STRATIFIED ALGEBRAS

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Les algèbres stratifiées

Résumé. Le concept de l'algèbre stratifiée est en relation directe avec la théorie des représentations des algèbres de Lie complexes semi-simples de dimension finie et est une généralisation naturelle du concept de l'algèbre quasi-héréditaire: les algèbres quasi-héréditaires ne sont autres que les algèbres stratifiées de dimension globale finie. De plus, les autres caractérisations des algèbres quasi-héréditaires, aussi bien homologiques que numeriques, résultent immediatement des caractérisations respectives des algèbres stratifiées. Bien que la motivation originale pour l'étude des algèbres stratifiées vienne de la généralisation de la catégorie $\mathcal{O}$ de Bernstein–Gelfand–Gelfand qui été introduite dans le travail de Futorny et Mazorchuk, ce concept entre dans le contexte générale présenté de faon indépendante par Cline, Parshall et Scott. Un exposé préliminaire sur les résultats de set article a été présenté aut Séminaire d'Algèbre à Paris en Mai, 1997.

1. Definitions and notation

Let A be a basic connected finite dimensional K-algebra and $A_A = \oplus_{i=1}^n P(i) = \oplus_{i=1}^n e_iA$ its (right) regular representation. Denote by $\mathbf{e} = (e_1, e_2, \ldots, e_n)$ the complete sequence of its indecomposable orthogonal idempotents and put $\varepsilon_i = \sum_{t=i}^n \varepsilon_t$, $1 \le i \le n$, $\varepsilon_{n+1} = 0$. Thus the symbol $(A, \mathbf{e})$ will denote the algebra A with the fixed order $\mathbf{e}$ of the idempotents e_i, or equivalently, with the fixed order $\big(S(1), S(2), \ldots, S(n)\big)$ of the (right) simple A-modules $S(i) = P(i)\big/\operatorname{rad} P(i) = e_iA\big/e_i \operatorname{rad} A$. Let us remark that this way we have also fixed the order of the left indecomposable projective A-modules $P^\circ(i) = Ae_i$ and the left simple A-modules $S^\circ(i) = Ae_i\big/\operatorname{rad} Ae_i$.

Definition 1.1. For a given algebra $(A, \mathbf{e})$, the sequence of the *right standard A-modules*

$$\Delta = \big(\Delta(1), \Delta(2), \ldots, \Delta(n)\big)$$

and the sequence of *right proper standard A-modules*

$$\overline{\Delta} = \big(\overline{\Delta}(1), \overline{\Delta}(2), \ldots, \overline{\Delta}(n)\big)$$

1991 Mathematics Subject Classification. Primary 16G70, 16G20.

[1] Research partially supported by the Foundation for Hungarian Higher Education and Research and by Hungarian NFSR grant no. T023434

[2] Research partially supported by NSERC of Canada grant no. A-7257

[3] Research partially supported by Hungarian NFSR grant no. T016432

is defined by

$$\Delta(i) = e_i A / e_i \operatorname{rad} A \varepsilon_{i+1} A, \quad 1 \le i \le n, \quad \text{and}$$
$$\overline{\Delta}(i) = e_i A / e_i \operatorname{rad} A \varepsilon_i A, \quad 1 \le i \le n,$$

respectively. The sequence of *left standard modules* Δ° and of *left proper standard modules* $\overline{\Delta}^\circ$ is defined similarly. Moreover, define the *right costandard A-modules* $\nabla(i)$ and the *right proper costandard A-modules* $\overline{\nabla}(i)$ by:

$$\nabla(i) = \operatorname{Hom}_K(\Delta^\circ(i), K), \quad 1 \le i \le n, \quad \text{and}$$
$$\overline{\nabla}(i) = \operatorname{Hom}_K(\overline{\Delta}^\circ(i), K), \quad 1 \le i \le n.$$

Although most of our statements will have their dual counterparts, formulated for the opposite algebra, we shall usually refrain from stating both statements explicitly.

