Semantic vs Syntactic Properties of Graph Polynomials, I:

On the Location of Roots of Graph Polynomials

Johann A. Makowsky*

* Faculty of Computer Science,
Technion - Israel Institute of Technology,
Haifa, Israel
janos@cs.technion.ac.il

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Partially joint work with E.V. Ravve and N.K. Blanchard

Graph polynomial project:
http://www.cs.technion.ac.il/∼janos/RESEARCH/gp-homepage.html

Reference

- Johann A. Makowsky, Elena V. Ravve

  On the Location of Roots of Graph Polynomials

  Special issue of the Erdős Centennial
  Electronic Notes in Discrete Mathematics, Volume 43 (2013), Pages 201-206

- Johann A. Makowsky, Elena V. Ravve, Nicolas K. Blanchard

  On the location of roots of graph polynomials

  European Journal of Combinatorics, Volume 41 (2014), Pages 1-19

Overview

- Semantic properties of graph polynomials

- Definability of graph polynomials in Second Order Logic SOL

- Many examples

- Roots of graph polynomials

- What do we learn?

Semantic Properties of Graph Polynomials

Graph polynomials

Let $\mathcal{R}$ be a (polynomial) ring.

A function $P : \mathcal{G} \to \mathcal{R}$ is a

graph parameter

if for any two isomorphic graphs $G_1, G_2 \in \mathcal{G}$ we have $P(G_1) = P(G_2)$.

It is a

graph polynomial

if for each $G \in \mathcal{G}$ it is a polynomial.

In this lecture we study univariate graph polynomials P with $\mathcal{R} = \mathbb{Z}[X]$ or $\mathbb{C}[X]$.

A complex number $z \in \mathbb{C}$ is a P-root if there is a graph $G \in \mathcal{G}$ such that $P(G, z) = 0$.

Similar graphs and similarity functions

Two graphs G_1, G_2 are similar if they have the same number of vertices, edges and connected components, i.e.,

- $|V(G_1)| = n(G_1) = n(G_2) = |V(G_2)|$,
- $|E(G_1)| = m(G_1) = m(G_2) = |E(G_2)|$, and
- $k(G_1) = k(G_2)$.

A graph parameter or graph polynomial is a similarity function if it is invariant and similarity.

(i) The nullity $\nu(G) = m(G) - n(G) + k(G)$ and the rank $\rho(G) = n(G) - k(G)$ of a graph G are similarity polynomials with integer coefficients.

(ii) Similarity polynomials can be formed inductively starting with similarity functions $f(G)$ not involving indeterminates, and monomials of the form $X^{g(G)}$ where X is an indeterminate and $g(G)$ is a similarity function not involving indeterminates. One then closes under pointwise addition, subtraction, multiplication and substitution of indeterminates X by similarity polynomials.

Distinctive power of graph polynomials, I

Two graph polynomials are usually compared via their distinctive power.

A graph polynomial $Q(G, X)$ is less distinctive than $P(G, Y)$, $Q \preceq P$,
if for every two similar graphs G_1 and G_2

$$P(G_1, X) = P(G_2, X) \text{ implies } Q(G_1, Y) = Q(G_2, Y).$$

We also say the $P(G; X)$ determines $Q(G; X)$ if $Q \preceq P$.

Two graph polynomials $P(G, X)$ and $Q(G, Y)$ are
equivalent in distinctive power (d.p-equivalent)
if for every two similar graphs G_1 and G_2

$$P(G_1, X) = P(G_2, X) \text{ iff } Q(G_1, Y) = Q(G_2, Y).$$

The same definition also works for graph parameters and multivariate graph polynomials.

Distinctive power of graph polynomials, II

$\mathbb{C}^\infty$ denotes the set of finite sequences of complex numbers.
We denote by $cP(G) \in \mathbb{C}^\infty$ the sequence of coefficients of $P(G, X)$.

