APPROXIMATIONS OF ALGEBRAS BY STANDARDLY STRATIFIED ALGEBRAS

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Abstract. The paper has its origin in an attempt to answer the following question: Given an arbitrary finite dimensional associative K-algebra A, does there exist a quasi-hereditary algebra B such that the subcategories of all A-modules and all B-modules, filtered by the corresponding standard modules are equivalent. Such an algebra will be called a *quasi-hereditary approximation* of A. The question is answered in the appropriate language of standardly stratified algebras: For any K-algebra A, there is a uniquely defined basic algebra $B = \Sigma(A)$ such that B_B is Δ-filtered and the subcategories $\mathcal{F}(\Delta_A)$ and $\mathcal{F}(\Delta_B)$ of all Δ-filtered modules are equivalent; similarly there is a uniquely defined basic algebra $C = \Omega(A)$ such that C_C is $\bar{\Delta}$-filtered and the subcategories $\mathcal{F}(\bar{\Delta}_A)$ and $\mathcal{F}(\bar{\Delta}_C)$ of all $\bar{\Delta}$-filtered modules are equivalent. These subcategories play a fundamental role in the theory of stratified algebras. Since, in general, it is difficult to localize these subcategories in the category of all A-modules, the construction of $\Sigma(A)$ and $\Omega(A)$ often helps to describe them explicitly. By applying consecutively the operators Σ and Ω for an algebra, we get a sequence of standardly stratified algebras which, after a finite number of steps, stabilizes in a properly stratified algebra. Thus, all standardly stratified algebras are partitioned into (generally infinite) trees, indexed by properly stratified algebras (as their roots).

1. Introduction

Let $(A, \mathbf{e})$ be a finite dimensional K-algebra with a (linearly) ordered complete set $\mathbf{e} = (e_1, \ldots, e_n)$ of primitive orthogonal idempotents. Let $\Delta_A = (\Delta(1), \Delta(2), \ldots, \Delta(n))$ and $\bar{\Delta}_A = (\bar{\Delta}(1), \bar{\Delta}(2), \ldots, \bar{\Delta}(n))$ be the respective sequences of (right) standard and properly standard A-modules. Hence, we have the well-defined (full) subcategories $\mathcal{F}(\Delta_A)$ and $\mathcal{F}(\bar{\Delta}_A)$ of all Δ_A-filtered and $\bar{\Delta}_A$-filtered A-modules, of the category mod-A of all finite dimensional (right) A-modules, respectively.

The concept of standardly stratified algebra (i. e. of Δ- and of $\bar{\Delta}$-filtered algebra) has its origin in the concept of a quasi-hereditary algebra introduced by Cline–Parshall–Scott [CPS] in order to deal with highest weight categories as they arise in the representation theory of semisimple complex Lie algebras and algebraic

1991 Mathematics Subject Classification. Primary 16E30.

[1] Research partially supported by Hungarian NFSR grant no. T043034

[2] Research partially supported by NSERC of Canada grant no. A-7257

Part of this work was done in the framework of the Canadian–Hungarian exchange project CAN-04/8

groups. The subcategories $\mathcal{F}(\Delta_A)$ and $\mathcal{F}(\bar{\Delta}_A)$ of mod-A of all Δ- and $\bar{\Delta}$-filtered modules of such algebras play a fundamental role in the theory.

In [DR] Dlab and Ringel established a simple characterization of the category $\mathcal{F}(\Delta_A)$ of a quasi-hereditary algebra in terms of a "standardizable" set of an abelian K-category. Their method, consisting of presenting the quasi-hereditary algebra as the endomorphism algebra of the direct sum of the relevant indecomposable Ext-projective objects, has been reformulated and applied in a number of papers (e.g. [ES], [MMS1], [MMS2]).

Note that one of the corollaries of their result is the following statement: *Given an arbitrary algebra whose standard and proper standard modules coincide, there is a unique basic quasi-hereditary algebra A_q such that $\mathcal{F}(\Delta_A)$ and $\mathcal{F}(\Delta_{A_q})$ are equivalent via an exact functor.*

Here and throughout the paper we shall assume that the equivalence functors between $\mathcal{F}(\Delta_A)$ and $\mathcal{F}(\Delta_B)$ are exact, meaning that sequences of Delta-filtered modules which are short exact in mod-A or mod-B are mapped into short exact sequences in the other module category.

In the present paper we are going to use this method to extend this result to standardly stratified algebras (Theorem 2.2 and 2.3) and to investigate two equivalences $\overset{\Delta}{\sim}$ and $\overset{\bar{\Delta}}{\sim}$ in the class of all algebras $(A, \mathbf{e})$: we shall say that $(A, \mathbf{e}) \overset{\Delta}{\sim} (A', \mathbf{e}')$ if and only if $\mathcal{F}(\Delta_A) \approx \mathcal{F}(\Delta_{A'})$ and $(A, \mathbf{e}) \overset{\bar{\Delta}}{\sim} (A', \mathbf{e}')$ if and only if $\mathcal{F}(\bar{\Delta}_A) \approx \mathcal{F}(\bar{\Delta}_{A'})$, the equivalence in both cases being induced by an exact functor. The respective equivalence classes are, up to fully described exceptions, infinite (cf. Theorem 3.3. and 3.5). The main point is the fact that every $\overset{\Delta}{\sim}$-equivalence class is represented by a unique basic Δ-filtered algebra and every $\overset{\bar{\Delta}}{\sim}$-equivalence class by a unique basic $\bar{\Delta}$-filtered algebra (cf. Theorem 2.2 and Theorem 2.3).

This process allows us to define two operators Σ and Ω on the class of all algebras $(A, \mathbf{e})$ with a given ordering on the simple types. The range of these operators will be the union of the class $\mathcal{A}(\Delta)$ of all basic Δ-filtered algebras and the class $\mathcal{A}(\bar{\Delta})$ of all basic $\bar{\Delta}$-filtered algebras. Recall that for a basic algebra $(A, \mathbf{e}) \in \mathcal{A}(\Delta)$ means that the regular representation A_A belongs to $\mathcal{F}(\Delta_A)$ and $(A, \mathbf{e}) \in \mathcal{A}(\bar{\Delta})$ means that $A_A \in \mathcal{F}(\bar{\Delta})$. Thus the class $\mathcal{A}(\Delta) \cap \mathcal{A}(\bar{\Delta})$ consists of all properly stratified algebras in the sense of [D2].

We define $\Sigma(A)$ as the unique algebra such that $\Sigma(A) \in \mathcal{A}(\Delta)$ and $A \overset{\Delta}{\sim} \Sigma(A)$. Similarly we define $\Omega(A)$ by $\Omega(A) \in \mathcal{A}(\bar{\Delta})$ and $A \overset{\bar{\Delta}}{\sim} \Omega(A)$. Note that Σ acts as the identity operator on $\mathcal{A}(\Delta)$ while Ω acts as the identity operator on $\mathcal{A}(\bar{\Delta})$. We shall investigate the action of the operators Σ and Ω, mostly on $\mathcal{A}(\Delta) \cup \mathcal{A}(\bar{\Delta})$.

In particular, we shall show that for every algebra A with n (non-isomorphic) simple modules

$$(\Omega\Sigma)^{n-1}(A) = \Sigma(\Omega\Sigma)^{n-1}(A)$$

(see Theorem 4.1). Defining a partial order $\preceq$ on $\mathcal{A}(\Delta) \cup \mathcal{A}(\bar{\Delta})$ by taking $A' \preceq A$ if and only if A' can be obtained from A by successive applications of the operators Σ and Ω, the class $\mathcal{A}(\Delta) \cup \mathcal{A}(\bar{\Delta})$ becomes a (disjoint) union of rooted trees whose roots are in one-to-one correspondence with the properly stratified algebras. In other words, the orbits of the action of the semigroup generated by the operators Σ and Ω carry a natural tree structure and they are indexed by properly stratified algebras.

The results of this paper were reported at the conference ICTAMI 2005 in Alba Iulia by I. Ágoston on Sept. 16, 2005 and at the Representation theory seminar of University of Bielefeld by V. Dlab on September 26, 2005.

2. Δ and $\bar{\Delta}$ equivalence of algebras

Throughout the paper we shall assume that A is a finite dimensional basic algebra over a field K. We shall fix in A a complete set of primitive orthogonal idempotents: $\mathbf{e} = (e_1, \ldots, e_n)$ such that $1 = e_1 + \cdots + e_n$, together with its ordering inherited from the natural ordering of the index set. The indecomposable projective (right) modules will be denoted by $P(i) \simeq e_iA$, and the corresponding simple tops by $S(i) = P(i)/\operatorname{rad} P(i)$, while the *standard modules* (with respect to the given order) are $\Delta(i) = e_iA\big/e_iA(e_{i+1} + \cdots + e_n)A$ and the *proper standard modules* are $\bar{\Delta}(i) = e_iA\big/e_i \operatorname{rad} A(e_i + \cdots + e_n)A$ for $1 \le i \le n$. Thus the standard module $\Delta(i)$ is the largest quotient of $P(i)$ such that the composition multipicity $[\Delta(i) : S(j)]$ is 0 for $j > i$, while $\bar{\Delta}(i)$ is the largest quotient of $\Delta(i)$ such that $[\bar{\Delta}(i) : S(i)] = 1$.

Recall that in some of the earlier papers $(A, \mathbf{e})$ is said to be *standardly stratified* if the right regular module A_A belongs to $\mathcal{F}(\Delta_A)$ while in others it is said to be standardly stratified if $A_A \in \mathcal{F}(\bar{\Delta}_A)$. Let us reiterate that $\mathcal{F}(\Delta_A)$ (or $\mathcal{F}(\bar{\Delta}_A)$) is the full subcategory of mod-A consisting of modules X with a filtration $X = X_0 \supseteq X_1 \supseteq \ldots \supseteq X_\ell \supseteq X_{\ell+1} = 0$ such that for every $0 \le j \le \ell$ the quotient $X_j/X_{j+1} \simeq \Delta(i)$ (or $X_j/X_{j+1} \simeq \bar{\Delta}(i)$) for some $1 \le i \le n$. By a result of [D1] $A_A \in \mathcal{F}(\bar{\Delta}_A)$ if and only if $A^{opp}_{A^{opp}} \in \mathcal{F}(\Delta_{A^{opp}})$. In this spirit, in order to streamline our formulations, we shall use throughout the paper the terminology of Δ-*filtered algebras* (i. e. when $A_A \in \mathcal{F}(\Delta_A)$) and $\bar{\Delta}$-*filtered algebras* (i. e. when $A_A \in \mathcal{F}(\bar{\Delta}_A)$). Those algebras that are either Δ-filtered or $\bar{\Delta}$-filtered will be then called *standardly stratified*. We believe that this terminology is more appropriate and hope that it will be generally accepted.

