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Summary of the Ph.D. dissertation

Connectivity augmentation algorithms

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Introduction

The main subject of the thesis is connectivity augmentation: we would like to make a given graph k-connected by adding a minimum number of new edges. There are four basic problems in this field, since one might consider both edge- and node-connectivity augmentation in both graphs and digraphs. Whereas the minimum cost version of all of these problems is NP-complete, the minimum cardinality versions turned out to be polynomial time solvable in three of these cases. Undirected edge-connectivity augmentation was solved by Watanabe and Nakamura in 1987 [19], directed edge-connectivity by Frank in 1992 [8], and directed node-connectivity by Frank and Jordán in 1995 [10].

The thesis wishes to contribute to three out of these four problems: directed- and undirected node-connectivity and undirected edge-connectivity augmentation. Although directed edge-connectivity augmentation is not being considered, the last chapter is devoted to a constructive characterization result related to directed edge-connectivity. Let us summarize the main results of the thesis.

- *We present the first combinatorial polynomial time algorithm for directed node-connectivity augmentation.* For this problem, Frank and Jordán gave a min-max formula in 1995; however, it remained an open problem to develop a combinatorial algorithm. We present two, completely different combinatorial algorithms. Chapter 2 contains one for the special case of augmenting connectivity by one (a joint work with András Frank), and Chapter 4 presents another for augmenting the connectivity of arbitrary digraphs (a joint work with András Benczúr Jr.). The latter result also gives a new, algorithmic proof of the general theorem of Frank and Jordán on covering positively crossing supermodular functions on set pairs.

- *We present a min-max formula and a combinatorial polynomial time algorithm for augmenting undirected node-connectivity by one.* The complexity status of undirected node-connectivity augmentation of arbitrary graphs is still open; already the special case of augmenting by one has attracted considerable attention. The formula proved in Chapter 3 was conjectured by Frank and Jordán in 1994.

- *We establish a constructive characterization of (k, ℓ)-edge-connected digraphs.* This result of Chapter 6, a joint work with Erika Renáta Kovács, settles a conjecture of Frank from 2003. The theorem gives a common generalization of a number of previously known characterizations, and naturally fits into the framework defined by splitting off and orientation theorems.

- *We present partial results concerning partition constrained undirected local edge-connectivity augmentation.* In Chapter 5, we discuss some classical results concerning undirected edge-connectivity augmentation in a unified framework, based on the technique of edge-flippings. For the partition constrained problem we formulate a conjecture and give a partial proof.

Directed node-connectivity augmentation

The solution of directed node-connectivity augmentation by Frank and Jordán [10] is via a much more general theorem on covering positively crossing supermodular functions on set pairs. The following definitions are needed in order to formulate this theorem.

Let us call $K = (K^-, K^+)$ a **set pair** if K^- and K^+ are disjoint nonempty subsets of the ground set V. Let $\mathcal{S}$ denote the set of all set pairs. We say that a directed edge[1] $xy \in V^2$ **covers** the set pair K if $x \in K^-$, $y \in K^+$. Two set pairs $K = (K^-, K^+)$ and $L = (L^-, L^+)$ are called **independent** if $K^- \cap L^- = \emptyset$ or $K^+ \cap L^+ = \emptyset$. This is equivalent to the property that no edge in V^2 covers both K and L. Two non-independent set pairs are called **dependent**. A set $\mathcal{F}$ of set pairs is **independent**, if its members are pairwise independent.

A natural partial order on $\mathcal{S}$ can be defined as follows: $K \preceq L$ if $K^- \subseteq L^-$ and $K^+ \supseteq L^+$. The pairs K and L are **comparable** if $K \preceq L$ or $L \preceq K$. Two dependent, but not comparable set pairs are called **crossing**. A set $\mathcal{F}$ of set pairs is called **cross-free** if it contains no crossing set pairs, that is, any two dependent elements are comparable.

For dependent K and L, let us define the set pairs $K \wedge L = (K^- \cap L^-, K^+ \cup L^+)$ and $K \vee L = (K^- \cup L^-, K^+ \cap L^+)$. The non-negative integer valued function p on $\mathcal{S}$ is called **positively crossing supermodular** if

$$p(K) + p(L) \le p(K \wedge L) + p(K \vee L)$$

whenever $K, L \in \mathcal{S}$, K and L are dependent and $p(K), p(L) > 0$.

