots + \lambda_i w(J_i, t_i) = m \big\} \Big|.$$

For any collection $\lambda_1, \dots, \lambda_i, J_1, \dots, J_i, t_1, \dots, t_i$ that satisfies the prescribed equalities, we have to choose λ_u values $x_{\alpha_{u,1}}, \dots, x_{\alpha_{u,\lambda_u}}$, without order, from a fixed basis of $H_{t_u}(BO(k))$; this can be done in $\binom{\dim H_{t_u}(BO(k))+\lambda_u-1}{\lambda_u}$ ways. Hence the sum in question is equal to

$$\sum_{i\geq 1} \sum_{(\vec{J},\vec{t},\vec{\lambda})\in\mathcal{B}(i,r,m)} \prod_{u=1}^{i} \binom{\dim H_{t_u}(BO(k)) + \lambda_u - 1}{\lambda_u}.$$

Substituting $\dim H_m(BO(k)) = q_m$ and $BG_r = B(S_r \wr O(k))$ yields the claimed formula. $\square$

Proof of Theorem 24. By Lemma 27 we have

$$\dim Bord_I(n) = \dim \mathfrak{N}_{n+k}(X_I) = \dim \mathfrak{N}_{n+k}(X_{I'}) + \dim \tilde{\mathfrak{N}}_{n+k}(T\tilde{\xi}_r) =$$
$$= \dim Bord_{I'}(n) + \dim \mathfrak{N}_{n+k-rk}(BG_r).$$

Substituting the formula of Lemma 28 and applying [10, Theorem 1.9] finishes the proof. $\square$

3.5 Orientability

The results demonstrated above lend themselves to extension to settings with additional orientability requirements imposed on the mappings. We will consider the case of coorientable maps (between not necessarily orientable manifolds) and the case of maps between oriented manifolds. Denoting the requirement of an oriented virtual normal bundle by a superscript ^{SO} on the involved objects as before, the generalized Pontryagin-Thom construction establishes the isomorphisms

$$\begin{aligned} Bord_I^{SO}(n) &\cong \mathfrak{N}_{n+k}(X_I^{SO}) \\ Bord_I^{or}(n) &\cong \Omega_{n+k}(X_I^{SO}). \end{aligned}$$

The symmetry groups G_1^{SO} and G_f^{SO} are the subgroups of G_1 and G_f, respectively, on which the orientation of the virtual normal bundle is preserved:

$$G_1^{SO} \cong \{A \in O(k) | \det A = 1\} = SO(k)$$

and (see (3.2))

$$\begin{aligned} G_f^{SO} &\cong \{(\varepsilon, A) \in \mathbb{Z}_2 \times O(k) | (\varepsilon \det A)^{-1} \det A \det \varepsilon A = \varepsilon^{k-1} \det A = 1\} = \\ &= \begin{cases} O(k) & \text{for } k \text{ even,} \\ \mathbb{Z}_2 \times SO(k) & \text{for } k \text{ odd.} \end{cases} \end{aligned}$$

Therefore we need to substitute the dimensions

$$\begin{aligned} q_m^{SO} &= \dim H_m(BG_1^{SO}) = \\ &= \left| \left\{ (m_1, \ldots, m_l) \middle| m = \sum_{j=1}^{l} m_j, 2 \le m_l \le \cdots \le m_1 \le k \right\} \right| \end{aligned}$$

for q_m in Lemmas 25 and 28 and

$$\dim H_m(BG_f^{SO}) = \begin{cases} q_m \text{ for } k \text{ even}, \\ q_0^{SO} + \cdots + q_m^{SO} \text{ for } k \text{ odd} \end{cases}$$

for the sum $q_0 + ... + q_m$ in the proof of Lemma 25. The proofs of Lemmas 26 and 27 can be repeated without modification, so we get the following statements:

Lemma 29. *In the long exact sequence of the triple* $(X_f^{SO}, \Gamma^{SO}, MSO(k))$ *the boundary map* $\partial_n^{SO} : H_n(X_f^{SO}, \Gamma^{SO}) \to H_{n-1}(\Gamma^{SO}, MSO(k))$ *satisfies*

$$\dim \operatorname{im} \partial_n^{SO} = \sum_{r=0}^{\left\lfloor \frac{n-2k-1}{2} \right\rfloor} q_r^{SO}$$

and

$$\dim \ker \partial_n^{SO} = q_{n-2k-1} - \sum_{r=0}^{\left\lfloor \frac{n-2k-1}{2} \right\rfloor} q_r^{SO}$$

for k even, while

$$\dim \ker \partial_n^{SO} = \sum_{r=\left\lfloor \frac{n-2k-1}{2} \right\rfloor + 1}^{n-2k-1} q_r^{SO}$$

for k odd.