The algebra $(A, \mathbf{e})$ is *quasi-hereditary* if and only if it is Δ-filtered and $\Delta(i) = \bar{\Delta}(i)$ for all $1 \le i \le n$. Note that quasi-hereditary algebras are those Δ-filtered algebras which have finite global dimension. For elementary properties of standard modules, quasi-hereditary algebras and standardly stratified algebras we refer to [DR], [ADL] and [CPS].

Theorem 2 of [DR] provides a full characterization of the category $\mathcal{F}(\Delta_A)$ for a quasi-hereditary algebra A by listing some characterizing homological properties of the standard modules. This characterization also leads to an explicit construction: given a subcategory $\mathcal{C}$ of modules satisfying these requirements we can construct a unique quasi-hereditary algebra $(A, \mathbf{e})$ such that its $\mathcal{F}(\Delta_A)$ is equivalent to $\mathcal{C}$.

It turns out that by making several adjustments and by taking care of some technicalities, we can establish a similar characterization in the case of standardly stratified algebras (see Proposition 2.1). In fact, such a generalization can be found also in the paper [ES] (although with slightly different emphasis and not explicitly referring to the corresponding 'standardization theorem' of [DR]). As a consequence, given an algebra A, there is a uniquely defined representative in the class of all basic Δ-filtered algebras B whose categories $\mathcal{F}(\Delta_B)$ are equivalent to $\mathcal{F}(\Delta_A)$ (Theorem 2.2). In a similar spirit, we can establish the existence of a uniquely defined representative in the class of all basic $\bar{\Delta}$-filtered algebras C whose categories $\mathcal{F}(\bar{\Delta}_C)$ are equivalent to $\mathcal{F}(\bar{\Delta}_A)$ for a given algebra A (Theorem 2.3).

Let us recall here the above mentioned characterization of the category $\mathcal{F}(\Delta)$ over a quasi-hereditary algebra (cf. Theorem 2 of [DR]). Given a subcategory $\mathcal{C}$ of a module category mod-A, this subcategory $\mathcal{C}$ is equivalent to $\mathcal{F}(\Delta_B)$ for some quasi-hereditary algebra $(B, \mathbf{e})$ if and only if $\mathcal{C} = \mathcal{F}(\Theta)$, for a finite set of indecomposable objects $\Theta = \{\, \Theta(i) \in \mathcal{C} \,|\, 1 \le i \le n \,\}$ satisfying the following conditions:

(1) $\operatorname{Hom}_A(\Theta(i), \Theta(j)) = 0$ for $1 \le j < i \le n$;
(2) $\operatorname{Ext}^1_A(\Theta(i), \Theta(j)) = 0$ for $1 \le j < i \le n$;
(3) $\operatorname{Ext}^1_A(\Theta(i), \Theta(i)) = 0$ for $1 \le i \le n$;
(4) $\operatorname{Hom}_A(\Theta(i), \Theta(i))$ is a division algebra for $1 \le i \le n$.

Note that the indecomposability of the objects in Θ actually follows from condition (4). However we prefer assuming indecomposability in our formulation since for characterizing Δ-filtered modules of Δ-filtered algebras we just omit the condition (4). In [DR] the elements of Θ are called *standardizable objects* of $\mathcal{C}$. Let us note here that standardizable objects may be identified within the category as the only objects which do not admit a non-trivial filtration within this category.

It is a well-known fact that standard modules over a quasi-hereditary algebra satisfy these conitions. To prove the sufficiency of these conditions one can show first that there are enough Ext-projective objects in the category $\mathcal{C}$. In fact, there are precisely n indecomposable (non-isomorphic) Ext-projective modules. Denoting by M their direct sum, $B = \operatorname{End}_A(M)$ is basic quasi-hereditary algebra and $\operatorname{Hom}_A(M, -)$ defines a categorical equivalence between $\mathcal{C} = \mathcal{F}(\Theta)$ and $\mathcal{F}(\Delta_B)$. (Let us point out that the endomorphisms of right A-modules will be written from the left.) Since for a quasi-hereditary algebra $\mathcal{F}(\Delta)$ contains the projective modules (and they can be identified as the Ext-projective objects of the category), the algebra itself is uniquely determined by $\mathcal{F}(\Delta)$ as the endomorphism algebra of the direct sum of the indecomposable Ext-projective objects. (Note that in [ES], using a dual approach and dealing with Ext-injective objects instead of Ext-projectives such systems, consisting of standardizable objects and the indecomposable Ext-injectives were called *stratifying systems*.)

The differences between quasi-hereditary algebras and Δ-filtered algebras stem from the fact that standard modules of Δ-filtered algebras are not necessarily Schurian, i. e. condition (4) above is not, in general, satisifed. If we retain the remaining conditions, we get a characterization of $\mathcal{F}(\Delta_A)$ for Δ-filtered algebras.

Proposition 2.1. *Let $\mathcal{C}$ be a full subcategory of* mod-A *of an arbitrary finite dimensional algebra. Then $\mathcal{C}$ is equivalent to $\mathcal{F}(\Delta_B)$ of a Δ-filtered algebra $(B, \mathbf{e})$ via an exact functor if and only if $\mathcal{C} = \mathcal{F}(\Theta)$ for a finite set of indecomposable objects $\Theta = \{\, \Theta(i) \in \mathcal{C} \,|\, 1 \le i \le n \,\}$ satisfying the conditions* (1), (2) *and* (3) *above. Moreover, the algebra B is unique up to Morita equivalence.*

Proof. For the proof, we refer to Theorem 2 of [DR]. The only major difference is that in the recursive construction of the Ext-projective objects $P_\Theta(i)$, the resulting module does not have to be indecomposable, but it will have a unique indecomposable direct summand containing $\Theta(i)$ in its top. Note that, in general, the Ext-projective modules will not be local. (See Example 2.9 at the end of this section). □

It is easy to see that the set of standard modules of any algebra A satisfies the above conditions (1)-(3). Thus an immediate consequence of Proposition 2.1 is the following theorem.

THEOREM 2.2. *Let $(A, \mathbf{e})$ be a finite dimensional algebra. Then there exists a unique basic Δ-filtered algebra $(B, \mathbf{f})$ such that the categories $\mathcal{F}(\Delta_A)$ and $\mathcal{F}(\Delta_B)$ are equivalent via an exact functor. In this case the number of isomorphism types of simple A-modules and simple B-modules is the same.*

Unlike standard modules, proper standard modules are Schurian. Thus, they satisfy the condition (4). On the other hand, in general, proper standard modules have self-extensions, i. e. they fail to satisfy (3). However, we can formulate a statement parallel to Theorem 2.2.

THEOREM 2.3. *Let $(A, \mathbf{e})$ be a finite dimensional algebra. Then there exists a unique basic $\bar{\Delta}$-filtered algebra $(C, \mathbf{g})$ such that the categories $\mathcal{F}(\bar{\Delta}_A)$ and $\mathcal{F}(\bar{\Delta}_C)$ are equivalent via an exact functor. In this case the number of isomorphism types of simple A-modules and simple C-modules is the same.*

Proof. Let us follow the line of proof of Theorem 2 in [DR], by constructing enough Ext-projective objects in $\mathcal{F}(\bar{\Delta}_A)$, namely n indecomposable modules $N(i)$, $1 \leq i \leq n$, such that:

(i) $N(i) \in \mathcal{F}(\bar{\Delta}(i), \bar{\Delta}(i+1), \dots, \bar{\Delta}(n))$;
(ii) there exists an epimorphism $N(i) \to \bar{\Delta}(i)$ and
(iii) $N(i)$ is Ext-projective in $\mathcal{F}(\bar{\Delta}_A)$, i. e. $\mathrm{Ext}^1_A(N(i), \bar{\Delta}(\ell)) = 0$ for all $1 \leq \ell \leq n$.

The modules $N(i)$ will be defined recursively, step by step, constructing a sequence of A-modules $Q(i,j)$, $i \leq j \leq n$, such that each $Q(i,j)$ satisfies the following conditions:

$(i)'$ $Q(i,j) \in \mathcal{F}(\bar{\Delta}(i), \bar{\Delta}(i+1), \dots \bar{\Delta}(j))$;
$(ii)'$ there exists an epimorphism $Q(i,j) \to \bar{\Delta}(i)$;
$(iii)'$ $\mathrm{Ext}^1_A(Q(i,j), \bar{\Delta}(\ell)) = 0$ for $1 \leq \ell \leq j$.

Obviously, $N(i) = Q(i,n)$ will then satisfy the conditions (i), (ii) and (iii).

Let us start the construction by defining $Q(i,i)$ to be the maximal quotient of $\Delta(i)$ belonging to $\mathcal{F}(\bar{\Delta}(i))$. Due to the fact that $\mathrm{Ext}^1_A(\bar{\Delta}(i), \bar{\Delta}(\ell)) = 0$ for all $\ell < i$, only the condition $\mathrm{Ext}^1_A(Q(i,i), \bar{\Delta}(i)) = 0$ requires a proof. Applying, for $1 \leq \ell \leq i-1$, the functor $\mathrm{Hom}_A(-, S(\ell))$ to the exact sequence

$$0 \to Z \to \Delta(i) \to Q(i,i) \to 0 \tag{2.3.1}$$

we see that $\mathrm{Hom}_A(Z, S(\ell))$=0 and thus, due to the maximality of $Q(i,i)$, we get that $\mathrm{Hom}(Z, \bar{\Delta}(i))) = 0$. Consequently, applying $\mathrm{Hom}_A(-, \bar{\Delta}(i))$ to (2.3.1), we conclude that $\mathrm{Ext}^1_A(Q(i,i), \bar{\Delta}(i)) = 0$, as required.