Let us observe that, for any K-algebra A and any order $\mathbf{e}$, $\Delta(n)$ is projective and $\overline{\Delta}(1)$ is a simple A-module. Clearly, $[\overline{\Delta}(i) : S(i)] = 1$ and thus $D_i = \operatorname{End}_A \overline{\Delta}(i) \simeq \operatorname{End}_A S(i) \simeq e_i A e_i / e_i \operatorname{rad} A e_i \simeq \operatorname{End}_A S^\circ(i)$ is a division algebra for all $1 \le i \le n$. Let us write $d_i = [D_i : K]$. In fact, one can see immediately that $\Delta(i) = \overline{\Delta}(i)$ (and consequently, $\Delta^\circ(i) = \overline{\Delta}^\circ(i)$) if and only if $\operatorname{End}_A \Delta(i) \simeq e_i A e_i / e_i A \varepsilon_{i+1} A e_i = D_i$, i. e. if and only if $e_i A \varepsilon_{i+1} A e_i = e_i \operatorname{rad} A e_i$.

Definition 1.2 Let Φ be a fixed set of (right) A-modules from mod-A, the category of all right A-modules of finite length. A module $X \in$ mod-A is said to be *Φ-filtered* if there is a chain of submodules

$$X = X_1 \supseteq X_2 \supseteq \ldots \supseteq X_t \supseteq X_{t=1} = 0$$

such that $X_s / X_{s+1} \in \Phi$ for all $1 \le s \le t$. Denote by $\mathcal{F}(\Phi)$ the full subcategory of mod-A of all Φ-filtered A-modules.

We will consider mainly the subcategories $\mathcal{F}(\Delta)$ and $\mathcal{F}(\overline{\Delta})$ of mod-A. Every $X \in \mathcal{F}(\Delta)$ has a canonical Δ-filtration given by a refinement of the *trace filtration*

$$X = X^{(1)} \supseteq X^{(2)} \supseteq \ldots \supseteq X^{(i)} = \operatorname{trace}_{\varepsilon_i A} X = X \varepsilon_i A \supseteq X^{(n)} \supseteq X^{(n+1)} = 0;$$

here, for each $1 \le i \le n$, the module $X^{(i)} / X^{(i+1)}$ is a direct sum of $\Delta(i)$-s (cf. [DK]). Similarly, any $X \in \mathcal{F}(\overline{\Delta})$ has a canonical $\overline{\Delta}$-filtration given by a refinement of the *proper trace filtration* which itself is a refinement of the trace filtration of X:

$$\ldots X^{(i)} = X^{(i,0)} \supseteq X^{(i,1)} \supseteq \ldots \supseteq X^{(i,j)} = \operatorname{trace}_{\varepsilon_i A} \operatorname{rad} X^{(i,j-1)} =$$
$$= X(\varepsilon_i \operatorname{rad} A \varepsilon_i)^j A \supseteq \ldots \supseteq X^{(i,h_i)} = X^{(i+1)} \supseteq \ldots;$$

here, for each $1 \le i \le n$, the module $X^{(i,j)} / X^{(i,j+1)}$ is a direct sum of $\overline{\Delta}(i)$'s for every $0 \le j \le h_i - 1$.

Definition 1.3. The algebra $(A, \mathbf{e})$ is said to be *stratified of type* $\mathbf{s} = (s_1, s_2, \ldots, s_n)$, where $s_i = \pm 1$, if each factor $A\varepsilon_i A / A \varepsilon_{i+1} A$ of the trace filtration of A_A belongs to $\mathcal{F}(\Delta^{-s_i})$; here $\Delta^{+1} = \Delta$ and $\Delta^{-1} = \overline{\Delta}$. In particular, $(A, \mathbf{e})$ is *standardly stratified* if it is a stratified algebra of type $\mathbf{1} = (+1, +1, \ldots, +1)$ (cf. [CPS]).