For a multiset F consisting of edges in V^2 and a set pair $K \in \mathcal{S}$, let $\delta_F(K)$ denote the number of edges in F covering K. We say that the edge set F **covers** the function p, if $\delta_F(K) \ge p(K)$ for every set pair $K \in \mathcal{S}$. Let τ_p denote the minimum size of an edge set covering p, and let $\nu_p = \max\{\sum_{K \in \mathcal{F}} p(K) : \mathcal{F} \text{ independent}\}$. $\nu_p \le \tau_p$ clearly holds, since an edge may cover at most one member of an independent system. The theorem of Frank and Jordán states that this in fact holds with equality:

Theorem 1 (Frank and Jordán, 1995 [10])**.** *Given a ground set V and a positively crossing supermodular function p on the set pairs, $\tau_p = \nu_p$.*

Applications of this theorem include both directed node- and edge-connectivity augmentation, ST-edge-connectivity augmentation, Győri's theorem on generators of path-systems and maximum K_{tt}-free t-matchings in bipartite graphs.

[1] By V^2 we denote the set of all directed edges on a ground set V, while $\binom{V}{2}$ stands for the set of all undirected edges on V.

Concerning directed node-connectivity augmentation of a digraph $D = (V, A)$ with a target value k, let us call a set pair $K \in \mathcal{S}$ a **one-way pair** if $\delta_D(K) = 0$, that is, there are no edges in D covering K. We denote by $\mathcal{O} = \mathcal{O}_D$ the set of one-way pairs. Let us define $s(K) := |V - (K^- \cup K^+)|$.

Theorem 2. *For a digraph $D = (V, A)$, the minimum number of edges whose addition makes D k-node-connected equals the maximum value of $\sum_{i=1}^{\ell}(k - s(K_i))$ over pairwise independent one-way pairs $K_1, \ldots, K_\ell$.*

Assume now that the digraph D is already $(k-1)$-node-connected; this problem will be referred to as augmenting connectivity by one. In this case, $s(K) \geq k-1$ for all one-way pairs. We call a one-way pair **strict** if $s(K) = k-1$ and denote their set by $\mathcal{O}^1 = \mathcal{O}^1_D$. The theorem simplifies to the following form:

Theorem 3. *For a $(k-1)$-node-connected digraph $D = (V, A)$, the minimum number of edges whose addition makes D k-node-connected equals the maximum number of pairwise independent strict one-way pairs.*

The orginial proof of Theorem 1 was not algorithmic. Although the original paper contained a polynomial time algorithm as well, this heavily relied on the ellipsoid method, leaving the problem of finding a combinatorial polynomial time algorithm open. The first result towards this direction was given by Enni in 1999 [5] for 1-ST-edge-connectivity augmentation. For fixed k, Frank and Jordán themselves gave a combinatorial algorithm in 1999 [11] for directed connectivity augmentation - that is, the running time is the product of a polynomial of n and an exponential function of k.

In Chapter 2 of the thesis, we present a combinatorial algorithm for augmenting connectivity by one. In Chapter 4, we develop a completely different algorithm for augmenting the connectivity of arbitrary graphs. This is via a new, algorithmic proof of Theorem 1.

Augmenting connectivity by one

In Chapter 2, we present two algorithms for augmenting connectivity by one. The first one uses a simple dual oracle, while the second also gives a new proof of Theorem 3. The dual oracle relies on the following theorem. By **skeleton**, we mean a maximum cross-free system.

Theorem 4 (Frank, V. [V1])**.** *For a skeleton $\mathcal{K} \subseteq \mathcal{O}^1$ the maximum number of pairwise independent one-way pairs is equal in $\mathcal{K}$ and $\mathcal{O}^1$, that is, $\nu(\mathcal{K}) = \nu(\mathcal{O}^1) = \nu(D)$.*

$\nu(\mathcal{K})$ for a skeleton $\mathcal{K}$ can be determined via Dilworth's theorem. Hence to calculate the dual optimum value, we only need to construct a skeleton. Although arbitrary maximal cross-free system suffices, finding one is nontrivial due to the possibly exponential size of $\mathcal{O}^1$. The main concept in the skeleton building subroutine is that of **stable cross-free systems**.