Proceeding by induction, assume that $Q(i, j-1)$for some $i < j \leq n$ has already been constructed. For convenience we write $Q(i, j-1) = Q$ and consider the universal extension U_1 of Q by $\bar{\Delta}(j)$:

$$0 \to X_1 = \bigoplus_{d_1} \bar{\Delta}(j) \to U_1 \to Q \to 0. \tag{2.3.2}$$

Here $d_1 = \dim_{D_j} \mathrm{Ext}^1_A(Q, \bar{\Delta}(j))$, where $D_j = \mathrm{End}_A(\bar{\Delta}(j))$. (The universality of the extension means that the pushout sequences along the projection maps $X_1 \to \bar{\Delta}(j)$ form a basis for $\mathrm{Ext}^1_A(Q, \bar{\Delta}(j))$.) Clearly, in addition to the conditions $(i)'$ and $(ii)'$, U_1 satisfies, by recursion, $\mathrm{Ext}^1_A(U_1, \bar{\Delta}(\ell)) = 0$ for all $1 \leq \ell \leq j-1$. In general, however, $\mathrm{Ext}^1_A(U_1, \bar{\Delta}(j)) \neq 0$; denote its D_j-dimension by d_2 and construct the universal extension U_2 of $\bar{\Delta}(j)$ by U_1:

$$0 \to X_2 = \bigoplus_{d_2} \bar{\Delta}(j) \to U_2 \to U_1 \to 0.$$

This sequence yields the following derived exact sequence:

$$0 \to \overline{X}_2 \to U_2 \to Q \to 0,$$

where $\overline{X}_2 \in \mathcal{F}(\bar{\Delta}(j))$ is an extension of X_2 by X_1. If $\mathrm{Ext}^1_A(U_2, \bar{\Delta}(j)) \neq 0$, we continue this process. In t steps we get – again by means of constructing the universal extensions

$$0 \to X_t \to U_t \to U_{t-1} \to 0 \tag{2.3.3}$$

of $\bar{\Delta}(j)$ by U_{t-1} – the corrseponding sequence:

$$0 \to \overline{X}_t \to U_t \to Q \to 0.$$

Note that, in each step of this procedure, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & X_t & \to & \overline{X}_t & \to & \overline{X}_{t-1} & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & X_t & \to & U_t & \to & U_{t-1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & Q & = & Q & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{2.3.4}$$

Here, by recursion $(i)'$, $(ii)'$ and $\mathrm{Ext}^1_A(U_t, \bar{\Delta}(\ell)) = 0$ hold for $1 \leq \ell \leq j-1$.

We are going to show that after a finite number of steps, the process of constructing the universal extensions will stabilize, i. e. that $\mathrm{Ext}^1_A(U_{t_0}, \bar{\Delta}(j)) = 0$ for some t_0.

Indeed, we can show by induction that $\mathrm{Hom}_A(\overline{X}_t, \bar{\Delta}(j)) \simeq \mathrm{Ext}^1_A(Q, \bar{\Delta}(j))$. The statement clearly holds for $\overline{X}_1 = X_1$ by the universality of the extension (2.3.2). For arbitrary $t > 1$ we can apply the functor $\mathrm{Hom}_A(-, \bar{\Delta}(j))$ to the diagram in (2.3.4) to get the following commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccc}
 & & & & & & \uparrow \\
0 \to \mathrm{Hom}(\overline{X}_{t-1},\bar\Delta(j)) & \xrightarrow{\alpha} & \mathrm{Hom}(\overline{X}_t,\bar\Delta(j)) & \to & \mathrm{Hom}(X_t,\bar\Delta(j)) & \xrightarrow{\beta} & \mathrm{Ext}^1(\overline{X}_{t-1},\bar\Delta(j)) \\
\uparrow & & \uparrow & & \| & & \uparrow \delta \\
0 \to \mathrm{Hom}(U_{t-1},\bar\Delta(j)) & \to & \mathrm{Hom}(U_t,\bar\Delta(j)) & \to & \mathrm{Hom}(X_t,\bar\Delta(j)) & \xrightarrow{\gamma} & \mathrm{Ext}^1(U_{t-1},\bar\Delta(j)) \\
 & & & & & & \uparrow \\
 & & & & & & \mathrm{Ext}^1(Q,\bar\Delta(j)) \\
 & & & & & & \uparrow \varphi \\
 & & & & & & \mathrm{Hom}(\overline{X}_{t-1},\bar\Delta(j)) \\
 & & & & & & \uparrow
\end{array}
$$

Here γ is an isomorphism since (2.3.3) is a universal extension, furthermore φ is an isomorphism by induction. Thus we get that δ is injective and so is β. This implies that α is an isomorphism which, in view of the induction hypothesis, yields the statement.

Observe that the isomorphism $\mathrm{Hom}_A(\overline{X}_t, \bar{\Delta}(j)) \simeq \mathrm{Ext}^1_A(Q, \bar{\Delta}(j))$ implies that $\mathrm{Hom}_A(\overline{X}_t, \bar{\Delta}(j)) \simeq \mathrm{Hom}_A(X_1, \bar{\Delta}(j))$ for each t. Note also that $\overline{X}_t$ is an extension of a module in $\mathcal{F}(\bar{\Delta}(j))$ by $X_1 = \bigoplus_{d_1} \bar{\Delta}(j)$. Hence, the previous isomorphism implies that $\overline{X}_t$ is a homomorphic image of $\oplus_{d_1} \Delta(j)$, and thus its dimension is bounded. Since $\dim X_1 < \dim \overline{X}_2 < \ldots < \dim \overline{X}_t$ we get that the sequence of the universal extensions must, after a finite number of steps, stabilize, i. e. $\mathrm{Ext}^1_A(U_{t_0}, \bar{\Delta}(j)) = 0$ for some t_0. We set $Q(i,j) = U_{t_0}$.

Thus, using this recursion we have constructed the Ext-projective objects $N(i)$ in $\mathcal{F}(\bar{\Delta}_A)$. To show that the modules $N(i)$ are indecomposable, we need the following lemma.

Lemma 2.4. *$\mathcal{F}(\bar{\Delta}_A)$ is closed under taking direct summands.*

Proof. Let M be an element of $\mathcal{F}(\bar{\Delta}_A)$, and suppose that $M = U \oplus V$. Since $\mathrm{Ext}_A(\bar{\Delta}_A(n), \bar{\Delta}_A(i)) = 0$ for $i \neq n$, $Me_nA = Ue_nA \oplus Ve_nA \in \mathcal{F}(\bar{\Delta}_A(n))$ and $M/Me_nA \simeq U/Ue_nA \oplus V/Ve_nA \in \mathcal{F}(\bar{\Delta}_A(1), \ldots, \bar{\Delta}_A(n-1))$. So it suffices to prove the statement for $Me_nA \in \mathcal{F}(\bar{\Delta}_A(n))$, and apply induction on the factor module. For simplicity assume that $M = Me_nA$. Then, $M \in \mathcal{F}(\bar{\Delta}_A(n))$ implies that $0 \neq \mathrm{Hom}_A(M, \bar{\Delta}_A(n)) = \mathrm{Hom}_A(U, \bar{\Delta}_A(n)) \oplus \mathrm{Hom}_A(V, \bar{\Delta}_A(n))$ so one of the summands, say, $\mathrm{Hom}_A(U, \bar{\Delta}_A(n))$ is nontrivial. But the top of U, and thus the top of any nonzero homomorphic image of U is filtered by $S(n)$, so a nonzero homomorphism from U to $\bar{\Delta}_A(n)$ must be an epimorphism. This means that $M/(U_1 \oplus V)$ is isomorphic to $\bar{\Delta}_A(n)$ for some $U_1 \leq U$, and thus $U_1 \oplus V$ is $\bar{\Delta}_A(n)$-filtered because $\mathcal{F}(\bar{\Delta}_A(n))$ is closed under kernels of epimorphisms (cf. [ADL]). Recursively we can prove that both U and V are $\bar{\Delta}_A(n)$-filtered. □

Now we can prove the indecomposability of $N(i)$, by showing that in the recursive construction of $N(i)$, every module $Q(i,j)$ is indecomposable. The initial module $Q(i,i)$ is a quotient of the local module $\Delta(i)$, hence it is indecomposable. Suppose now that $Q(i,j-1)$ is indecomposable for some $i<j\le n$. We constructed $Q(i,j)$ as an extension of a $\bar{\Delta}_A(j)$-filtered module X by $Q(i,j-1)$:

$$0 \to X \to Q(i,j) \to Q(i,j-1) \to 0,$$

and we also know that in the long exact sequence

$$\cdots \to \operatorname{Hom}_A(Q(i,j),\bar{\Delta}_A(j)) \xrightarrow{\beta} \operatorname{Hom}_A(X,\bar{\Delta}_A(j)) \xrightarrow{\alpha} \operatorname{Ext}^1_A(Q(i,j-1),\bar{\Delta}_A(j)) \to \cdots$$

the morphism α is an isomorphism. Thus $\beta = 0$.

Now suppose that $Q(i,j) = U \oplus V$ is a proper decomposition of $Q(i,j)$. Since $X = Q(i,j)e_jA = Ue_jA \oplus Ve_jA$, we have $Q(i,j-1) \simeq U/Ue_jA \oplus V/Ve_jA$. The indecomposability of $Q(i,j-1)$ implies that one of the components in the latter decomposition is 0. We may assume that $U \subseteq X$. The previous lemma implies that U is $\bar{\Delta}_A(j)$-filtered. But then an epimorphism from U to $\bar{\Delta}_A(j)$ gives a homomorphism in $\operatorname{Hom}_A(Q(i,j),\bar{\Delta}_A(j))$, which has a nonzero restriction to X. This is a contradiction, since $\beta = 0$.

This proves that each $Q(i,j)$ and thus each $N(i)$ must be indecomposable for $1 \le i \le n$.

Put $N = \overset{n}{\underset{i=1}{\oplus}} N(i)$ and $C = \operatorname{End}_A(N)$.

To show that C is a basic $\bar{\Delta}$-filtered algebra and that the functor $\operatorname{Hom}_A(N,-)$ induces an equivalence between $\mathcal{F}(\bar{\Delta}_A)$ and $\mathcal{F}(\bar{\Delta}_C)$ we can follow almost word by word the rest of the proof of Theorem 2 in [DR]. This task is left to the reader. □

In view of Theorems 2.2 and 2.3, we can introduce the following definitions.