Lemma 30. *The long exact sequence of the pair* $(X_f^{SO}, MSO(k))$ *splits at the boundary mapping.*

Lemma 31. *For $I = I_r$ and $I' = I_{r-1}$, the long exact sequence of the pair* $(X_I^{SO}, X_{I'}^{SO})$ *in unoriented bordism splits at the boundary mapping. Consequently,* $\mathfrak{N}_*(X_I^{SO}) \cong \mathfrak{N}_*(X_{I'}^{SO} \vee T\tilde{\xi}_r^{SO}) \cong \mathfrak{N}_*(X_{I'}^{SO}) \oplus \tilde{\mathfrak{N}}_*(T\tilde{\xi}_r^{SO})$.

Lemma 32.

$$\dim H_m(BG_r^{SO}) = \sum_{i\geq 1} \sum_{(\vec{J},\vec{t},\vec{\lambda})\in\mathcal{B}(i,r,m)} \prod_{u=1}^{i} \binom{q_{t_u}^{SO}+\lambda_u-1}{\lambda_u}.$$

Theorem 33. *For k even*

$$\dim Bord_{I_2}^{SO}(n) = \sum_{s=0}^{n+k} \sum_{j=0}^{\lfloor \frac{n-k-s-1}{2} \rfloor} q_j^{SO} q_{n-k-s-j}^{SO} d_s + \sum_{s=0}^{n+k} q_{n-k-s-1} d_s -$$
$$- \sum_{s=0}^{n+k} \sum_{r=0}^{\lfloor \frac{n-k-s-1}{2} \rfloor} q_r^{SO} d_s + \sum_{s=0}^{n+k} q_{n-s}^{SO} d_s + d_{n+k}.$$

For k odd

$$\dim Bord_{I_2}^{SO}(n) = \sum_{s=0}^{n+k} \sum_{j=0}^{\lfloor \frac{n-k-s-1}{2} \rfloor} q_j^{SO} q_{n-k-s-j}^{SO} d_s + \sum_{s=0}^{n+k} \sum_{r=\lceil \frac{n-k-s}{2} \rceil}^{n-k-s-1} q_r^{SO} d_s +$$
$$+ \sum_{s=0}^{n+k} q_{n-s}^{SO} d_s + d_{n+k}.$$

Theorem 34. *For $I = I_r$ and $I' = I_{r-1}$, the dimensions of the singular bordism groups of cooriented maps satisfy*

$$\dim Bord_I^{SO}(n) = \dim Bord_{I'}^{SO}(n) + \sum_{i=0}^{n+k} d_{n+k-i} \dim H_{i-rk}(BG_r^{SO}),$$

where

$$\dim H_m(BG_r^{SO}) = \sum_{i\geq 1} \sum_{(\vec{J},\vec{t},\vec{\lambda})\in\mathcal{B}(i,r,m)} \prod_{u=1}^{i} \binom{q_{t_u}^{SO}+\lambda_u-1}{\lambda_u}.$$