General connectivity augmentation

This result, contained in [V4], extends the previous work of Benczúr [2] on augmenting directed connectivity by one. We give an equivalent reformulation of Theorem 1 in terms of posets. The problem we investigate is covering a certain type of weighted poset by a minimum number of intervals. In a poset, we mean by an interval the set of elements between a minimal and maximal one, and call two elements **dependent** if contained in a common interval and **independent** otherwise. We define the **strong interval property** of posets which involves operations $\vee$ and $\wedge$ on dependent elements.

The notion of a **positively crossing supermodular function** p on such a poset is analogous to the one on set pairs: for all dependent x and y with $p(x) > 0$ and $p(y) > 0$ we require $p(x) + p(y) \leq p(x \wedge y) + p(x \vee y)$.

Consider a multiset $\mathcal{I}$ of intervals. We say that $\mathcal{I}$ **covers** the function p or $\mathcal{I}$ is a **cover** of p if for every x, at least $p(x)$ intervals in $\mathcal{I}$ contain x. Our theorem is an analogoue of Theorem 1 for posets. In fact, it can be shown that the two theorems are equivalent.

Theorem 5 (V. and Benczúr [V4])**.** *For a poset $(\mathcal{P}, \preceq)$ with the strong interval property and a positively crossing supermodular function p, the minimum number of intervals covering p is equal to the maximum of the sum of p values of pairwise independent elements of $\mathcal{P}$.*

Our algorithm uses a primal-dual scheme for finding covers of the poset. For an initial (possibly greedy) cover the algorithm searches for witnesses for the necessity of each element in the cover. If any two (weighted) witnesses are independent, the solution is optimal. As long as this is not the case, the witnesses are gradually exchanged by smaller ones. Each witness change defines an appropriate change in the solution; these changes are finally unwound in a shortest path manner to obtain a solution of size one less.

This scheme works for arbitrary posets, however, when applied for node-connectivity augmentation we have to be careful, since the size of the poset $\mathcal{S}$ of set pairs can be exponential. In this case, the basic steps can be implemented by maximum flow computations and breadth-first search algorithms.

Undirected node-connectivity augmentation

It is still open whether undirected node-connectivity augmentation is solvable in polynomial time. The previously known best result is due to Jackson and Jordán from 2005 [14]: they gave a polynomial time algorithm for finding an optimal augmentation of arbitrary graphs for any fixed k.

In Chapter 3, we present a min-max formula and a polynomial algorithm[2] for the special case of augmenting connectivity by one, conjectured by Frank and Jordán in 1994.

[2]The running time bound is $O(kn^7)$, that is, polynomial in both k and n.

In the $(k-1)$-node-connected graph $G = (V, E)$, a subpartition $X = (X_1, \ldots, X_t)$ of V with $t \geq 2$ is called a **clump** if $|V - \bigcup X_i| = k - 1$ and $d(X_i, X_j) = 0$ for any $i \neq j$. The sets X_i are called the **pieces** of X while $|X|$ denotes t, the number of pieces. If $t = 2$ then X is a **small clump**, while for $t \geq 3$ it is a **large clump**. An edge $uv \in \binom{V}{2}$ **connects** X if u and v lie in different pieces of X. Two clumps are said to be **independent** if there is no edge $uv \in \binom{V}{2}$ connecting both.

A **bush** $\mathcal{B}$ is a set of pairwise distinct small clumps, so that each edge in $\binom{V}{2}$ connects at most two of them. A **shrub** is a set consisting of pairwise independent (possibly large) clumps. For a bush $\mathcal{B}$, let $def(\mathcal{B}) = \left\lceil \frac{|\mathcal{B}|}{2} \right\rceil$, and for a shrub $\mathcal{S}$, let $def(\mathcal{S}) = \sum_{K \in \mathcal{S}} (|K| - 1)$.