Remark. Note that the type of a stratified algebra is not uniquely defined; in fact, s_1 can be both $+1$ and -1 for any stratified algebra. Moreover, those algebras which are both of type $\mathbf{s}$ and $-\mathbf{s}$, and consequently both of type $+\mathbf{1}$ and $-\mathbf{1}$ are said to be *fully standardly stratified algebras* and are of particular interest.

2. Stratified algebras

The first part of the following lemma is related to a characterization of heredity ideals in [DR].

LEMMA 2.1. *Let e be an indecomposable idempotent and ${}_A(AeA)$ be projective. Then the multiplication map*

$$Ae \underset{eAe}{\otimes} eA \overset{\text{mult.}}{\longrightarrow} AeA$$

is bijective, ${}_{eAe}eA$ is projective and ${}_A(AeA) \in \mathcal{F}(Ae)$. Observe that ${}_{eAe}(eA)$ is projective if and only if $eA \in \mathcal{F}\big(\overline{\Delta}(e)\big)$ and $[AeA : Ae] = \text{length}\,\overline{\Delta}(e)$, where $\overline{\Delta}(e) = eA\big/e\,\mathrm{rad}\,AeA$.

Following an idea of Section 2 in [D], we can derive the following results.

THEOREM 2.2. *An algebra $(A, \mathbf{e})$ is stratified of type $\mathbf{s}$ if and only if the opposite algebra $(A^{opp}, \mathbf{e})$ is stratified of type $-\mathbf{s}$. In particular, every right projective A-module is $\overline{\Delta}$-filtered if and only if every left projective A-module is Δ-filtered.*

THEOREM 2.3. *If $(A, \mathbf{e})$ is a stratified algebra of type $\mathbf{s} = (s_1, s_2, \ldots, s_n)$, then $(A, \mathbf{e})$ has a stratification in the sense of* [CPS]. *In particular,*

$$\dim_K A = \sum_{i=1}^{n} \frac{1}{d_i} \dim_K \Delta^{\circ s_i}(i) \cdot \dim_K \Delta^{-s_i}(i),$$

where $\Delta^{+1}(i) = \Delta(i)$, $\Delta^{\circ +1}(i) = \Delta^{\circ}(i)$, $\Delta^{-1}(i) = \overline{\Delta}(i)$, $\Delta^{\circ -1}(i) = \overline{\Delta}^{\circ}(i)$, if and only if $(A, \mathbf{e})$ is stratified of type $(s_1, s_2, \ldots, s_n)$.

THEOREM 2.4 (cf. [D]). *Let $(A, \mathbf{e})$ be a stratified algebra. Then the following statements are equivalent:*

(*i*) *gl.dim $A < \infty$;*
(*ii*) *$\Delta(i) = \overline{\Delta}(i)$ for all $1 \leq i \leq n$;*
(*iii*) *$(A, \mathbf{e})$ is quasi-hereditary.*

The following is a general form of the Bernstein–Gelfand–Gelfand law.

THEOREM 2.5. *Let $(A, \mathbf{e})$ be a stratified algebra of type $\mathbf{s} = (s_1, s_2, \ldots, s_n)$. Then*

$$d_j\,[P^{\circ}(i) : \Delta^{\circ s_j}(j)] = d_i\,[\Delta^{-s_j}(j) : S(i)] \quad \textit{for all} \;\; 1 \leq i, j \leq n.$$

In particular, if $(A, \mathbf{e})$ is a standardly stratified algebra, then

$$d_j\,[P^{\circ}(i) : \Delta^{\circ}(j)] = d_i\,[\overline{\Delta}(j) : S(i)] \quad \textit{for all} \;\; 1 \leq i, j \leq n.$$

Moreover, if all $d_i = d_j = 1$ (as it is the case when K is algebraically closed) then we get the "classical" BGG-reciprocity relation

$$[P^{\circ}(i) : \Delta^{\circ}(j)] = [\overline{\Delta}(j) : S(i)]$$

and the Cartan matrix $C(A)$ of A has a product decomposition

$$C(A) = \overline{\Delta}^{tr} \cdot \Delta^{\circ}.$$

Let us conclude this section with a canonical construction of the shallow algebra over a given K-species $\mathcal{S} = (D_i,\, {}_iW_j \,|\, 1 \leq i, j \leq n)$ (cf. [ADL]) and a numerical characterization of shallow algebras.