Definition 2.5. (1) The algebras $(A,\mathbf{e})$ and $(B,\mathbf{f})$ are called Δ-equivalent if the respective full subcategories $\mathcal{F}(\Delta_A) \subseteq$ mod-A and $\mathcal{F}(\Delta_B) \subseteq$ mod-B are equivalent via an exact functor; in this case we write $(A,\mathbf{e}) \overset{\Delta}{\sim} (B,\mathbf{f})$ or simply $A \overset{\Delta}{\sim} B$.

(2) The algebras $(A,\mathbf{e})$ and $(B,\mathbf{f})$ are called $\bar{\Delta}$-equivalent if the respective full subcategories $\mathcal{F}(\bar{\Delta}_A) \subseteq$ mod-A and $\mathcal{F}(\bar{\Delta}_B) \subseteq$ mod-B are equivalent via an exact functor; in this case we write $(A,\mathbf{e}) \overset{\bar{\Delta}}{\sim} (B,\mathbf{f})$ or simply $A \overset{\bar{\Delta}}{\sim} B$.

In this way we get two equivalence relations on the class of all algebras (or rather, on Morita equivalence classes of algebras).

Definition 2.6. For an arbitrary algebra $(A,\mathbf{e})$ we define $\Sigma(A)$ to be the unique algebra satisfying:

(*i*) $(\Sigma(A),\mathbf{f})$ is Δ-filtered and basic;

(*ii*) $A \overset{\Delta}{\sim} \Sigma(A)$.

Similarly we define $\Omega(A)$ to be the unique algebra satisfying:

(*i*)′ $(\Omega(A),\mathbf{f})$ is $\bar{\Delta}$-filtered and basic;

(*ii*)′ $A \overset{\bar{\Delta}}{\sim} \Omega(A)$.

Thus, $A \overset{\Delta}{\sim} B$ if and only if $\Sigma(A) \simeq \Sigma(B)$ with the isomorphism preserving the corresponding orderings. In a similar fashion, $A \overset{\bar{\Delta}}{\sim} B$ if and only if $\Omega(A) \simeq \Omega(B)$.

The explicit construction in the proof of Proposition 2.1 and Theorem 2.3 gives us a bound on the dimension of these algebras:

PROPOSITION 2.7. *There exist functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ such that for any algebra A we have:*

$$\dim \Sigma(A) \leq f(\dim A) \qquad \textit{and} \qquad \dim \Omega(A) \leq g(\dim A).$$

Proof. We will not make any attempt to give an optimal bound: our estimate will be very rough and in most cases far from the best possible bound.

Since $\Sigma(A)$ and $\Omega(A)$ can be obtained as the endomorphism algebras of the direct sum of indecomposable Ext-projective modules in $\mathcal{F}(\Delta)$ and $\mathcal{F}(\bar{\Delta})$, respectively, it is enough to show that there is an upper bound on the dimension of these indecomposable Ext-projective modules, since their number is n, the number of isomorphism types of simple A-modules, and this is not greater than $\dim A = d$.

First we show that for modules of bounded dimension the dimension of their first extension group is also bounded. Let us take two A-modules, X and Y with their dimensions bounded by x and y, respectively. If $0 \to \Omega_1(X) \to P_0 \to X \to 0$ is a projective cover of X, then $\dim \Omega_1(X) \leq \dim P_0 \leq \dim A \cdot \dim X \leq dx$. The long exact sequence $\cdots \to \mathrm{Hom}_A(\Omega_1(X), Y) \to \mathrm{Ext}^1_A(X, Y) \to 0$ yields that $\dim \mathrm{Ext}^1_A(X, Y) \leq \dim \mathrm{Hom}_A(\Omega_1(X), Y) \leq dxy$.

Thus if Z is the universal extension of X with Y, i.e. we have $0 \to Y^k \to Z \to X \to 0$ with $k = \dim \mathrm{Ext}^1_A(X, Y)$, then $\dim Z \leq x + ky \leq x + dxy^2 = x(1 + dy^2)$.

We can apply this estimate to the recursive construction of the indecomposable Ext-projective modules $M_\Delta(i)$ in $\mathcal{F}(\Delta_A)$, to their direct sum M and to $\Sigma(A) = \mathrm{End}_A(M)$. We use the bound $\dim(\Delta(i)) \leq d$ to get:

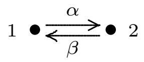
$$\dim M_\Delta(i) \leq d(1 + d^3)^{n-i} \leq d(1 + d^3)^n$$
$$\dim M \leq nd(1 + d^3)^n$$
$$\dim \Sigma(A) \leq n^2 d^2 (1 + d^3)^{2n}$$

Since the number of simple module types n is cleary not more than d, we get the desired function f.

In the recursive construction of the indecomposble Ext-projective modules $N_{\bar{\Delta}}(i)$ in $\mathcal{F}(\bar{\Delta}_A)$ we have seen that when one of the intermediate modules X is extended by a module filtered by $\bar{\Delta}(j)$-s then the latter module is the homomorphic image of the direct sum of k copies of $\Delta(j)$-s where $k = \dim \mathrm{Ext}^1_A(X, \bar{\Delta}(j)$. Hence we get the earlier recursive estimate for the dimension of the indecomposable Ext-projective objects: $\dim N_{\bar{\Delta}}(i) \leq d(1 + d^3)^n$. Thus we also get the same estimate for $\dim \Omega(A)$ as for $\dim \Sigma(A)$, namely: $\dim \Omega(A) \leq n^2 d^2 (1 + d^3)^{2n}$. This gives the function g. □

At the end of this section, let us give some examples for these constructions.

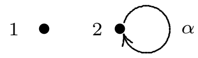
EXAMPLE 2.8. Let us consider the algebra $A = KQ_A/I_A$ whose quiver Q_A and right regular representation are as follows:

Q_A:  ; $I_A = \langle \alpha\beta \rangle$; $A_A = \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\\2\end{smallmatrix}$.

Then the direct sum M of the indecomposable Ext-projective objects in $\mathcal{F}(\Delta_A)$ is:

$$M_A = 1 \oplus \begin{smallmatrix}2\\1\\2\end{smallmatrix} ;$$

hence $\Sigma(A) = \mathrm{End}_A(M) = KQ_{\Sigma(A)}/I_{\Sigma(A)}$ is given by:

$Q_{\Sigma(A)}$:  ; $I_{\Sigma(A)} = \langle \alpha^2 \rangle$;

${}_{\Sigma(A)}\Sigma(A) = 1 \oplus \begin{smallmatrix}2\\2\end{smallmatrix}$; $\Sigma(A)_{\Sigma(A)} = 1 \oplus \begin{smallmatrix}2\\2\end{smallmatrix}$.

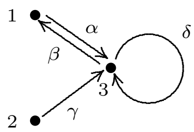
On the other hand, for the Ext-projective object N in $\mathcal{F}(\bar{\Delta}_A)$ we get

$$N_A = 1 \oplus \begin{smallmatrix}2\\1\end{smallmatrix} ;$$

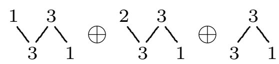
hence $\Omega(A) = \mathrm{End}_A(N) = KQ_{\Omega(A)}$ is given by:

$Q_{\Omega(A)}$: ; ${}_{\Omega(A)}\Omega(A) = \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 2$; $\Omega(A)_{\Omega(A)} = 1 \oplus \begin{smallmatrix}2\\1\end{smallmatrix}$.

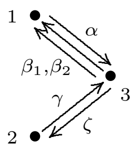
EXAMPLE 2.9. Let us take the algebra $A = KQ_A/I_A$ whose quiver Q_A and right regular representation are as follows:

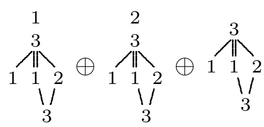
Q_A:  ; $I_A = \langle \delta^2, \delta\beta, \alpha\beta, \gamma\delta, \beta\alpha \rangle$; $A_A = \begin{smallmatrix}1\\3\\3\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\\1\end{smallmatrix} \oplus \begin{smallmatrix} & 3 & \\ 1 & & 3\end{smallmatrix}$.

Then the direct sum M of the indecomposable Ext-projective objects in $\mathcal{F}(\Delta_A)$ is:

$M_A =$ 

and its endomorphism ring $\Sigma(A) = \mathrm{End}_A(M) = KQ_{\Sigma(A)}/I_{\Sigma(A)}$ is given by:

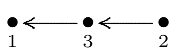
$Q_{\Sigma(A)}$:  ; $I_{\Sigma(A)} = \langle \beta_1\alpha, \beta_2\alpha - \zeta\gamma, \beta_2\alpha\beta_1, \beta_2\alpha\beta_2, \beta_2\alpha\zeta \rangle$;

$\Sigma(A)_{\Sigma(A)} =$  .

The Ext-projective object N in $\mathcal{F}(\bar{\Delta})$ is given by

$$N_A = 1 \oplus \begin{smallmatrix}2\\3\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\end{smallmatrix}$$

and its endomorphism algebra $\Omega(A) = KQ_{\Omega(A)}$ is as follows:

$Q_{\Omega(A)}$:  ; $\Omega(A)_{\Omega(A)} = 1 \oplus \begin{smallmatrix}2\\3\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\end{smallmatrix}$.

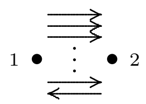
3. The size of equivalence classes

In this section we will look more closely at the equivalence classes with respect to the relations $\overset{\Delta}{\sim}$ and $\overset{\bar{\Delta}}{\sim}$.

It may happen that the categories $\mathcal{F}(\Delta)$ or $\mathcal{F}(\bar{\Delta})$ fully determine the algebra, more precisely the whole module category. For example, when all standard A modules are simple — note that this fact can be recognized within $\mathcal{F}(\Delta_A)$ since this means that the standardizable objects are Schurian and there are no non-trivial homomorphisms between different standardizable objects — then $\mathcal{F}(\Delta_A)$ is the full module category. Thus any algebra Δ-equivalent to A must be Morita equivalent to A. A similar situation arises when the proper standard modules are simple.

In the above situations the corresponding $\overset{\Delta}{\sim}$ or $\overset{\bar{\Delta}}{\sim}$ class has only one (basic) element. On the other hand the following two examples show that some equivalence classes are infinite.