Next we consider maps between oriented manifolds. Attempting to calculate the oriented bordism group

$$Bord_I^{or}(n) \cong \Omega_{n+k}(X_I^{SO})$$

as before runs into the problem of [10, Theorem 1.9] not having a complete analogue for the homology theory Ω_* (the rest of the proofs is all inherited from the case of cooriented mappings). However, [10] proves that

$$\Omega_*(X) \otimes \mathbb{Q} \cong H_*(X;\mathbb{Q}) \otimes \Omega_*,$$

so at least $\Omega_*(X) \otimes \mathbb{Q}$ can be recovered from $H_*(X;\mathbb{Q})$. Note that the spaces X_τ^{SO} do possess odd torsion for general sets of fold multisingularities τ: Theorem 9 shows 3-primary torsion in

$$\pi_{3k+1}(X_{fold}^{SO}) \cong \pi_{3k+1}\left(X^{SO}_{\{\Sigma^0,2\Sigma^0,3\Sigma^0,\Sigma^{1,0},\Sigma^{1,0}+\Sigma^0\}}\right)$$

for infinitely many odd $k > 1$, so [10] cannot provide $\Omega_*(X_\tau^{SO})$ completely even for some classes of fold maps. Below we sketch an algorithm for obtaining the rational homology groups $H_m(X_I^{SO};\mathbb{Q})$; combined with the formula above and the well-known fact that the free component of Ω_* is [48]

$$\Omega_* \otimes \mathbb{Q} \cong \mathbb{Q}[\mathbb{C}P^{2r} | r \geq 0],$$

this gives us the free component of the bordism group of oriented manifolds $Bord_I^{or}(n)$ (namely $\Omega_{n+k}(X_I^{SO}) \otimes \mathbb{Q}$).

From now on, all (co)homology groups are those with rational coefficients. The groups $H_m(X_I^{SO})$ can be obtained from Lemma 31 as direct sums of some of the groups $H_m(T\tilde{\xi}_r^{SO}) \cong H^m(T\tilde{\xi}_r^{SO})$ and $H_m(X_f^{SO}) \cong H^m(X_f^{SO})$. Additionally, the groups $H^m(T\tilde{\xi}_r^{SO})$ have already been calculated by Szűcs [42], so it is enough to calculate the remaining groups $H^m(X_f^{SO})$.

We can repeat Lemma 30 with rational coefficients word by word. To obtain the analogue of Lemma 29, we proceed as in the unoriented case, reducing the problem to the calculation of the boundary map

$$\partial_n^{SO,\mathbb{Q}} : \tilde{H}_n(T\tilde{\xi}_f^{SO}) \to \tilde{H}_{n-1}(T\tilde{\xi}_2^{SO})$$

(which is induced by the restriction of the representation ρ of the unoriented case) by calculating its dual ∂_n^*.

For k odd, $\tilde{\xi}_2$ is not orientable, its oriented cover is $\gamma_k^{SO} \times \gamma_k^{SO}$ and the deck transformation switches the two component bundles. The involution induced on cohomology inverts the sign of the Thom class U and keeps all the p_j classes invariant. Hence all invariant cohomology classes in $H^*(T(\gamma_k^{SO} \times \gamma_k^{SO}))$, which correspond to the cohomology classes in $H^*(T\tilde{\xi}_2)$, are sums of classes of the form $U(p_I \otimes p_J - p_J \otimes p_I)$ for some multiindexes I and J. On the other hand, $G_f^{SO} \cong \mathbb{Z}_2 \times SO(k)$. Since $H^*(B\mathbb{Z}_2) \cong H^*(point)$, we can restrict our attention to the subgroup $SO(k)$ (and thus consider the orientable cover of $\tilde{\xi}_f^{SO}$) without changing the rational (co)homology groups and the involved mappings. Lifted to the orientable cover, the mapping ρ between the bases of the involved bundle maps is the diagonal map $BSO(k) \to BSO(k) \times BSO(k)$, so the mapping it induces in cohomology is the cup product

$$\begin{aligned} H^m(BSO(k) \times BSO(k)) &\to H^m(BSO(k)), \\ \sum_i \alpha_i \otimes \beta_i &\mapsto \sum_i \alpha_i \beta_i. \end{aligned}$$

In particular, $p_I \otimes p_J - p_J \otimes p_I$ is mapped to 0, so ∂_n^* is the zero mapping, implying that $\partial_n^{SO,\mathbb{Q}}$ vanishes as well.