Definition 2.6. A basic algebra $(A, \mathbf{e})$ is said to be a *shallow stratified algebra* of type $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ if, for every $1 \le i \le n$, $\operatorname{rad} \Delta^{\circ s_i}(i)$ and $\operatorname{rad} \Delta^{-s_i}(i)$ are semisimple.

Remark. A shallow stratified algebra $(A, \mathbf{e})$ is always lean (in the sense of [ADL]) and clearly satisfies $\operatorname{rad}^3 A = 0$.

Now, given an ordered K-species $\mathcal{S}$, let us define the canonical shallow stratified algebra $S_{\mathcal{S},\mathbf{s}}$ of type $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ by $S_{\mathcal{S},\mathbf{s}} = T\mathcal{S}/I$, where I is generated by all ${}_iW_j \otimes {}_jW_k$ with $j < \max\{i,k\}$ or $i + k + s_j(k - i) = 2j$. Thus, the canonical shallow algebra of type $(+1, +1, \ldots, +1)$ and of type $(-1, -1, \ldots, -1)$ are $T\mathcal{S}/\langle {}_iW_j \otimes {}_jW_k \mid j < \max\{i,k\} \text{ or } j = k\rangle$ and $T\mathcal{S}/\langle {}_iW_j \otimes {}_jW_k \mid j < \max\{i,k\} \text{ or } i = j\rangle$.

Theorem 2.7. *Let $\mathcal{S} = \mathcal{S}_A$ be the ordered species of a stratified algebra $(A, \mathbf{e})$ of type $\mathbf{s}$. Then*

$$\dim_K A \ge \dim_K S_{\mathcal{S},\mathbf{s}}.$$

Moreover, if $\dim_K A = \dim_K S_{\mathcal{S},\mathbf{s}}$, then A is shallow.

There is a similar construction of a canonical fully standardly stratified shallow algebra $S_{\mathcal{S}}$ over an ordered species $\mathcal{S}$:

$$S_{\mathcal{S}} = T\mathcal{S}/\langle {}_iW_j \otimes {}_jW_k \mid j < \max i, k \text{ or } i = j = k\rangle.$$

Here, $\operatorname{rad}^4 S_{\mathcal{S}} = 0$ and an analogue of Theorem 2.7 holds.

As an illustration of the preceding constructions, let us remark that the shallow algebras over a complete quiver on n vertices (i.e. an oriented graph with an arrow from i to j for all pairs $1 \le i, j \le n$, including the loops from i to i) have the following dimensions:

$$\dim S_{Q,\mathbf{s}} = \frac{n(n+1)(n+2)}{3} \quad \text{for all types } \mathbf{s} \quad \text{and}$$
$$\dim S_Q = \frac{n(n+1)(2n+1)}{3}.$$

3. Standardly stratified algebras

In this section we sketch the proof of the main result characterizing standardly stratified algebras. For situations where $\Delta = \overline{\Delta}$, this result reduces to a characterization of quasi-hereditary algebras (cf. for example Theorem A.2.6 of [DK]).