Example 3.1. Let A_k for $k \geq 1$ be the algebras whose quiver and right regular representation are as follows:



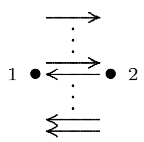
and $\quad (A_k)_{A_k} = \begin{smallmatrix} & 1 & \\ 2 & 2 \cdots & 2 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}$

with k arrows heading from 1 to 2. Here the Ext-projective module M in $\mathcal{F}(\Delta_A)$ and its endomorphism algebra $\Sigma(A_k)$ are given by:

$$M_{A_k} = 1 \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \; ; \qquad {}_{\Sigma(A_k)}\Sigma_{A_k} = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus 2 \; ; \qquad \Sigma(A_k)_{\Sigma_{A_k}} = 1 \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \; ;$$

thus, $\Sigma(A_k)$ is independent of k, i.e. it is isomorphic for every algebra A_k. Note that $\mathcal{F}(\Delta_{A_k}) = \mathcal{F}(\bar{\Delta}_{A_k})$ and $\mathcal{F}(\Delta_{\Sigma(A_k)}) = \mathcal{F}(\bar{\Delta}_{\Sigma(A_k)})$; hence, $\Omega(A_k) = \Sigma(A_k)$.

Example 3.2. Let us now consider the algebras B_k for $k \geq 1$ whose quivers and right regular representation are as follows:



and $\quad (B_k)_{B_k} = \begin{smallmatrix} & 1 & \\ 2 & 2 \cdots & 2 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 1 & 1 \cdots & 1 \\ & 2 & \end{smallmatrix} \; ;$

here, there are k arrows $\alpha_1, \ldots, \alpha_k$ from 1 to 2 and $k+1$ arrows $\beta_0, \ldots, \beta_k$ from 2 to 1 satisfying the following relations: $\alpha_j \beta_\ell = 0$ for any $1 \leq j \leq k$ and $0 \leq \ell \leq k$ and $\beta_i \alpha_j = 0$ for $i \neq j$ and $\beta_i \alpha_i = \beta_j \alpha_j$ for any $1 \leq i, j \leq k$. Then an easy calculation shows that $\Delta_{B_k}(1)$ and $\Delta_{B_k}(2)$ are Ext-projective in $\mathcal{F}(\Delta_{B_k})$. By taking for M their direct sum, the algebra $\Sigma(B_k) = \mathrm{End}_{B_k}(M)$ does not depend on k and it can be described by the regular representations

$${}_{\Sigma(B_k)}\Sigma(B_k) = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \; ; \qquad \Sigma(B_k)_{\Sigma(B_k)} = 1 \oplus \begin{smallmatrix} & 2 \\ 1 & 2 \end{smallmatrix} \; .$$

It turns out that these two extreme cases exhaust all possibilities: apart from one element classes, the equivalence classes are always infinite.

THEOREM 3.3. *Let* $(A, \mathbf{e})$ *be an arbitrary algebra. Then the number of Morita equivalence classes of algebras which are* Δ*-equivalent to* A *is:*
(i) *one if all standard modules* $\Delta(i)$ *for* $2 \leq i \leq n$ *are simple;*
(ii) *infinite otherwise.*

Proof. The fact that a standard module $\Delta(i)$ is simple is clearly invariant under Δ-equivalence: it means that $\mathrm{Hom}_A(\Delta(j), \Delta(i)) = 0$ for $j \neq i$ and $\Delta(i)$ is Schurian. Furthermore, if all standard modules $\Delta_A(i)$ are simple for $2 \leq i \leq n$ then the algebra A must be Δ-filtered. Since every Δ-equivalence class contains a unique basic Δ-filtered algebra, we are done with case (i).

We have to show now that if at least one of the standard modules $\Delta(i)$ for $i \geq 2$ is not simple then there are infinitely many non-isomorphic basic algebras which are Δ-equivalent to A.

To this end let us first formulate a technical lemma, giving a general framework for the construction of these algebras.

LEMMA 3.4. *Let* U *be an* (A, A)*-bimodule,* $\Phi = \mathrm{Hom}_A(U_A, A_A)$ *and* $X = \mathrm{Tr}_A(U_A)$ *the trace of* U_A *in* A_A *(i. e.* $X = \sum\{\,\mathrm{Im}\,\varphi \mid \varphi \in \Phi\,\}$*). Thus* Φ *and* X *also carry a natural* (A, A)*-bimodule structure. Assume that* $XU = UX = 0$*. Then the* (A, A)*-bimodule*

$$\tilde{A} = A \oplus U \oplus \Phi$$

can be given an associative algebra structure as follows: multiplication by elements of A *is given by the* (A, A)*-bimodule structure;* $U\Phi = UU = \Phi\Phi = 0$*; finally* $\varphi \cdot u = \varphi(u)$ *for* $\varphi \in \Phi$ *and* $u \in U$*. Furthermore* U *is a right ideal of* $\tilde{A}$ *such that*

$$\mathrm{End}(\tilde{A}/U)_{\tilde{A}} \simeq A.$$

Proof. First, the assumption $XU = UX = 0$ implies that $XX = 0$ and $\Phi X = X\Phi = 0$. Using these relations, it is easy to verify that the multiplication

$$(a, u, \varphi) \cdot (a', u', \varphi') = (aa' + \varphi(u'), au' + ua', a\varphi' + \varphi a')$$

is associative.

Clearly, since $(0, u, 0)(a', u', \varphi') = (0, ua', 0)$, U is a right ideal of $\tilde{A}$.

Moreover, every endomorphism of the (right) $\tilde{A}$-module $\tilde{A}/U$ is induced by left multiplication by an element (a_0, u_0, φ_0) of $\tilde{A}$ such that $(a_0, u_0, \varphi_0)U \subseteq U$. As a consequence, in view of $(a_0, u_0, \varphi_0)(0, u, 0) = (\varphi_0(u), a_0 u, 0)$ for all $u \in U$, we have $\varphi_0 = 0$. But then, modulo U,

$$(a_0, u_0, 0)(a, 0, \varphi) = (a_0 a, u_0 a, a_0 \varphi) \sim (a_0 a, 0, a_0 \varphi) = (a_0, 0, 0)(a, 0, \varphi).$$

Thus, $\mathrm{End}_{\tilde{A}}(\tilde{A}/U) \simeq A$.

□

Returning to the proof of Theorem 3.3, we define the (A,A)-bimodule $U = \oplus S^\circ(1)\otimes_K e_iA/e_i \operatorname{rad} A(e_1+\ldots+e_i)A$ (here $\oplus S^\circ(1)$ is the direct sum of any finite number of copies of the left A-module $S^\circ(1)$). Define $\tilde{A}$ as in Lemma 3.4. We are going to prove that $\Sigma(\tilde{A}) = A$.

Note that the conditions that $\Delta(i)$ is not simple but $\Delta(j)$ is simple for all $j > i$ imply that $e_i \operatorname{rad} A(e_1+\ldots+e_i) \neq 0$ and $e_j \operatorname{rad} Ae_k = 0$ for all $j > i$ and $k \leq j$.

First we verify that the relations in the construction of $\tilde{A}$ in Lemma 3.4 are satisfied, i.e. that $X := Tr_A(U) \subseteq \operatorname{rad} A$ and, as a consequence, $XU = UX = 0$. Indeed, the definition of U yields $U = Ue_iA$ and $U \operatorname{rad} A(e_1+\ldots+e_i) = 0$, so $X = Xe_iA$ and $X \operatorname{rad} A(e_1+\ldots+e_i) = 0$. Now for $j \neq i$, $e_jX = e_jXe_iA \subseteq e_i \operatorname{rad} A$. To prove that e_iX is also in $\operatorname{rad} A$, we first observe that $e_iX \operatorname{rad} A(e_1+\ldots+e_i) = 0$; but $e_i \operatorname{rad} A(e_1+\ldots+e_i) \neq 0$, so $e_i \notin e_iX$. Thus e_iX is a proper submodule of the local module e_iA, hence $e_iX \subseteq e_i \operatorname{rad} A$. This finishes the proof of the first statement. The rest follows from $XU \subseteq (\operatorname{rad} A)U = 0$ and $UX = UXe_iA \subseteq U(\operatorname{rad} A)e_iA = 0$.

Second, let us show that $\tilde{A}/U$ is Δ-filtered. Observe that the condition that $\Delta(j)$ are simple for $j > i$ means that $e_j \operatorname{rad} Ae_k = 0$ for $j > i$ and $k \leq j$, and that this property is inherited by the algebra $\tilde{A}$: $(e_{i+1}+\ldots+e_n)U = 0$ and $(e_{i+1}+\ldots+e_n)\Phi U = (e_{i+1}+\ldots+e_n)\Phi Ue_iA \subseteq (e_{i+1}+\ldots+e_n)Ae_iA = 0$ implies $(e_{i+1}+\ldots+e_n)\Phi = 0$, and thus $e_j \operatorname{rad} \tilde{A}e_k \subseteq e_j \operatorname{rad} Ae_k = 0$ for $j > i$ and $k \leq j$.

It is easy to check that $\tilde{A}/\tilde{A}(e_{i+1}+\ldots+e_n)\tilde{A}$ is isomorphic to the algebra that we obtain by the same construction from $A/A(e_{i+1}+\ldots+e_n)A$. So it is sufficient to prove that $\tilde{A}/U$ is Δ-filtered in the case when $i = n$.

In this case, since A is Δ-filtered, Ae_nA is $\Delta(n)$-filtered, i.e. $Ae_nA \simeq \oplus e_nA$. The ismorphism naturally induces an isomorphism from $Ae_n\Phi = \operatorname{Hom}(U_A, Ae_nA_A)$ to the direct sum of copies of $e_n\Phi = \operatorname{Hom}(U_A, e_nA_A)$ as right A-modules. On the other hand, $\tilde{A}e_n\tilde{A}/U = (Ae_nA + Ae_n\Phi + Ue_nA)/U = (Ae_nA + Ae_n\Phi)/U$, while $e_n\tilde{A} = e_nA + e_n\Phi$, so this proves that $\tilde{A}e_n\tilde{A}/U$ is Δ-filtered . To finish the proof we only need to observe that $\tilde{A}/U + \tilde{A}e_n\tilde{A} = \tilde{A}/\tilde{A}e_n\tilde{A} \cong A/Ae_nA$, since $U+\Phi \subseteq Ae_nA$, and this shows that the $\Delta(j)'s$ of $\tilde{A}$ for $j < n$ are the same as those of A and $\tilde{A}/U$ is Δ-filtered.