Proposition 1
Two graph polynomials $P(G, X_1, \ldots X_r)$ and $Q(G, Y_1, \ldots, Y_s)$ are equivalent in distinctive power (d.p-equivalent) *($P \sim_{d.p.} Q$) iff there are two functions $F_1, F_2 : \mathbb{C}^\infty \to \mathbb{C}^\infty$ such that for every graph G*

$$F_1(n(G), m(G), k(G), cP(G)) = cQ(G) \text{ and}$$
$$F_2(n(G), m(G), k(G), cQ(G)) = cP(G)$$

Proposition 1 shows that our definition of equivalence of graph polynomials is mathematically equivalent to the definition proposed by
C. Merino and S. Noble in 2009.

Computability

The functions F_1, F_2 in Proposition 1 need not be computable in any sense, even if the coefficients of $P(G)$ and $Q(G)$ are integers.

A graph polynomial $P(G;X)$ with coefficients in a ring $\mathcal{R}$ is computable (in a suitable model of computation for $\mathcal{R}$) if

(i) the function $cP : \mathcal{G} \to \bigcup_n \mathcal{R}^n$ computing the coefficients of $P(G;X)$ is computable, and

(ii) the decision problem, given $s \in \bigcup_n \mathcal{R}^n$ is there a graph with $cP_(G) = s$ is decidable.

Theorem 2
Let $P(G;X)$ and $Q(G;X)$ be two computable graph polynomials which are d.p.-equivalent. Then there are F_1, F_2 as in Proposition 1 which are computable.

In this case we say that $P(G;X)$ and $Q(G;X)$ are computably d.p.-equivalent.

Prefactor and subtsitution equivalence, I

- We say that $P(G;\bar{X})$ is prefactor reducible to $Q(G;\bar{X})$ and we write
$$P(G;\bar{Y}) \preceq_{prefactor} Q(G;\bar{X})$$
if there are similarity functions
$$f(G;\bar{X}), g_1(G;\bar{X}), \ldots, g_r(G;\bar{X})$$
such that
$$P(G;\bar{Y}) = f(G;\bar{X}) \cdot Q(G; g_1(G;\bar{Y}), \ldots, g(G;\bar{Y})).$$
- We say that $P(G;\bar{X})$ is substitutions reducible to $Q(G;\bar{X})$, and we write
$$P(G;\bar{Y}) \preceq_{subst} Q(G;\bar{X})$$
if $f(G;\bar{X}) = 1$ for all values of $\bar{X}$.
- $P(G;\bar{X})$ and $Q(G;\bar{X})$ are prefactor (substitution) equivalent if the relationship holds in both directions.

It follows that if $P(G;\bar{X})$ and $Q(G;\bar{X})$ are prefactor (substitution) equivalent then they are computably d.p.-equivalent.

Semantic properties of graph parameters

A semantic property is a class of graph parameters (polynomials) closed under d.p.-equivalence.

Let $p(G)$ be a graph parameter with values in $\mathbb{N}$, and $P(G;X)$ be a graph polynomial.

- The degree of $P(G;X)$ equals $p(G)$ is not a semantic property of $P(G;X)$.

  Using Proposition 1 we see that $P(G;X)$ and $P(G;X^2)$ are d.p.-equivalent, but they have different degrees.

- $P(G;X)$ determines $p(G)$ is a semantic property of $P(G;X)$.

Semantic vs syntactic properties of graph polynomials, I

Semantically meaningless properties:

(i) $P(G, X)$ is monic for each graph G, i.e., the leading coefficient of $P(G; X)$ equals 1.

Multiplying each coefficient by a fixed polynomial gives an equivalent graph polynomial.

(ii) The leading coefficient of $P(G, X)$ equals the number of vertices of G.

However, proving that two graphs G_1, G_2 with $P(G_1, X) = P(G_2, X)$ have the same number of vertices is semantically meaningful.

