DOCTORAL SCHOOL IN MATHEMATICS

DOCTORAL PROGRAM IN PURE MATHEMATICS

TOPICS FOR THE COMPREHENSIVE EXAM

Contents

1. Real analysis

Main subject: *Choose* 12 *out of the following* 19 *topics.*
Secondary subject: *Choose* 3 *out of topics* 1.1 *to* 1.9.

1.1. Classical measure and integral theory

Measure spaces. Integral in measure spaces. Absolute continuity and singularity. Product spaces. L_p spaces. Haar measure. Generalized integrals.

1.2. Differentiation in higher dimensions

Implicit function theorem, inverse function theorem. Integral formulas: Gauss's theorem, Green's theorem, Stokes' theorem. Integration of differential forms. Sard's theorem. Whitney extension theorem. Lipschitz functions. Kirszbraun theorem. Rademacher's theorem.

1.3. Derivatives

Properties of the derivatives. Monotonicity theorems. The density topology. Approximate continuity. Contingence theorem. Denjoy–Young-Saks theorem.

1.4. Geometric measure theory I.

Covering theorems. Hausdorff measure and dimension. Further dimension concepts. The isoperimetric and the isodiametric inequality. Fractals and dimension formulas. Differentiation of measures and integrals. The maximal operator and its applications.

1.5. Geometric measure theory II.

Regular and irregular s-sets. Projection theorems. Theorems about dimensions of products. Frostman's lemma, energy, capacity. Besicovitch sets. Applications of fractals in other areas (e.g. number theory, dynamical systems, random fractals, physical applications).

1.6. Category theorem

Complete metric spaces. Polish spaces. Baire sets. Category theorem. Typical properties of continuous functions. Banach–Mazur game. Erdős–Sierpinski duality theorem.

1.7. Descriptive set theory

Borel sets. $\mathcal{A}$-operation, analytical sets. Universal sets. Separation theorem. Graph theorem. Baire classes of functions. Luzin and Suslin spaces.

1.8. Dynamical systems

One-parameter groups and semigroups of measure preserving transformations. Concrete systems: homomorphisms of the unit circle, linear map and the baker's map, Gauss's map, endomorphisms of an interval, algebraic automorphisms of the torus. Invariant measures. Krylov–Bogolyubov theorem. Symbolic dynamics.

1.9. Ergodic theory

Poincaré's recurrence theorem. Ergodic theorems. Ergodic, mixing maps and their spectral characterization. Entropy and topological entropy. Isomorphisms of dynamical systems. Ergodicity of Bernoulli sequences.

1.10. Complex analysis

Complex differentiation. Power series and Laurent series. General integral theorems. Isolated singularity, residue. Riemann mapping theorem, extension to the boundary. Harmonic functions, Dirichlet problem for a circle. Capacity: Chebyshev constant, Green's function and its connection to Hausdorff measure.

1.11. Fourier series

Dirichlet kernel, Fejér kernel, summation. L_2 theory, Carleson's theorem, Riesz–Fischer theorem. Bochner's theorem. Pointwise convergence. Multiplicators. Sidon sets. Series with gaps. Singular integrals.

1.12. Fourier integral

Convolution. Fourier inversion theorem. Wiener's approximation theorem. Fourier transform of a complex measure. Parseval's theorem. Poisson summation formula. Fourier integral on the complex plane, Paley–Wiener theorem. Laplace integral, inverse formulas, applications.

1.13. Classes of harmonic and regular functions

Poisson integral of functions in L_p. Hardy classes. Conjugate harmonic functions, theorem of Marcell Riesz. Interpolation of operators in L_p. Theorem of the brothers Riesz.

1.14. Approximation theory

Weierstrass's theorems. Berstein polynomials. Jackson's and Timan's theorem. The best interpolating polynomials, Chebyshev polynomial. Approximation of derivatives of polynomials. Inverse theorems. Lagrange interpolation, projective operators. Stone–Weierstrass theorem, Daniell–Stone theorem.

1.15. Topological vector spaces

Locally convex vector space as a vector space with a topology generated by a family of seminorms. Metrizability. Dual space, Banach–Alaouglu theorem. Baire category theorem. Banach–Steinhaus theorem, Banach's open mapping theorem and closed graph theorem. Hahn Banach theorem. Separation of convex sets. Krein–Milman theorem.

1.16. Operators and Banach algebras

Adjoint operators. Riesz–Schauder theory of compact operators. Fredholm alternative. Riesz representation theorems. Convolution and measure algebra. Unitary representation. Gelfand–Raikov theorem. Irreducible decomposition.