For k even, $G_f^{SO} \cong O(k)$, and both $\tilde{\xi}_f^{SO}$ and $\tilde{\xi}_2^{SO}$ are orientable, so ∂_n^* is identified by the Thom isomorphism with the homomorphism $H^{n-2k-1}(B\rho)$. The mapping $B\rho : BO(k) \to BSO(k) \times BSO(k)/\mathbb{Z}_2$ fits into the following commutative diagram:

$$\begin{array}{ccc} BSO(k) & \xrightarrow{\Delta} & BSO(k)\times BSO(k) \\ \downarrow & & \downarrow \\ BO(k) & \xrightarrow{B\rho} & BSO(k)\times BSO(k)/\mathbb{Z}_2 \end{array}$$

In rational cohomologies, the vertical arrows induce monomorphisms, the top arrow induces the cup product, so the homomorphism $H^*(B\rho)$ is the cup product mapping. The deck transformation of the right vertical arrow switches and reverses the orientations of the two branches of $\gamma_k^{SO}\times\gamma_k^{SO}$, so in cohomology the Pontryagin classes $1\otimes p_I$ and $p_I\otimes 1$ map to each other, while the Euler classes $1\otimes\chi$ and $\chi\otimes 1$ map to each other's negative. Hence for any $\alpha\in H^*(BO(k))$, a linear combination of Pontryagin classes, the class $\alpha\otimes 1+1\otimes\alpha\in H^*(BSO(k)\times BSO(k))$ is invariant under the involution induced by the deck transformation, and $H^*(B\rho)$ maps it to 2α. This proves that $H^*(B\rho)$ is surjective, hence ∂_n^* is also surjective, and $\partial_n^{SO,\mathbb{Q}}$ is injective.

Thus for k odd $\partial_n^{SO,\mathbb{Q}}=0$, for k even $\dim\ker\partial_n^{SO,\mathbb{Q}}=0$ and the calculation of $Bord_f^{or}(n)$ goes on in the same way as in the previous cases. Using the notation $q''_m=\dim H_m(BO(k))$ (so q''_{4r} is the number of partitions of r into natural numbers not exceeding $k/2$) and $q'_m=\dim H_m(BSO(k))$, with the convention $q''_m=0$ and $q'_m=0$ when m is not a natural number, we get that for k odd

$$\begin{aligned}\dim H_n(X_f^{SO}) &= \dim H_n(T\tilde{\xi}_2^{SO})+\dim\ker\partial_n^{SO,\mathbb{Q}}+\dim H_n(MSO(k))= \\ &= \dim H_n(T\tilde{\xi}_2^{SO})+\dim H_{n-2k-1}(BSO(k))+\dim H_{n-k}(BSO(k))= \\ &= \sum_{j=0}^{\lceil\frac{n-2k}{2}\rceil-1} q'_j q'_{n-2k-j}+\frac{q'_{\frac{n-2k}{2}}\left(q'_{\frac{n-2k}{2}}-1\right)}{2}+q'_{n-2k-1}+q'_{n-k},\end{aligned}$$

and for k even

$$\begin{aligned}\dim H_n(X_f^{SO}) &= \dim H_n(T\tilde{\xi}_2^{SO}) - \dim \operatorname{im} \partial_{n+1}^{SO,\mathbb{Q}} + \dim H_n(MSO(k)) = \\ &= \dim H_n(T\tilde{\xi}_2^{SO}) - \dim H_{n-2k-1}(BO(k)) + \dim H_{n-k}(BSO(k)) = \\ &= \sum_{j=0}^{\lceil \frac{n-2k}{2} \rceil - 1} q'_j q'_{n-2k-j} + \frac{q'_{\frac{n-2k}{2}} \left(q'_{\frac{n-2k}{2}} + 1 \right)}{2} - q''_{n-2k-1} + q'_{n-k}.\end{aligned}$$

From these, it is straightforward to obtain formulas for the group

$$Bord_{I_2}^{or}(n) \otimes \mathbb{Q} \cong \Omega_{n+k}(X_f^{SO}) \otimes \mathbb{Q},$$

which is isomorphic to the homogeneous degree $(n+k)$ part of the polynomial ring $H^*(X_f^{SO})[z_1, z_2, \dots]$, where the variable z_j has degree $4j$ and $\dim H^*(X_f^{SO}) = \dim H_*(X_f^{SO})$.

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