(iii) The graph polynomials $P(G; X)$ and $Q(G; X)$ coincide on a class $\mathcal{C}$ of graphs, i.e. for all $G \in \mathcal{C}$ we have $P(G; X) = Q(G; X)$.

The semantic content of this situation says that if we restrict our graphs to $\mathcal{C}$, then $P(G; X)$ and $Q(G; X)$ have the same distinguishing power.

The equality of $P(G; X)$ and $Q(G; a)X$ is a syntactic conincidence or reflects a clever choice in the definitions $P(G; X)$ and $Q(G; X$.

Semantic vs syntactic properties of graph polynomials, II

Clever choices of can be often achieved.

Let $\mathcal{C}$ be class of finite graphs closed under graph isomorphisms.

Proposition 3
Assume that $P(G;X)$ and $Q(G;X)$ have the same distinguishing power on a class of graphs $\mathcal{C}$. Then there is $P' \sim_{d.p.} P$ such that the graph polynomials $P'(G;X)$ and $Q(G;X)$ coincide on a class $\mathcal{C}$ of graphs.

If, additionally, $\mathcal{C}, P(G;X)$ and $Q(G;X)$ are computable, then $P'(G;X)$ can be made computable, too.

Proposition 3 also holds when we replace computable by definable in SOL, as we shall see later.

Prominent graph polynomials

Spectral graph theory, I

Let $G = (V(G), E(G))$ be a loopless graph without multiple edges.

- A_G is the adjacency matrix of a graph G.
- D_G is the diagonal matrix with $(D_G)_{i,i} = d(i)$, the degree of the vertex i.
- $L_G = D_G - A_G$ is the Laplacian of G.

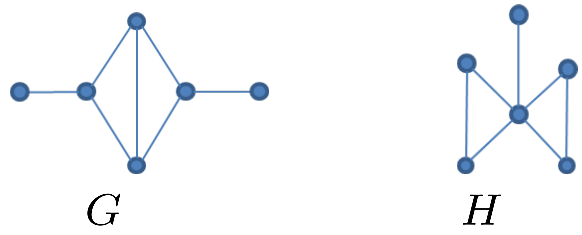
In spectral graph theory two computable graph polynomials are considered:

- The characteristic polynomial $P_A(G; X)$ of G defined as
$$P_A(G; X) = \det(X \cdot \mathbb{I} - A_G)$$
- and the Laplacian polynomial $P_L(G; X)$ of G defined as
$$P_L(G; X) = \det(X \cdot \mathbb{I} - L_G)$$

Here $\mathbb{I}$ denotes the unit element in the corresponding matrix ring.

Spectral graph theory, II

G and H below are similar.



We have

$$P_A(G;X) = P_A(H;X) = (X-1)(X+1)^2(X^3 - X^2 - 5X + 1),$$

but G has eight spanning trees, and H has six.

Therefore, $P_L(G;X) \neq P_L(H;X)$, as one can compute the number of spanning trees from $P_L(G;X)$.

Spectral graph theory, III

On the other hand, the graphs below G' and H' are similar, but G' is not bipartite, whereas, H' is.

G' H'

As P_A determines bipartiteness, we have $P_A(H';X) \neq P_A(G',X)$, but one can easily check that $P_L(H';X) = P_L(G';X)$.

Conclusion:

The characteristic polynomial and the Laplacian polynomial are d.p.-incomparable.

However, if restricted to k-regular graphs, they are d.p.-eqivalent.

Matching polynomials, I

There are two versions of the univariate matching polynomial:
The matching defect polynomial (or acyclic polynomial)

$$dm(G;X) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k m_k(G) X^{n-2k},$$

and the matching generating polynomial

$$gm(G;X) = \sum_{k=0}^{n} m_k(G) X^k$$

The relationship between the two is given by

$$dm(G;X) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k m_k(G) X^{n-2k} = X^n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k m_k(G) X^{-2k} =$$

and

$$= X^n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} m_k(G)((-1) \cdot X^{-2})^k = X^n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} m_k(G)(-X^{-2})^k = X^n gm(G;(-X^{-2}))$$

The matching polynomials, II

It follows that

- Both matching polynomials are computable.
- gm and dm are d.p.-equivalent.
- However, $gm(G;X)$ is invariant under addition or removal of isolated vertices, whereas $dm(G;X)$ counts them.