A **grove** is a set consisting of some (possibly zero) bushes and one (possibly empty) shrub, so that the clumps belonging to different bushes are independent, and a clump belonging to a bush is independent from all clumps belonging to the shrub. For a grove Π consisting of the shrub $\mathcal{B}_0$ and bushes $\mathcal{B}_1, \ldots, \mathcal{B}_\ell$, let $def(\Pi) = \sum_i def(\mathcal{B}_i)$. For a $(k-1)$-node-connected graph $G = (V, E)$, let $\tau(G)$ denote the minimum number of edges whose addition makes G k-connected, and let $\nu(G)$ denote the maximum value of $def(\Pi)$ over all groves Π.

Theorem 6 (V. [V3])**.** *For a $(k-1)$-node-connected graph $G = (V, E)$ with $|V| \geq k + 1$, $\nu(G) = \tau(G)$.*

Both the proof and the algorithm are extensions of those in Chapter 2 for the directed case. Cross-free systems and skeletons can also be defined in the appropriate sense, and the analogue of Theorem 4 will also be true. For a skeleton, instead of Dilworth's theorem, we use Fleiner's theorem [6] on covering symmetric posets by symmetric chains.

Constructive characterizations

By a constructive characterization of a graph property $\mathcal{P}$ we mean a set of operations preserving property $\mathcal{P}$, so that each graph with property $\mathcal{P}$ can be obtained by a sequence of such operations starting from a small set of basic instances. Such characterizations are often useful for proving further properties of graphs with property $\mathcal{P}$.

Well-known examples are the ear-decompositions of 2-edge and 2-node-connected graphs. The following theorem gives a constructive characterization of $2k$-edge-connected digraphs.

Theorem 7 (Lovász, 1976 [16])**.** *An undirected graph is $2k$-edge-connected if and only if it can be obtained from a single node by iteratively applying the following two operations:*

(i) add a new edge (possibly a loop),

(ii) subdivide k existing edges and identify the subdividing nodes with a single node z.

Later Mader gave a similar characterization for $2k+1$-edge-connected graphs [17]. Theorem 7 immediately implies the weak version of Nash-Williams' orientation theorem, stating that an undirected graph has a k-edge-connected orientation if and only if it is $2k$-edge-connected. The analogous directed characterization is very similar.

Theorem 8 (Mader, 1982 [18]). *A directed graph is k-edge-connected if and only if it can be obtained from a single node by iteratively applying the two operations in Theorem 7 (in the directed sense).*

In this theorem and in Theorem 7 as well, operation (ii) will be called **pinching** k edges with z. By pinching 0 edges we simply mean the addition of a node. A main ingredient of the proof is Mader's directed splitting theorem [18]. Using Nash-Williams' weak orientation theorem, Theorem 7 can easily be derived from Theorem 8.

(k,ℓ)-edge-connectivity is a natural common generalization of k-edge-connectivity and rooted k-edge-connectivity of digraphs. We say that the digraph $D=(V,A)$ is (k,ℓ)**-edge-connected** for some integers $0\le\ell\le k$ and a root node $r_0\in V$, if for each node $v\ne r_0$, there exist k edge-disjoint paths from r_0 to v and ℓ edge-disjoint paths from v to r_0. Note that (k,k)-edge-connectivity coincides with k-edge-connectivity while $(k,0)$-edge-connectivity means rooted k-edge-connectivity. An undirected graph $G=(V,E)$ is called (k,ℓ)**-partition connected** if for any partition of the nodes into $t\ge 2$ classes, there are at least $k(t-1)+\ell$ edges connecting distinct classes. These two concepts are linked by the following theorem:

Theorem 9 (Frank, 1980 [7]). *For integers $0\le\ell\le k$, an undirected graph G has a (k,ℓ)-edge-connected orientation if and only if G is (k,ℓ)-partition connected.*

Mader's directed splitting theorem also extends to (k,ℓ)-edge-connectivity, namely:

Theorem 10 (Frank, 1999 [9]). *Let $D=(U+z,A)$ be a digraph (k,ℓ)-edge-connected in U (with $r_0\in U$) and $\rho(z)=\delta(z)$. Then there exists a complete splitting at z resulting in a (k,ℓ)-edge-connected graph.*

The main result of this chapter was conjectured by Frank in 2003.