Theorem 3.1. *Let $(A, \mathbf{e})$ be a K-algebra. Then the following statements are equivalent:*

(i) $(A, \mathbf{e})$ *is standardly stratified.*
(ii) $\operatorname{Ext}^2_A\big(\overline{\Delta}(i), \nabla(j)\big) = 0$ *for all* $1 \le i, j \le n$.
(iii) $\mathcal{F}(\overline{\Delta}) = \big\{ X \mid \operatorname{Ext}^1_A\big(X, \nabla(j)\big) = 0$ *for all* $1 \le j \le n \big\}$.
(ii)' $\operatorname{Ext}^k_A\big(\overline{\Delta}(i), \nabla(j)\big) = 0$ *for all* $1 \le i, j \le n$ *and all* $k \ge 1$.
(iii)' $\mathcal{F}(\overline{\Delta}) = \Big\{ X \mid \operatorname{Ext}^k_A\big(X, \nabla(j)\big) = 0$ *for all* $1 \le j \le n$ *and all* $k \ge 1 \Big\}$.

In the proof we will need the following lemma (cf. Proposition A.2.3 of [DK]).

LEMMA 3.2. *The category $\mathcal{F}(\overline{\Delta})$ is closed under kernels of epimorphisms.*

Proof. Let $0 \to X \to Y \to Z \to 0$ be an exact sequence with $Y, Z \in \mathcal{F}(\overline{\Delta})$. First of all, it is clearly enough to prove that $X \in \mathcal{F}(\overline{\Delta})$ if $Y \in \mathcal{F}(\overline{\Delta})$ and $Z \simeq \overline{\Delta}(j)$ for some $1 \leq j \leq n$; the repeated application of this result will yield the general result. To prove this statement, we shall proceed by induction on the length of the $\overline{\Delta}$-filtration of Y, the initial steps being trivial. (Note that the length of the $\overline{\Delta}$-filtration is an invariant of any module in $\mathcal{F}(\overline{\Delta})$.)

Thus assume that Y has a $\overline{\Delta}$-filtration $0 \subseteq Y_1 \subseteq Y_2 \subseteq \cdots \subseteq Y_\ell$ which is the refinement of the proper standard filtration of the module Y. In particular, if $Y/X \simeq \overline{\Delta}(j)$ and $Y_1 \simeq \overline{\Delta}(k)$ then $j \leq k$.

If $Y_1 \subseteq X$ then we get the following exact sequence: $0 \to X/Y_1 \to Y/Y_1 \to \overline{\Delta}(j) \to 0$, and here the length of the $\overline{\Delta}$-filtration of Y/Y_1 is smaller than that of Y. Hence, by induction, we get that $X/Y_1 \in \mathcal{F}(\overline{\Delta})$ and this immediatey yields that $X \in \mathcal{F}(\overline{\Delta})$.

If $Y_1 \not\subseteq X$ then the natural projection $Y \to Y/X = \overline{\Delta}(j)$ gives a non-zero map $\varphi : Y_1 = \Delta(k) \to \overline{\Delta}(j)$. But the existence of such a map implies that $j = k$ and in this case φ is an isomorphism. Hence $X \cap Y_1 = 0$, thus X embeds into Y/Y_1. A simple dimension argument shows that this embedding is an isomorphism, too, i. e. $X \simeq Y/Y_1 \in \mathcal{F}(\overline{\Delta})$. □

Proof of Theorem 3.1. In what follows, we shall freely use the following simple facts: $\mathrm{Ext}_A^1\left(\overline{\Delta}(i), \nabla(j)\right) = 0$ for $1 \leq i, j \leq n$, and furthermore $\mathrm{Ext}_A^1\left(S(i), \nabla(j)\right) = 0$, whenever $i \leq j$.

Let us observe first that the implications $(ii)' \Rightarrow (ii)$, $(iii)' \Rightarrow (iii)$ and $(iii) \Rightarrow (i)$ are trivial, while the implications $(i) \Rightarrow (ii)'$ and $(iii) \Rightarrow (iii)'$ can be proved using Lemma 3.2 and a clear induction.