Furthermore we have

Theorem 4 (Godsil and Gutmann)
A graph G is a forest iff $dm(G,X) = P_A(G;X)$.

This is a syntactic theorem. One cannot replace $dm(G;X)$ by $gm(G;X)$.

It holds for $P_L(G;X)$ only if one restricts it to k-regular forests.

Definability of Graph Polynomials

in Second Order Logic SOL

Graph polynomials definable in Second Order Logic SOL, I

There are **too many** d.p.-equivalent graph polynomials.

For example, let $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ be two injective functions and

let $P(G, X) = \sum_i a_i(G)X^i$ a graph polynomial.

Then $Q(G, X) = \sum_i a_{f(i)(G)} X^{g(i)}$ is equivalent to $P(G, X)$.

SOL-definable generating functions:

Let $\phi(U)$ be an SOL-formula in the language of graphs with a free relation variable U. Let

$$a_i(G) = \mid \{U \subseteq V : (G, U) \models \phi(U) \text{ and } |U| = i\} \mid$$

be uniformly defined numeric graph parameters.

Then

$$\sum_i a_i(G)X^i = \sum_{U:\phi(u)} X^{|U|}$$

is a the simplest form of an SOL-definable graph polynomial.

Graph polynomials definable in Second Order Logic SOL, II

We can form many d.p.-equivalent graph polynomials such as

$$\sum_i a_i(G) X^i = \sum_{U:\phi(u)} X^{|U|} \tag{1}$$

$$\sum_i a_i(G)(-1)^i X^i = \sum_{U:\phi(u)} (-1)^{|U|} X^{|U|} \tag{2}$$

$$\sum_i a_i(G) X^{|V(G)|-i} = \sum_{U:\phi(u)} X^{|V(G)-U|} \tag{3}$$

$$\sum_i a_i(G) \binom{X}{i} = \sum_{U:\phi(u)} \binom{X}{|U|} \tag{4}$$

$$\sum_i a_i(G) X^{\underline{i}} = \sum_{U:\phi(u)} X^{\underline{|U|}} \tag{5}$$

Simple SOL-definable graph polynomials

The graph polynomial $dm(G;X) = \sum_i m_i(G) \cdot X^i$, can be written also as

$$dm(G;X) = \sum_{M \subseteq E(G)} \prod_{e \in E} X$$

where M ranges over all matchings of G.
To be a matching is definable by a formula $\phi(I)$ of Second Order Logic SOL
.

A **simple** SOL-**definable graph polynomial** $P(G,X)$ is a polynomial of the form

$$P(G,X) = \sum_{A \subseteq V(G)^r : \phi(A)} \prod_{v \in A} X$$

where A ranges over all subsets of $V(G)^r$ satisfying $\phi(A)$

and $\phi(A)$ is a SOL-formula.

General SOL-definable graph polynomials

For the general case

- One allows several indeterminates $X_1, \ldots, X_t$.
- One gives an inductive definition.
- One allows an ordering of the vertices.
- One requires the definition to be **invariant under the ordering**, i.e., different orderings still give the same polynomial.
- This also allows to define the modular counting quantifiers $C_{m,q}$ "there are, modulo q exactly m elements..."

The general case includes the Tutte polynomial, the cover polynomial, and virtually all graph polynomials from the literature.

Graph polynomials definable in Second Order Logic SOL, III

Let $P(G,X)$ be a SOL-definable graph polynomial and

let $S(G,X)$ be and SOL-definable similarity function.