Theorem 11 (Kovács, V. [V2]). *For $0\le\ell\le k-1$, a directed graph $D=(V,A)$ is (k,ℓ)-edge-connected with root $r_0\in V$ if and only if it can be built up from the single node r_0 by the following two operations.*

(i) add a new edge,

(ii) for some i with $\ell\le i\le k-1$, pinch i existing edges with a new node z, and add $k-i$ new edges entering z and leaving existing nodes.

Using Theorem 9, this immediately leads to the next theorem:

Theorem 12. *For $0\le\ell\le k-1$, an undirected graph $G=(V,E)$ is (k,ℓ)-partition-connected if and only if it can be built up from a single node by iteratively applying the two operations in Theorem 11 (in the undirected sense).*

Besides $\ell = 0$, the special cases $\ell = 1$ (Frank and Szegő [13]), and $\ell = k-1$ (Frank and Király [12]) were known beforehand. The proof of the latter case serves as a starting point of our argument, however, the proof is significantly more complicated. Among other tools, we formulate a new, abstract splitting off result.

Local edge-connectivity augmentation

In case of edge-connectivity, we can also cope with the problem of **local edge-connectivity augmentation**, that is, we may have a different connectivity requirement for each pair of nodes: $r(u,v) = r(v,u)$ for the nodes $u, v \in V$. We say that $G = (V,E)$ is **r-edge-connected** if $\lambda(u,v) \geq r(u,v)$ for any $u,v \in V$. Let us define $R(X) := \max\{r(u,v) : u \in X, v \notin X\}$ if $\emptyset \neq X \subsetneq V$ and $R(\emptyset) = R(V) = 0$. Let $p(X) := (R(X) - d_G(X))^+$. A set $C \subseteq V$ is called a **marginal set** if $R(C) \leq 1$ and $d_F(C) = 0$.

Theorem 13 (Frank, 1992 [8]). *Assume we are given a graph $G = (V,E)$ and a connectivity requirement function r so that G has no marginal sets.*[3] *Then the minimum number of edges whose addition makes G r-edge-connected equals the maximum value of $\left\lceil \frac{1}{2} p(\mathcal{X}) \right\rceil$ over subpartitions $\mathcal{X}$ of V.*

The nontrivial direction is proved via Mader's undirected splitting off theorem [17]. A related result (which we do not formulate here) is the theorem of Benczúr and Frank from 1999 [3] on covering positively crossing symmetric supermodular functions.

Both Theorem 13 and the Benczúr-Frank theorem can easily be derived from their degree-prescribed versions. $m : V \to \mathbb{Z}_+$ is called a **degree prescription** if $m(V)$ is even; F is a **m-prescribed edge-set** if $d_F(v) = m(v)$ for every $v \in V$. In the thesis we present new proofs of the degree-prescribed versions of these two theorems, using the edge-flipping technique instead of the standard method of splitting off. For $xy, uv \in F$, by **flipping** (xy, uv) we mean replacing F by $F' = F - \{xy, uv\} + \{xv, uy\}$. The two proofs are discussed in a unified framework; they share a large part in common using only the **symmetric positively skew supermodular property** of the demand functions, that is, $p(X) = p(V - X)$ for every $X \subseteq V$, and if $p(X), p(Y) > 0$ then at least one of the following hold:

$$p(X) + p(Y) \leq p(X \cup Y) + p(X \cap Y) \tag{1a}$$
$$p(X) + p(Y) \leq p(X - Y) + p(Y - X). \tag{1b}$$

The main problem discussed in Chapter 5 is **partition-constrained local edge-connectivity augmentation**. Besides the connectivity requirement r, we are also given a partition $\mathcal{Q} = (Q_1, \dots, Q_t)$ of V. We want to find a minimum cardinality augmenting edge set consiting

[3]The original theorem of Frank is slightly stronger by forbidding only marginal components.

only of **$\mathcal{Q}$-legal** edges, that is, edges between different classes of $\mathcal{Q}$. For global edge-connectivity ($r \equiv k \geq 2$), this problem was solved by Bang-Jensen, Gabow, Jordán and Szigeti in 1999 [1].