1.17. Ordinary differential equations

Existence, uniqueness, continuability. Measurable and analytical right side. Continuity and differentiability of the characteristic function. Equations depending on parameters. Straightening theorem. Linear systems. Classification of stationary points of linear equations with constant coefficients. Basic concepts of stability theory. Linearisation, Lyapunov functions.

1.18. Partial differential equations

Initial value problem, boundary value problem, initial boundary value problem, physical examples. Classification of semilinear partial differential equations and their canonical form in case of constant coefficients. Classical and generalized Cauchy problem for hyperbolic and parabolic equations with constant coefficients.

1.19. Distributions and Sobolev spaces

Algebraic operations, derivatives of distributions. Convolution and direct product of distributions. Fourier transform in space $\mathcal{S}$ and among tempered distributions, in L_1 and L_2. Fundamental solution of linear partial differential equations with constant coefficients, examples. Extension of the elements of $H^1(\Omega)$. Equivalent norms in $H_0^1(\Omega)$. Embedding of $H^1(\Omega)$ and $H_0^1(\Omega)$ into $L_2(\Omega)$. Trace of the elements of $H^1(\Omega)$.

2. Complex analysis

Main subject: *Choose* 12 *out of the following* 21 *topics.*
Secondary subject: *Choose* 2 *out of topics* 2.1 *to* 2.9 *and* 2 *further topics from* 2.10 *to* 2.21.

2.1. Elements of complex analysis

2.1.1 Functions in a single complex variable Complex differentiability. Power and Laurent series. Isolated singularities. Integral theorems. Saddle point method. The argument and maximum modulus principles, Schwarz's reflection principle. Conformal maps. Picard's theorem. Sequences of regular functions. Harmonic functions.

2.1.2 Functions in several complex variables Separately holomorphic functions. Multivariable integral theorems and power series, Reinhardt domains. Non-existence of isolated singularities. Inequivalence of the ball and the polydisc.

2.2. Geometric function theory

Phragmén–Lindelöf type theorems. Capacity, Green function, connection with the Hausdorff measure. Conformal radius. Univalent functions, area theorem, Koebe distortion theorems, coefficients. Extremal length, the moduli of rectangles and annuli. Quasiconformal maps, quasisymmetric functions, quasiconformal curves.

2.3. Function spaces

The Poisson integral of L_p functions. Hardy spaces. Conjugate harmonic functions, the theorem of Marcell Riesz. Interpolation of operators on L_p spaces. The theorem of the Riesz brothers. Characterization of Hardy spaces via the maximal operator. Canonical representation of entire functions of finite order, Borel's theorem. Meromorphic functions on the plane, Nevanlinna's 1st and 2nd main theorems.

2.4. Complex dynamics

Julia and Fatou sets. Smooth Julia sets. Attractor, superattractor and parabolic fixed points, Cremer points and Siegel discs. Holomorphic fixed point formula. Notable dense subsets of the Julia set. Herman rings. Wandering domains. Iteration of polynomials. Mandelbrot set. Newton iteration. Hyperbolic maps.

2.5. Modular forms

Poincaré model of the hyperbolic plane. The group $SL_2(\mathbb{R})$ and its discrete subgroups. Classical holomorphic modular forms and Maass forms, number theoretic applications. Maass forms and the spectrum of the Laplace–Beltrami operator. Discrete spectrum for compact quotients. Spectral decomposition for $SL_2(\mathbb{Z})$.

2.6. Riemann surfaces

Abstract definition. Coverings, universal covering. Differential forms. Dirichlet problem, the Perron method. Conformal maps of simply connected surfaces. Uniformization. Fundamental domain. Function elements and germs. Extension with branch points.

2.7. Complex manifolds

Elliptic functions and curves. Closed Riemann surfaces, algebraic functions and curves. Ricci flow on Riemann surfaces. Uniformization via the Ricci flow. Projective algebraic varieties, Chow's theorem, the Kodaira embedding theorem.

2.8. Sheaves

Cousin's 1st and 2nd problem, sheaves, sheaf cohomology, Dolbeault and de Rham isomorphism theorems. Algebraic properties of holomorphic function germs: Weierstrass theorems, divisors, Chern classes, Oka–Serre theorem, coherent analytic sheaves, Oka's coherence theorem, Cartan's theorems A and B for domains in $\mathbb{C}^n$.

2.9. Complex convexity

Removing compact singularities. Holomorphic convexity, domains of holomorphy, pseudoconvexity, Levi convexity, Levi problem, polynomial convexity. Oka–Weil theorem. Inhomogeneous Cauchy–Riemann equations in one or several variables, Dolbeault cohomology. Hörmander's L_2 method for solving the $\overline{\partial}$-equation on pseudoconvex domains.