Then the following polynomials are SOL-definable and d.p.-equivalent:

- $S(G,X) + P(G,X)$
- $S(G,X) \cdot P(G,X)$

In the second case $S(G;X)$ is called in the literature a prefactor.

The two matching polynomials are related to each other
by a substitution and by a prefactor.

$$dm(G;X) = X^n \cdot gm(G;(-X^{-2}))$$

(Almost) all graph polynomials

from the literature

are SOL-**definable!**

Computability of SOL-definable graph polynomials

Proposition 5
Every SOL*-definable graph polynomial* $P(G; X)$
with coefficients in a ring $\mathcal{R}$
is computable in a model of computation suitable for $\mathcal{R}$*.*

For a detailed discussion of the model of computation, cf.

T. Kotek, J.A. Makowsky and E.V. Ravve,
A Computational Framework for the Study of Partition Functions and Graph Polynomials
Proceedings of the 12th Asian Logic Conference,
Wellington, New Zealand, 15 - 20 December 2011
Edited by: Rod Downey, Jörg Brendle, Robert Goldblatt and Byunghan Kim.
DOI: 10.1142/9789814449274_0012

Roots of Graph Polynomials

P-roots

It is an established topic to study the locations of the roots of graph polynomials.

For a fixed graph polynomial $P(G, X)$ typical statements about roots are:

(i) For every G the roots of $P(G, X)$ are real.

(ii) For every G all real roots of $P(G, X)$ are positive (negative) or the only real root is 0.

(iii) For every G the roots of $P(G, X)$ are contained in a disk of radius $\rho(p(G))$ where $p(G)$ is some numeric graph parameter (degree, girth, clique number, etc).

(iv) For every G the roots of $P(G, X)$ are contained in a disk of constant radius.

(v) The roots of $P(G, X)$ are dense in the complex plane.

(vi) The roots of $P(G, X)$ are dense in some absolute region.

Studying P-roots

We now overview polynomials P for which P-roots have been studied.

- Spectra of graphs, chromatic polynomial, matching polynomial, independence polynomial.
  Studying the location of their roots is motivated by applications in chemistry, statistical mechanics.
- Edge cover polynomial and domination polynomial.
  Studying the location of their roots is motivated by analogy only.
- All these polynomials are SOL-definable.
- All are univariate.

Spectral graph theory

Let $G(V,E)$ be a simple undirected graph with $|V| = n$, and
Let A_G and L_G be the (symmetric) adjacency resp. Laplacian matrix of G.

The characteristic polynomial of G is defined as

$$P_A(G,\lambda) = \det(\lambda \cdot 1 - A_G)$$

and the Laplacian polynomial of G is defined s

$$P_L(G,\lambda) = \det(\lambda \cdot 1 - L_G)$$

Theorem 6
The roots of $P_A(G,\lambda)$ and $P_L(G,\lambda)$ are all real.

There is a rich literature.

A.E. Brouwer and W. H. Haemers
Spectra of Graphs
Springer 2010.

The (vertex) chromatic polynomial

Let $G = (V(G), E(G))$ be a graph, and $\lambda \in \mathbb{N}$.

A **λ-vertex-coloring** is a map

$$c : V(G) \to [\lambda]$$

such that $(u, v) \in E(G)$ implies that $c(u) \neq c(v)$.

We define $\chi(G, \lambda)$ to be the number of λ-vertex-colorings

Theorem 7 (G. Birkhoff, 1912)
$\chi(G, \lambda)$ is a polynomial in $\mathbb{Z}[\lambda]$.

Proof:

(i) $\chi(E_n) = \lambda^n$ where E_n consists of n isolated vertices.

(ii) For any edge $e = E(G)$ we have $\chi(G - e, \lambda) = \chi(G, \lambda) + \chi(G/e, \lambda)$.

The Four Color Conjecture

Birkhoff wanted to prove the **Four Color Conjecture** using techniques from real or complex analysis.