Finally, we show that $\tilde{A}/U$ is the direct sum of indecomposable Ext-projectives in the category of Δ-filtered right $\tilde{A}$-modules:

Since $\tilde{A}/U$ is the direct sum of local modules with tops $S(1),\ldots,S(n)$, the only thing left to prove is that $\operatorname{Ext}^1(\tilde{A}/U, \Delta_{\tilde{A}}(j)) = 0$ for all j. If we apply the $\operatorname{Hom}(-, \Delta_{\tilde{A}}(j))$ functor on the short exact sequence $0 \to U \to \tilde{A} \to \tilde{A}/U \to 0$, then we see that $\operatorname{Ext}^1(\tilde{A}/U, \Delta_{\tilde{A}}(j)) = 0$ if and only if the morphism $\operatorname{Hom}(\tilde{A}, \Delta_{\tilde{A}}(j)) \to \operatorname{Hom}(U, \Delta_{\tilde{A}}(j))$ is surjective. This condition is easily satisfied for $j \neq i$ because in that case $U = Ue_iA$ (and the simplicity of $\Delta_{\tilde{A}}(j)$ for $j > i$) implies that $\operatorname{Hom}(U, \Delta_{\tilde{A}}(j)) = 0$.

In the case when $j = i$, we can assume again that $i = n$. Under this condition $\Delta_{\tilde{A}}(n) = e_n\tilde{A} = e_nA + e_n\Phi$, and $\Phi = \Phi e_1$, while $Ue_1 = 0$, so $\operatorname{Hom}(U, \Delta_{\tilde{A}}(n)) = \operatorname{Hom}(U, e_nA) = e_n\Phi$. Let $\varphi \in \operatorname{Hom}(U, e_nA)$, and define $\alpha \in \operatorname{Hom}(\tilde{A}, e_n\tilde{A})$ with $\alpha(\tilde{a}) = \varphi\tilde{a}$. Since $\varphi\tilde{A} \subseteq e_n\Phi\tilde{A} \subseteq e_n\tilde{A}$, we get that $\alpha \in \operatorname{Hom}(\tilde{A}, \Delta_{\tilde{A}}(n))$, and $\alpha(u) = \varphi u = \varphi(u)$, so α is an extension of φ. This proves that the morphism $\operatorname{Hom}(\tilde{A}, \Delta_{\tilde{A}}(n)) \to \operatorname{Hom}(U, \Delta_{\tilde{A}}(n))$ is surjective, thus implying that $\tilde{A}/U$ is an Ext-projective module in the construction of $\Sigma(\tilde{A})$. Now, applying Lemma 3.4, this shows that $\Sigma(\tilde{A}) \cong A$.

□

Let us now formulate the parallel statement for $\bar{\Delta}$-equivalence.

THEOREM 3.5. *Let* $(A, \mathbf{e})$ *be an arbitrary algebra. Then the number of Morita equivalence classes of algebras which are* $\bar{\Delta}$*-equivalent to* A *is*
(i) *one if all standard modules* $\bar{\Delta}(i)$ *for* $2 \le i \le n$ *are simple;*
(ii) *infinite otherwise.*

Proof. The proof of case (i) is similar to that of the corresponding case of Theorem 3.3.

To prove case (ii), we could slightly modify the construction in the proof of Theorem 3.3. For later use, however, we shall give now a different construction showing that if at least one of the modules $\bar{\Delta}(i)$ for $i \ge 2$ is non-simple then there are infinitely many algebras in the $\bar{\Delta}$-equivalence class of A. (Recall that $\bar{\Delta}(1)$ is always a simple module.)

Thus, let i be such that $\bar{\Delta}(i)$ is not simple and $\bar{\Delta}(j)$ is simple for all $j > i$. Let us define the following (A, A)-bimodule: $L = Ae_i \otimes_K S(i)$. Finally let $\tilde{A}$ be defined as the trivial extension of A by L, i. e.

$$\tilde{A} = L \rtimes A = \left\{ \begin{pmatrix} a & \ell \\ 0 & a \end{pmatrix} \middle| \, a \in A, \ \ell \in L \right\}.$$

(Note that for path algebras this means adding one extra loop α at vertex i and an additional defining relation $\alpha^2 = 0$.) We want to show that A and $\tilde{A}$ are $\bar{\Delta}$-equivalent. Then, repeating the construction we can get infinitely many non-isomorphic basic algebras which are all $\bar{\Delta}$-equivalent to A.

Note that there is a natural action of $\tilde{A}$ on all A-modules and the modules $S(i)$ for $1 \le i \le n$ give a natural set of representatives of all simple $\tilde{A}$-modules. Furthermore, L is an ideal in A contained in $\operatorname{rad} A$, isomorphic as a right $\tilde{A}$-module to a direct sum of simple modules of type $S(i)$. This implies that for each indecomposable projective $\tilde{A}$-module $P_{\tilde{A}}(j)$, we get an exact sequence of $\tilde{A}$-modules

$$0 \to K(j) \to P_{\tilde{A}}(j) \to P_A(j) \to 0,$$

where $P_A(j)$ is the corresponding indecomposable projective A-module; moreover, $K(j) \simeq \oplus S(i)$.

Now, let us observe that the proper standard A-modules $\bar{\Delta}_A(j)$ are — as $\tilde{A}$-modules — isomorphic to the proper standard modules $\bar{\Delta}_{\tilde{A}}(j)$ for $1 \le j \le n$. This holds because the choice (the maximality) of i implies that L has a trivial intersection with the idecomposable projectives $e_j\tilde{A}$ for $j > i$, while for $j \le i$ the kernel of the epimorphism $e_j\tilde{A} \to \bar{\Delta}_{\tilde{A}}(j)$ contains $L \cap e_j\tilde{A}$ since L is a direct sum of $S(i)$-s, contained in the radical of A.

This also implies that modules in $\mathcal{F}(\Delta_A)$ also belong to $\mathcal{F}(\Delta_{\tilde{A}})$. In particular the direct sum M of indecomposable Ext-projective modules in $\mathcal{F}(\Delta_A)$ belongs to $\mathcal{F}(\Delta_{\tilde{A}})$. To show that $\bar{\Delta}(A)$ and $\bar{\Delta}(\tilde{A})$ are isomorphic, it is enough to show that M remains Ext-projective in $\mathcal{F}(\Delta_{\tilde{A}})$.

To this end let us take the projective cover $P_A(M)$ of M over A and the projective cover $P_{\tilde{A}}(M)$ of M over $\tilde{A}$. Then we get the following diagram of $\tilde{A}$-modules:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & K' & \to & K'' & \to & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \tilde{K} & \to & P_{\tilde{A}}(M) & \to & M & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & K & \to & P_A(M) & \to & M & \to & 0 \end{array}$$

Here, as mentioned earlier, $K' \simeq K'' \simeq \oplus S(i)$, moreover the map $\tilde{K} \to K$ must be surjective. In view of our choice of i, there are no non-zero homomorphisms $S(i) \to \bar{\Delta}(j)$ for $1 \le j \le n$, and thus $\mathrm{Hom}_{\tilde{A}}(K, \bar{\Delta}(j)) \simeq \mathrm{Hom}_{\tilde{A}}(\tilde{K}, \bar{\Delta}(j))$ and $\mathrm{Hom}_{\tilde{A}}(P_A(M), \bar{\Delta}(j)) \simeq \mathrm{Hom}_{\tilde{A}}(P_{\tilde{A}}(M), \bar{\Delta}(j))$. Since $\mathrm{Ext}^1_A(M, \bar{\Delta}(j)) = 0$, the map $\mathrm{Hom}_A(P_A(M), \bar{\Delta}(j)) \to \mathrm{Hom}_A(K, \bar{\Delta}(j))$ is surjective. Using the previous isomorphisms we get that $\mathrm{Hom}_{\tilde{A}}(P_{\tilde{A}}(M), \bar{\Delta}(j)) \to \mathrm{Hom}_{\tilde{A}}(\tilde{K}, \bar{\Delta}(j))$ is also surjective. This means that $\mathrm{Ext}^1_{\tilde{A}}(M, \bar{\Delta}(j)) = 0$ and shows that M is Ext-projective in $\mathcal{F}(\bar{\Delta}_{\tilde{A}})$. The proof is completed. □

Let us observe that from the construction of $\tilde{A}$ it is easy to derive (say, by a dimension counting argument) that if the original algebra A is Δ-filtered then so is $\tilde{A}$. Thus, we get the following corollary.

Corollary 3.6. *If a $\bar{\Delta}$-equivalence class has more than one element and contains at least one Δ-filtered algebra then it contains infinitely many non-isomorphic basic Δ-filtered algebras.*

4. The orbit graph of the operators Σ and Ω

As before, all algebras in this section will be basic. Let us point out that the equivalence $(A, \mathbf{e}) \overset{\Delta}{\sim} (B, \mathbf{f})$ (or $(A, \mathbf{e}) \overset{\bar{\Delta}}{\sim} (B, \mathbf{f})$) implies the respective equivalence for the factor algebras $\mathrm{fact}_i(A) = A\big/A(e_{i+1} + \cdots + e_n)A$ and $\mathrm{fact}_i(B) = B\big/B(f_{i+1} + \cdots + f_n)B$ for all $1 \le i \le n$. This follows from the fact that in the equivalence between the categories of Δ-filtered (or $\bar{\Delta}$-filtered) modules over A and B, the modules filtered by $\Delta(j)$'s (or $\bar{\Delta}(j)$'s) with $j \le i$ correspond to each other. Consequently,

$$\Sigma(\mathrm{fact}_i(A)) \simeq \mathrm{fact}_i(\Sigma(A))$$

and

$$\Omega(\mathrm{fact}_i(A)) \simeq \mathrm{fact}_i(\Omega(A)).$$

Theorem 4.1. *Denote the number of the (non-isomorphic) simple A-modules by n. Then the algebra $(\Omega\Sigma)^{n-1}(A)$ is properly stratified.*

For the proof of the theorem we shall need the following lemma:

Lemma 4.2. *Let A be a Δ-filtered algebra such that the factor algebra* $\mathrm{fact}_{n-1}(A)$ *is $\bar{\Delta}$-filtered. Then $\Omega(A)$ is properly stratified.*

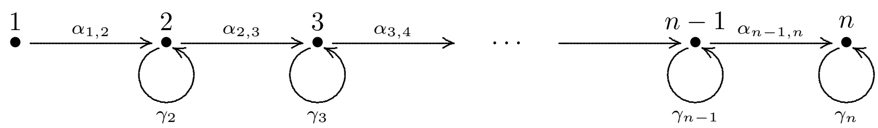
Proof. Let e_nA/e_nX be a maximal $\bar{\Delta}$-filtered factor of e_nA. Then $\mathrm{Hom}(e_nX, \bar{\Delta}(j)) = 0$ is true for all j: for $j < n$ it follows from $[\bar{\Delta}(j) : S(n)] = 0$, while for $j = n$, any nontrivial homomorphism from e_nX to $\bar{\Delta}(n)$ must be surjective, and thus bijective, so the existence of such a homomorphism would contradict the maximality of the factor e_nA/e_nX.