A natural lower bound is the one in Theorem 13, namely, $\alpha(G) = \max \left\lceil \frac{1}{2} p(\mathcal{X}) \right\rceil$ over subpartitions $\mathcal{X}$ of V. For a similar bound for each $1 \leq h \leq t$, let us call $\mathcal{X}$ a h-subpartition, if $\mathcal{X}$ is a subpartition of Q_h. Let $\beta_h(G) = \max p(\mathcal{X})$ over j-subpartitions $\mathcal{X}$. Let $\Psi_{\mathcal{Q}}(G)$ denote the maximum of $\alpha(G)$ and $\beta_h(G)$ for $h = 1, \ldots, t$. Bang-Jensen et al. [1] showed that the for the global connectivity case, the size of an optimal augmenting edge set is $\Psi_{\mathcal{Q}}(G)$ or $\Psi_{\mathcal{Q}}(G)+1$ depending on the existence of some special configurations in the graph. For partition constrained local edge-connectivity augmentation we first prove an approximation result.

Theorem 14. *Assume we are given a graph $G = (V, E)$, a partition $\mathcal{Q}$ of the nodes and a connectivity requirement r so that G contains no marginal sets. Then the minimum number of $\mathcal{Q}$-legal edges whose addition makes G r-edge-connected is at most $\Psi_{\mathcal{Q}}(G) + r_{\max}$.*

Here $r_{\max}$ denotes the maximum value of the function r. Recently, a weaker version of this theorem was also proved by Lau and Yung [15] (for two partition classes and $2r_{\max}$.)

We formulate a conjecture on the optimum value for $t = 2$. For this, we need the following complicated dual structure. A partition $\mathcal{H} = \{X^*, Y^*, C_1, C_2, \ldots, C_\ell\}$ of V forms a **hydra** with **heads** X^*, Y^* and **tentacles** C_i, if $d_G(C_i, C_j) = 0$ for every $1 \leq i < j \leq \ell$; and for any two disjoint index sets $\emptyset \neq I, J \subseteq \{1, \ldots, \ell\}$, (1a) holds with equality for $X^* \cup (\bigcup_{i \in I} C_i)$ and $X^* \cup (\bigcup_{j \in J} C_j)$, and also for $Y^* \cup (\bigcup_{i \in I} C_i)$ and $Y^* \cup (\bigcup_{j \in J} C_j)$.[4]

Let $\mathcal{Q} = \{Q_1, Q_2\}$ be the bipartition constraint. For a fixed value $h \in \{1, 2\}$, let $\mathcal{Z}$ be a h-subpartition which is a refinement of $\{C_1, \ldots, C_\ell\}$. The tentacle C_i is called **h-toxic**, if

$$p(C_i \cup X^*) - p(X^*) + \sum (p(Z) : Z \in \mathcal{Z}, Z \subseteq C_i)$$

is odd. Let Let χ'_h denote the number of h-toxic tentacles. Let us define

$$\tau'_h(G, r, \mathcal{Z}, \mathcal{H}) = \frac{1}{2} \left(\chi'_h + p(X^*) + p(Y^*) + p(\mathcal{Z}) \right).$$

Let $\tau'(G, r, \mathcal{Q})$ denote the maximum of $\tau'_h(G, r, \mathcal{Z}, \mathcal{H})$ over all choices of h, $\mathcal{H}$ and $\mathcal{Z}$ as above.

Conjecture 15. *Let us be given a graph $G = (V, E)$, a connectivity requirement function r and a partition $\mathcal{Q} = \{Q_1, Q_2\}$ of V so that G contains no marginal sets. Then the minimum size of a $\mathcal{Q}$-legal augmenting edge set equals the maximum of $\Psi_{\mathcal{Q}}(G)$ and $\tau'(G, r, \mathcal{Q})$.*

We also formulate a degree-prescribed version of this conjecture, which would easily imply this one. A partial proof is given, for the degree-prescribed version, using the edge-flipping technique.

[4] Although the definition contains an exponential number of conditions, we also provide an equivalent, polynomial-size characterization.

The thesis is based on the following papers

[V1] A. Frank and L. A. Végh. An algorithm to increase the node-connectivity of a digraph by one. *Discrete Optimization*, 5:677–684, 2008.

[V2] E. R. Kovács and L. A. Végh. The constructive characterization of (k,l)-edge-connected digraphs. *Combinatorica*. (accepted); available as EGRES Tech. Report TR-2008-14 at `http://www.cs.elte.hu/egres`.

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