Conjecture:(Birkhoff and Lewis, 1946)
If G is planar then $\chi(G, \lambda) \neq 0$ for $\lambda \in [4, +\infty) \subseteq \mathbb{R}$.

Theorem 8 (Birkhoff and Lewis, 1946)
For planar graphs G we have $\chi(G, \lambda) \neq 0$ for $\lambda \in [5, +\infty)$.

Still open: Are there planar graphs G such that

$\chi(G, \lambda) = 0$ for some $\lambda \in (4, 5)$?

More on chromatic roots, I

For real roots of χ we know:

Theorem 9 (Jackson, 1993, Thomassen, 1997)

For simple graphs G we have $\chi(G,\lambda) \neq 0$ for real $\lambda \in (-\infty, 0)$, $\lambda \in (0,1)$ and $\lambda \in (1, \frac{32}{27})$.
The only real roots $\leq \frac{32}{27}$ are 0 and 1.

The real roots of all chromatic polynomials are dense in $[\frac{32}{27}, \infty)$

More on chromatic roots, II

For complex roots of χ we know:

Theorem 10 (Sokal, 2004)

The complex roots are dense in $\mathbb{C}$.

The complex roots are bounded by $7.963907 \cdot \Delta(G) \leq 8 \cdot \Delta(G)$ where $\Delta(G)$ is the maximal degree of G.

We shall see that this is not a semantic property of the chromatic polynomial.

However, we have an interpretation in physics:

The chromatic polynomial corresponds to the zero-temperature limit of the antiferromagnetic Potts model. In particular, theorems guaranteeing that a certain complex open domain is free of zeros are often known as Lee-Yang theorems.

The above theorem says that no such domain exists.

More on chromatic roots, III

Theorem 11 (C. Thomassen, 2000)

If the chromatic polynomial of a graph has a real noninteger root less than or equal to

$$t_0 = \frac{2}{3} + \frac{1}{3}\sqrt[3]{26 + 6\sqrt{33}} + \frac{1}{3}\sqrt[3]{26 - 6\sqrt{33}} = 1.29559\ldots$$

Then the graph has no Hamiltonian path.

This result is best possible in the sense that it becomes false if t_0 is replaced by any larger number.

This is not a semantic property of the chromatic polynomial.

A semantic version would be:

The chromatic polynomial determines the existence of Hamiltonian paths..

The three matching polynomials

Let $m_i(G)$ be the number sets of independent edges of size i. We define

$$dm(G,x) = \sum_r (-1)^r m_r(G) x^{n-2r} \tag{6}$$

$$gm(G,x) = \sum_r m_r(G) x^r \tag{7}$$

$$M(G,x,y) = \sum_r m_r(G) x^r y^{n-2r} \tag{8}$$

We have $dm(G;x) = x^n gm(G;(-x)^{-2}) = M(G,-1,x)$ where $n =\mid V \mid$.

All three matching polynomials are d.p-equivalent.

Theorem 12 (Heilmann and Lieb 1972)
The roots of $dm(G,x)$ *are real and symmetrically placed around zero, i.e.,* $dm(G,x) = 0$ *iff* $dm(G,-x) = 0$

The roots of $gm(G,x)$ *are real and negative*

Independence polynomial

Let $in_i(G)$ be the number of independent sets of G of size i,
and the **independence polynomial**

$$I(G, X) = \sum_i in_i(G) X^i$$

Clearly there are no positive real independence roots.
For a survey see: V.E. Levit and E. Mandrescu,
The independence polynomial of a graph - a survey,
Proceedings of the 1st International Conference on Algebraic Informatics,
Thessaloniki, 2005, pp. 233-254.

J. Brown, C. Hickman and R. Nowakowski showed in Journal of Algebraic Combinatorics, 2004:

Theorem 13 (J. Brown, C. Hickman and R. Nowakowski, 2004)
The real roots are dense in $(-\infty, 0]$ *and the complex roots are dense in* $\mathbb{C}$.