Thus, we shall concentrate on proving $(ii) \Rightarrow (iii)$. We would like to show that $\mathcal{F}(\overline{\Delta}) \supseteq \left\{ X \mid \mathrm{Ext}_A^1\left(X, \nabla(j)\right) = 0 \text{ for all } 1 \leq j \leq n \right\} = \mathcal{F}$, the opposite inclusion being trivial.

Let Y be a module from $\mathcal{F}$. We shall prove by induction on the composition length of the module Y that $Y \in \mathcal{F}(\overline{\Delta})$.

Let i be the maximal index for which $[Y : S(i)] \neq 0$, and let $X \subseteq Y$ be a minimal submodule containing a compostion factor isomorphic to $S(i)$. Then clearly, X is local, containing $S(i)$ in its top only, with all composition fators having smaller index. Thus we have the following exact sequences:

$$0 \to X \to Y \to Z \to 0, \tag{3.1.1}$$

$$0 \to X' \to \overline{\Delta}(i) \to X \to 0. \tag{3.1.2}$$

We want to show first that both Z and X belong to $\mathcal{F}$. Let us consider the long exact sequence obtained from (3.1.1) by applying $\mathrm{Hom}_A\left(-, \nabla(j)\right)$:

$$\begin{array}{llllll} & & \cdots & \to & \mathrm{Hom}_A\left(X, \nabla(j)\right) & \to \\ \to & \mathrm{Ext}_A^1\left(Z, \nabla(j)\right) & \to \ \mathrm{Ext}_A^1\left(Y, \nabla(j)\right) & \to & \mathrm{Ext}_A^1\left(X, \nabla(j)\right) & \to \\ \to & \mathrm{Ext}_A^2\left(Z, \nabla(j)\right) & \to \qquad \cdots & & & \end{array}$$

Here $\mathrm{Ext}_A^1\left(Z, \nabla(j)\right) = 0$ is automatically satisfied for $j \geq i$ since all the composition factors of Z have index at most i, and $\mathrm{Ext}_A^1\left(S(k), \nabla(j)\right) = 0$ for $k \leq j$. On the other hand, if $j < i$, then $\operatorname{Hom}_A\big(X, \nabla(j)\big) = 0$ (since top $X \simeq S(i)$), which together with the assumption that $\operatorname{Ext}^1_A\big(Y, \nabla(j)\big) = 0$ yields that $\operatorname{Ext}^1_A\big(Z, \nabla(j)\big) = 0$. Thus, $Z \in \mathcal{F}$. Since $\ell(Z) < \ell(Y)$, by induction we get that $Z \in \mathcal{F}(\overline{\Delta})$. Furthermore, this implies, using condition (ii), that $\operatorname{Ext}^2_A\big(Z, \nabla(j)\big) = 0$ for every j, hence we obtain that $X \in \mathcal{F}$, as well.

To finish the proof, we have to show that $X \in \mathcal{F}(\overline{\Delta})$. It is clearly enough to show, that $X \simeq \overline{\Delta}(i)$, that is, $X' = 0$. But if $X' \neq 0$, then we would get a non-split extension of X with $S(j)$ for some $j < i$. The pushout of this extension along the natural embedding map $S(j) \to \nabla(j)$ would result in a non-split extension of X with $\nabla(j)$, a contradiction. (The fact that the resulting sequence does not split can be seen as follows: the middle term of the original non-split extension of X with $S(j)$ has a simple top $S(i)$ with $i > j$, hence it has no non-trivial homomorphisms into $\nabla(j)$). This shows that $X \in \mathcal{F}(\overline{\Delta})$, and hence that $Y \in \mathcal{F}(\overline{\Delta})$.

□

REMARK. There is an analogous characterization of stratified algebras of type $\mathbf{s} = (s_1, s_2, \dots, s_n)$; these are the algebras satisfying:

$$\operatorname{Ext}^2_A\big(\Delta^{-s_i}(i), \nabla^{s_i}(j)\big) = 0 \quad \text{for all } 1 \le i, j \le n.$$

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