Now, consider the ideal $I = Ae_nX$ of A. Since Ae_nA is a direct sum of the modules $\Delta(n) = e_nA$, the ideal I is a direct sum of the submodules e_nX. Thus $\mathrm{Hom}(I, \bar{\Delta}(j)) = 0$ for all $1 \leq j \leq n$. Since $\mathrm{Ext}^1(A, \bar{\Delta}(j)) = 0$, also $\mathrm{Ext}^1(A/I, \bar{\Delta}(j)) = 0$ for all $1 \leq j \leq$. Consequently, in view of the fact that A/I is a direct sum of n $\bar{\Delta}$-filtered factors of the projective modules $P(j) = e_jA$, A/I is the Ext-projective module used in the construction of $\Omega(A)$, i. e. $\Omega(A) \simeq \mathrm{End}_A(A/I) \simeq A/I$.

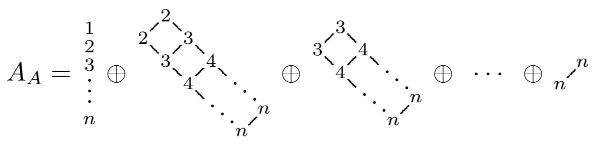
Since $\Omega(A)$ must be $\bar{\Delta}$-filtered, we only need to prove that A/I is Δ-filtered. The assumption of the lemma gives that A/Ae_nA is Δ-filtered, and we saw that $Ae_nA/I \simeq \oplus\, e_nA/e_nI$, so Ae_nA is $\Delta(n)$-filtered. This finishes the proof that $\Omega(A)$ is properly stratified. □

Proof of Theorem 4.1. Let us proceed by induction. The statement trivially holds for $n = 1$. Assume now that the statement holds for algebras with $n-1$ simple modules. Then $(\Omega\Sigma)^{n-2}(A\big/Ae_nA)$ is properly stratified. Thus, denoting $\Sigma(\Omega\Sigma)^{n-2}$ by Π, we have $\Pi(A\big/Ae_nA) \simeq \Pi(A)\big/\Pi(A)e_n\Pi(A)$ is $\bar{\Delta}$-filtered. Furthermore, $\Pi(A)$ is Δ-filtered by definition. Hence, applying the lemma to $\Pi(A)$, we get $\Omega\Pi(A) = (\Omega\Sigma)^{n-1}(A)$ is properly stratified. □

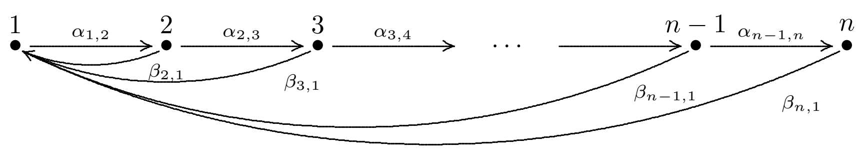
Example 4.3. The following example shows that the bound in Theorem 4.1 is optimal. Let A be the algebra given as $A = KQ_A/I_A$, where Q_A is given by:



and $I_A = \langle \gamma_i^2, \gamma_i\alpha_{i,i+1} - \alpha_{i,i+1}\gamma_{i+1} \,|\, 2 \leq i \leq n-1\rangle$. Thus the right regular representation of A can be described as follows:

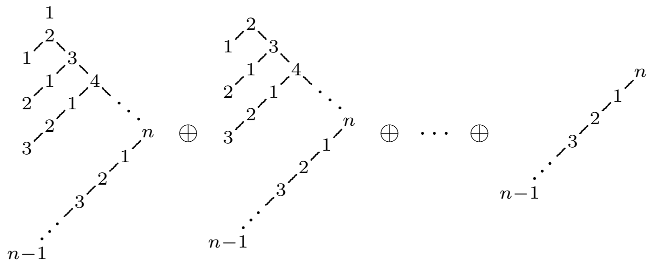


Then $(\Omega\Sigma)^{n-1}(A) = B = KQ_B/I_B$, where Q_B is given by:



and $I_B = \langle \beta_{i1}\alpha_{12}\alpha_{23}\cdots\alpha_{i-1,i} \,|\, 2 \le i \le n\rangle$ with regular decomposition:

$B_B =$



Here $\Sigma(\Omega\Sigma)^{n-2}(A)$ is Δ-filtered but not $\bar{\Delta}$-filtered. The last projective (i.e. the last standard module) is a uniserial module with a composition series of length $n+1$ as follows: $S(n), S(1), S(2), \ldots, S(n-1), S(n)$.

Let us now take the Cayley-graph of this action of the operators Σ and Ω, restricted to the class of all standardly stratified algebras (A, e). Thus, we define an arrow of type Σ from A to $\Sigma(A)$ and an arrow of type Ω from A to $\Omega(A)$.

For this graph, as an immediate consequence of Theorem 4.1 we get the following corollary.

Corollary 4.4. *The family of all basic standardly stratified algebras with n non-isomorphic simple modules is a disjoint union of oriented trees of algebras, indexed by properly stratified algebras as their roots. The height of these trees is bounded by* $2(n-1)$.

Note that although Theorem 4.1 is valid for general algebras, we have restricted our formulation of Corollary 4.4. to standardly stratified algebras, where it seems to be possible to describe also the proper preimages $\Sigma^{-1}(A)$ and $\Omega^{-1}(A)$ (i. e. the preimages not including the algebra itself), of a given algebra. In the family of all algebras this may be an impossible task. A more detailed description of the structure of this graph will be presented in a separate paper. Here we conclude our discussion with two remarks only, illustrating the complexity of the question.

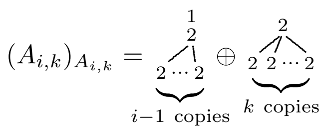
Corollary 3.6 immediately implies that if A is a standardly stratified algebra then the proper preimage $\Omega^{-1}(A)$ is either empty or it is infinite. (Note that we have excluded the algebra A from its proper preimage.) Namely, if $\Omega^{-1}(A)$ is non-empty then A is $\bar{\Delta}$-filtered and its $\bar{\Delta}$-equivalence class contains at least one Δ-filtered algebra, not isomorphic to A. Thus by Corollary 3.6 it contains infinitely many Δ-filtered elements, hence $\left|\Omega^{-1}(A)\right| = \infty$.

On the other hand the following example shows that the cardinality of $\Sigma^{-1}(A)$ can be equal to any natural number.

Example 4.5. The following examples of algebras show that the Δ-equivalence classes of algebras can contain an arbitrary finite number of $\bar{\Delta}$-filtered algebras.

Let $k \in \mathbb{N}$, $k \ge 1$ be given and consider the algebras $A_{i,k}$ defined for $1 \le i \le k$ as $A_{i,k} = KQ_{A_{i,k}}/I_{A_{i,k}}$ with $Q_{A_{i,k}}$ having two vertices, one arrow α from 1 to 2 and k loops at 2, denoted by $\beta_1, \ldots, \beta_k$, subject to the relations $I_{A_{i,k}} = \langle \beta_p\beta_q,\ \alpha\beta_r,\ \mid 1 \le$

$p, q \leq k,\ i \leq r \leq n\rangle$. Thus the right regular decomposition of $A_{i,k}$ can be described as follows:



Clearly each algebra $A_{i,k}$ is $\bar{\Delta}$-filtered, moreover $A_{i,k}$ is a homomorphic image of $A_{j,k}$ for $i \leq j$. In this way we can say that the standard modules for $A_{1,k}$ are also standard modules for each $A_{i,k}$ and $\dim \operatorname{Ext}^1_{A_{i,k}}(\Delta(1), \Delta(2)) = k$ for each $1 \leq i \leq k$. Hence the universal extension construction of $\Delta(2)$ by $\Delta(1)$ over $A_{1,k}$ gives the Ext-projective module for every algebra $A_{i,k}$, $1 \leq i \leq k$. This implies that $\Sigma(A_{i,k}) \simeq \Sigma(A_{j,k})$ for $1 \leq i, j \leq k$.

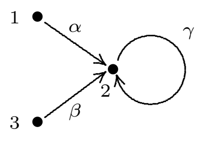
We want to show that there is no other $\bar{\Delta}$-filtered algebra A for which $\Sigma(A)$ is isomorphic to $\Sigma(A_{i,k})$. Suppose that A and $A_{1,k}$ are Δ-equivalent and A is $\bar{\Delta}$-filtered. Then it is easy to see that $\Delta_A(2)$ must not contain a simple module of type $S_A(1)$ in its socle, since this would give a nonzero homomorphism in $\operatorname{Hom}_A(\Delta_A(1), \Delta_A(2))$ although such a homomorphism does not exist in $\mathcal{F}(\Delta_{A_{i,k}})$. Since A is $\bar{\Delta}$-filtered, we get that $\Delta_A(2)$ is homogeneous, containing only simple factors of type $S(2)$. Now it is easy to see that the structure of $\Delta_A(2)$ is well described by its endomorphism ring $\operatorname{End}_A(\Delta_A(2))$ which is isomorphic to $\operatorname{End}_{A_{1,k}}(\Delta_{A_{i,k}}(2))$. Now, knowing the structure of $\operatorname{Ext}^1_A(\Delta_A(1), \Delta_A(2)$, we get that $\operatorname{rad} P_A(1)/\operatorname{rad}^2 P_A(1)$ is isomorphic to $S_A(2)$, hence $\operatorname{rad} P_A(1)$ is a homomorphic image of $\Delta_A(2)$. This implies that, depending on the composition length of $P_A(1)$, the algebra A must be isomorphic to one of the algebras $A_{i,k}$.