Edge cover polynomial

Let $e_i(G)$ be the number of edge coverings of G of size i, and the **edge cover polynomial**

$$E(G,X) = \sum_i e_i(G) X^i$$

Theorem 14 (P. Csikvári and M.R.Oboudi, 2011)
All roots of $E(G,X)$ are in the ball

$$\{z \in \mathbb{C} : |z| \leq \frac{(2+\sqrt{3})^2}{1+\sqrt{3}} = \frac{(1+\sqrt{3})^3}{4}\}.$$

Domination polynomial

Inspired by the rich literature on dominating sets, **S. Alikhani** introduced in his Ph.D. thesis the **domination polynomial**;

Let $d_i(G)$ be the number of dominating sets of G of size i, and the **domination polynomial**

$$D(G,X) = \sum_i d_i(G) X^i$$

It is easy to see that 0 is a domination root, and that there are no real positive domination roots.

J. Brown and J. Tufts (Graphs and Combinatorics, , 2013) showed:

Theorem 15 (J. Brown and J. Tufts)
The domination roots are dense in $\mathbb{C}$.

D.p.-Equivalence and the

Location of the Roots

of SOL-Definable Graph Polynomials

From now on all graph polynomials

are supposed to be SOL-definable.

Roots vs distinctive power, I

Let $s(G)$ be a similarity function.

Theorem 16 (MRB)

For every univariate graph polynomial $P(G;X) = \sum_{i=0}^{s(G)} h_i(G)X^i$

where $s(G)$ *and* $h_i(G), i = 0, \ldots s(G)$ *are graph parameters with values in* $\mathbb{N}$,

there exists a univariate graph polynomials $Q_1(G;X)$,

prefactor equivalent to $P(G;X)$ *such that for every* G

all real roots of $Q_1(G;X)$ *are*

positive (negative) or the only real root is 0.

Roots vs distinctive power, II

Let $s(G)$ be a similarity function.

Theorem 17 (MRB)

For every univariate graph polynomial

$$P(G;X) = \sum_{i=0}^{i=s(G)} h_i(G)X^i \in \mathbb{Z}[X], \text{ resp. } \mathbb{R}[X]$$

there is a d.p.-equivalent graph polynomial

$$Q_2(G;X) = \sum_{i=0}^{i=s(G)} H_i(G)X^i \in \mathbb{Z}[X], \text{ resp. } \mathbb{R}[X]$$

such that all the roots of $Q(G;X)$ are real.

Roots vs distinctive power, III

Let $P(G;X)$ as before.

Theorem 18 (MRB)

For every univariate graph polynomial $P(G;X)$

there exist univariate graph polynomials $Q_3(G;X)$

substitution equivalent to $P(G;X)$ such that

for every G the roots of $Q_3(G;X)$ are contained in a disk of constant radius.

If we want to have all roots real and bounded in $\mathbb{R}$,

we have to require d.p.-equivalence.

Roots vs distinctive power, IV

Let $P(G; X)$ as before.

Theorem 19 (MRB)

For every univariate graph polynomial $P(G; X)$

there exists a univariate graph polynomial $Q_4(G; X)$

prefactor equivalent to $P(G; X)$ such that

$Q_4(G; X)$ has only countably many roots,

and its roots are **dense in the complex plane***.*

If we want to have all roots **real and dense in** *$\mathbb{R}$,*

we have to require d.p.-equivalence.

The proofs use various tricks!

Proofs: Theorem 16

Let $P(G,X) = \sum_i c_i(G)X^i = \sum_{A \subset V(G)^r} X^{|A|}$ be SOL-definable. We want to show:

For every G all real roots of $P(G,X)$ are negative.

This is true, because all coefficients of $P(G,X)$ are non-negative integers, due to SOL-definability.

If we want to find $Q_1(G;X)$ d.p.-equivalent to $P(G;X)$ such that

for every G all real roots of $Q_1(G,X)$ are positive,

we put $Q_1(G,X) = P(G,-X) = \sum_i c_i(G)(-X)^i = \sum_i (-1)^i c_i(G)(X)^i$.

If we want to find $Q_1'(G;X)$ d.p.-equivalent to $P(G;X)$ such that

for every G the only real root of $Q_1(G,X)$ is 0,

we put $Q_1'(G,X) = P(G,X^2) = \sum_i c_i(G)(X)^{2i}$.

Q.E.D.

Proofs: Theorem 17

Let $P(G, X)$ as before be SOL-definable.

We want to find $Q_3(G; X)$ d.p.-equivalent to $P(G; X)$ such that all roots of $Q_2(G; X)$ are real.

We define $Q_2(G; X) = \prod_{i=0}^{s(G)} (X - h_i(G))$.

Q.E.D.

Proofs: Theorem 18

Let $P(G, X)$ be SOL-definable.

We want to show:

For every G the roots of $Q_3(G, X)$ are contained in a disk of constant radius.

To relocate the roots of $P(G, X)$ we use Rouché's Theorem in the following form:

Lemma 20
Let $P(X) = \sum_{i=0}^{d} h_i X^i$ and $P'(X) = A \cdot X^{2d}$ with $A \geq \max_i\{h_i : 0 \leq i \leq d-1\}$.

Let $Q_3(X) = P(X) + P'(X)$.

Then all complex roots ξ of $Q_3(X)$ satisfy $|\xi| \leq 2$.

Clearly, $P'(G, X)$ is SOL-definable and d.p. equivalent to $P(G, X)$. Q.E.D.

Reference: P. Henrici, Applied and Computational Complex Analysis, volume 1,

Wiley Classics Library, John Wiley, 1988.

Section 4.10, Theorem 4.10c

Proofs: Theorem 19

Lemma 21
There exist univariate similarity polynomials $D^i_{\mathbb{C}}(G;X), i = 1,2,3,4$ of degree 12 such that all its roots of $D^i_{\mathbb{C}}(G;X)$ are dense in the ith quadrant of $\mathbb{C}$.

We use this lemma and put

$$Q_4(G;X) = \left(\prod_{i=1}^{i=4} D^i(G;X)\right) \cdot P(G;X).$$

To get the real roots to be dense we proceed similarily.

Q.E.D.

Are the locations of P-roots semantically meaningfull?

Our results seems to suggest:

- The location of P-roots depends strongly on the syntactic presentation of P.
- We still don't understand the particular rôle syntactic presentation of graph polynomials have to play.
- d.p. equivalence garantees that the information conveyed by coefficients or roots is inherent in every presentation.
  The choice of presentation only serves in making it more or less visible.
- Although the location of chromatic roots is easily interpretable, the same is not true for edge cover or domination roots.
- The study of P-roots needs better justifications besides mere mathematical curiosity.

The rôle of recurrence relations

The chromatic polynomial, Tutte polynomial and the matching polynomial satisfy **recurrence relations** of the type

$$P(G, X) = \alpha \cdot P(G_{-e}, X) + \beta \cdot P(G_{/e} X) + \gamma \cdot P(G_{\dagger e}, X)$$

where G_{-e} is deletion of the edge e,
$G_{/e}$ is contraction of the edge e, and
$G_{\dagger e}$ is extraction of the edge e, and
$\alpha, \beta, \gamma \in \mathbb{Z}[X]$ are suitable polynomials.

It is conceivable, and the proofs use these relations, that the location of the corresponding P-roots are intrinsically related to these recurrence relations.

Note: It is not clear how recurrence relations **behave** under d.p. equivalence.

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Note: Ilia Averbouch, PhD Thesis, Haifa, February 2011

"Completeness and Universality Properties of Graph Invariants and Graph Polynomials",

http://www.cs.technion.ac.il/ janos/RESEARCH/averbouch-PhD.pdf

Thank you for your attention!