5. An example of Δ-equivalence

Let us conclude the paper by exhibiting the subcategories of Δ-filtered modules in one particular case. Compare the inclusions of the subcateogories $\mathcal{F}(\Delta_A)$ and $\mathcal{F}(\Delta_{\Sigma(A)})$ in the Auslander–Reiten quiver of A and $\Sigma(A)$.

Example 5.1. Let $A = KQ_A/I_A$ be the algebra given by the following quiver and right regular representation:

Q_A:



; $\quad I_A = \langle \alpha\gamma, \gamma^2 \rangle; \quad A_A = \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\2\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\\2\end{smallmatrix}$.

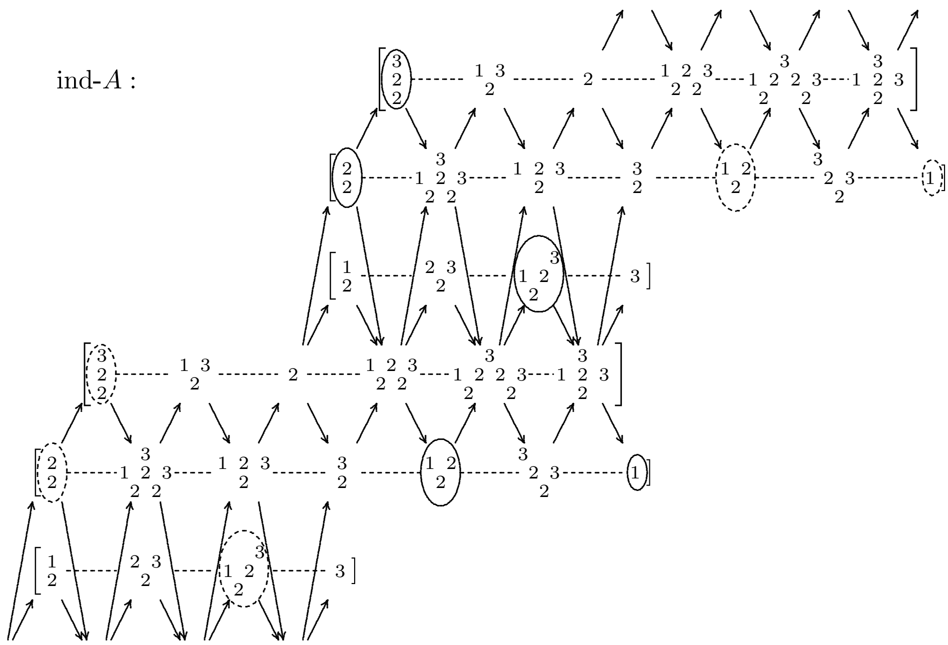
Thus A is a $\bar{\Delta}$-filtered algebra. The standard and proper standard modules are given by:

$$\Delta_A(1) = 1\,; \qquad \Delta_A(2) = \begin{smallmatrix}2\\2\end{smallmatrix}\,; \qquad \Delta_A(3) = \begin{smallmatrix}3\\2\\2\end{smallmatrix}\,;$$

$$\bar{\Delta}_A(1) = 1\,; \qquad \bar{\Delta}_A(2) = 2\,; \qquad \bar{\Delta}_A(3) = 3\,.$$

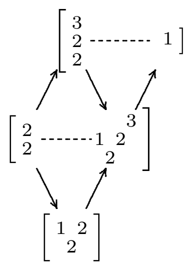
The Auslander–Reiten quiver of the indecomposable right A-modules is as follows (encircled are the elements of $\mathcal{F}(\Delta_A)$):

ind-A :



Thus there are 17 indecomposable A-modules in three τ-orbits, while there are five indecomposable modules in $\mathcal{F}(\Delta_A)$, forming the relative Auslander–Reiten quiver:

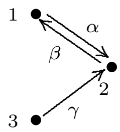
$\mathcal{F}_{ind}(\Delta_A)$:



The direct sum of indecomposable Ext-projective objects in $\mathcal{F}(\Delta_A)$ is given by

$$M_A = \begin{smallmatrix}1&&2\\&2&\end{smallmatrix} \oplus \begin{smallmatrix}2\\2\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\\2\end{smallmatrix}.$$

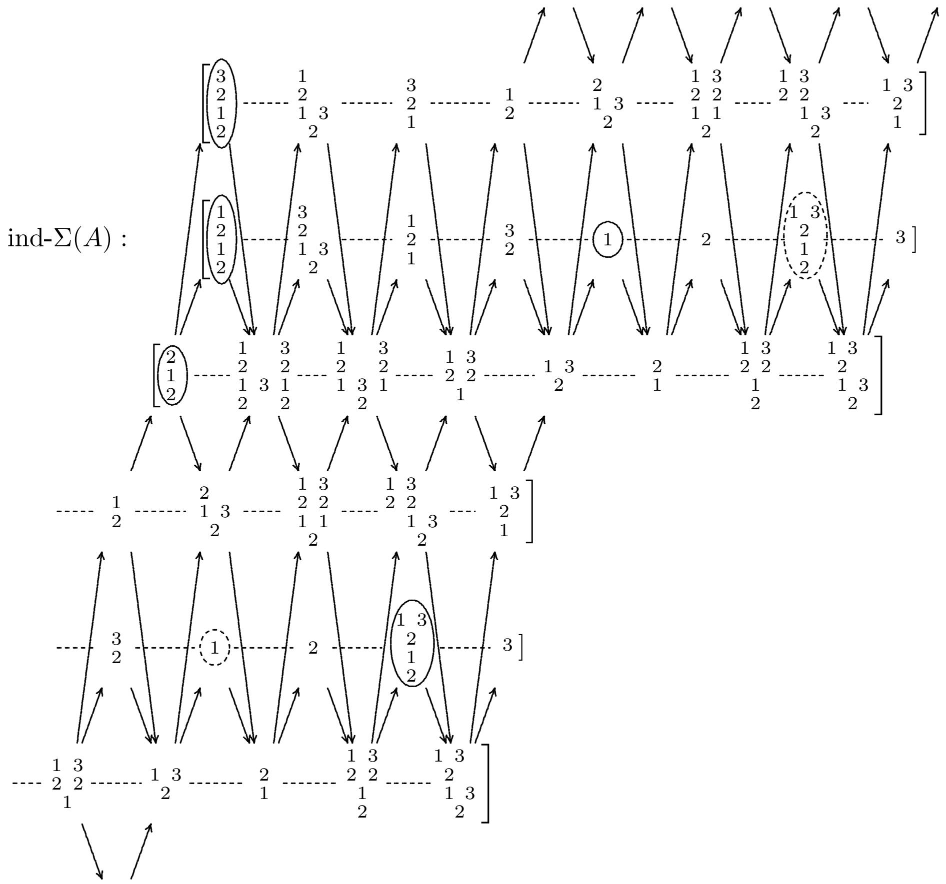
Here, $\Sigma(A) = \mathrm{End}_A(M)$ is given by $\Sigma(A) = KQ_{\Sigma(A)}/I_{\Sigma(A)}$ with the quiver and regular representation as follows:

$Q_{\Sigma(A)}$:  ; $\quad I_{\Sigma(A)} = \langle \beta\alpha\beta \rangle$; $\quad \Sigma(A)_{\Sigma(A)} = \begin{smallmatrix}1\\2\\1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\\2\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\\1\\2\end{smallmatrix}$.

Clearly, $\Sigma(A)$ is Δ-filtered. The standard modules are given by:

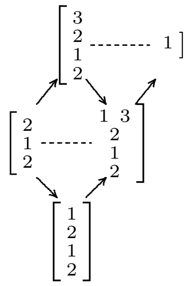
$$\Delta_{\Sigma(A)}(1) = 1; \qquad \Delta_{\Sigma(A)}(2) = \begin{smallmatrix}2\\1\\2\end{smallmatrix}; \qquad \Delta_{\Sigma(A)}(3) = \begin{smallmatrix}3\\2\\1\\2\end{smallmatrix}.$$

The Auslander–Reiten quiver of the indecomposable right A-modules is as follows (encircled are the elements of $\mathcal{F}(\Delta_{\Sigma(A)})$):



Thus there are 24 indecomposable $\Sigma(A)$-modules in three τ-orbits, and five indecomposable modules in $\mathcal{F}(\Delta_{\Sigma(A)})$, forming the relative Auslander–Reiten quiver:

$\mathcal{F}_{ind}(\Delta_A)$:



Acknowledgment. The authors would like to thank the referee for his/her useful advice.

References

[ADL] Ágoston, I., Dlab, V., Lukács, E., Stratified algebras, *C.R.Math.Rep. Acad.Sci.Canada* **20** (1998), 20–25.

[CPS] Cline, E., Parshall, B.J., Scott, L.L., *Stratifying endomorphism algebras*, Memoirs of the AMS **591**, 1996.

[D1] Dlab, V., Quasi-hereditary algebras revisited, *An. St. Univ. Ovidius Constantza* **4** (1996), 43–54.

[D2] Dlab, V., Properly stratified algebras, *C.R. Acad. Sci. Paris Sér 1 Math.* **331** (2000), 191–196.

[DR] Dlab, V., Ringel, C.M., The module theoretical approach to quasi-hereditary algebras, in: *Representations of Algebras and Related Topics*, London Math. Soc. Lecture Note Series **168**, Cambridge Univ. Press 1992, 200–224.

[ES] Erdmann, K., Sáenz, C., On standardly stratified algebras, *Comm. Algebra* **31** (2003), 3429–3446.

[MMS1] Marcos, E., Mendóza, O., Sáenz, C., Stratifying systems via relative simple modules, *J. Algebra* **280** (2002), 472–487.

[MMS2] Marcos, E., Mendóza, O., Sáenz, C., Stratifying systems via relative projective modules, *Comm. Algebra* **33** (2005), 1559 - 1573.

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