2.10. Classical measure and integral theory

Measure spaces. Integration in measure spaces. Absolute continuity and singularity. Product of measure spaces. L_p spaces. Haar measure. Generalized integrals.

2.11. Geometric measure theory

Covering theorems. Hausdorff measure and dimension. Other concepts of dimension. Isoperimetric and isodiametric inequalities. Fractals and dimension formulas. Differentiation of measures and integrals. The maximal operator and its applications.

2.12. Fourier series

Dirichlet and Fejér kernels and summation. L_2-theory, Carleson's theorem, Riesz–Fischer theorem. Bochner's theorem. Pointwise convergence. Multipliers. Sidon sets. Fourier series with gaps. Singular integrals.

2.13. Fourier integral

Convolution. Inversion formula. Wiener's approximation theorem. Fourier transform of a complex measure. Parseval formula. Poisson summation formula. Fourier integral on the complex plane, Payley–Wiener theorem. Laplace integral, inversion formulae, applications.

2.14. Approximation theory

Weierstrass theorems, Stone–Weierstrass theorem. Bernstein polynomials, positive operators. Theorems of Jackson and Timan. Optimal approximation with polynomials, Chebyshev polynomial. Estimates on the derivatives of polynomials. Inverse theorems. Lagrange interpolation, projection operators.

2.15. The classical theory of ordinary differential equations

Existence and uniqueness of solutions, Gronwall's inequality. Linear systems. Boundary value problems for second order linear differential equations. Stability concepts. Stability of a system of linear differential equations: stable, unstable and central subspace, linearization near a stationary point.

Ljapunov's method. Asymptotic behavior, boundary sets, attractor set, Poincaré–Bendixson theory. Stability of a periodic solution, Poincaré map.

2.16. Elements of distribution theory

Fourier transform of tempered distributions, in L_1 and L_2. Application for the construction of fundamental solutions. Sobolev spaces. Extension, equivalent norms in $H_0^1(\Omega)$. Trace operator. Theorems on compact embeddings. Characterization of $H^k(\mathbb{R}^n)$ and $H^k(\mathbb{R}^n_+)$ via Fourier transform. Smoothness of functions in a Sobolev space. Anisotropic Sobolev spaces.

2.17. Initial and boundary value problems for linear partial differential equations

Classical and generalized boundary and eigenvalue problems, and their variational formulations. Fredholm alternative for boundary value problems of general elliptic form. Uniqueness of the solution. Existence of solutions and their construction via the methods of Fourier and Galerkin.

2.18. Basics of functional analysis

Banach space and its dual. Operators between Banach spaces. Hahn–Banach theorem. Baire category theorem. Banach–Steinhaus theorem. Banach's open mapping and closed graph theorems. Riesz–Schauder theory of compact operators. Fredholm alternative.

2.19. Topological vector spaces

Constructions of topological vector spaces. Families of seminorms and locally convex spaces. Inductively constructed locally convex spaces. Strict inductive limits. Geometric form of the Hahn–Banach theorem. Separation theorems of Hahn and Banach.

2.20. Banach algebras

Properties of the spectrum and the resolvent. Character and Gelfand representation of a Banach algebra. Bochner's theorem. Holomorphic functional calculus in unital Banach algebras. Regular Banach algebras. Enveloping C^*-algebras and the abstract Stone theorem.

2.21. Topological groups

Characterization and projective construction of topological groups. Semi-metrizability. Uniformly continuous maps between topological groups. Transitive continuous representations. Wigner–Neumann theorem. Dual of a topological group. Construction of invariant Radon measures and continuous unitary representations. Existence and uniqueness of the Haar measure.

3. Differential equations

Main subject: *Choose* 6 *out of the following* 10 *topics.*
Secondary subject: *Choose* 3 *out of topics* 3–10.

3.1. Basics of functional analysis

Banach space and its dual. Operators between Banach spaces. Hahn–Banach theorem. Baire category theorem. Banach–Steinhaus theorem. Banach's open mapping and closed graph theorems. Riesz–Schauder theory of compact operators. Fredholm alternative.

3.2. Geometry of Hilbert spaces and operators between Hilbert spaces

The decomposition and the representation theorem of Riesz. Hilbert–Schmidt theorem for compact normal (self-adjoint) operators. Dunford's classic spectral theory for continuous normal (self-adjoint) operators.

3.3. The classical theory of ordinary differential equations

Existence and uniqueness of solutions, Gronwall's inequality. Linear systems. Boundary value problems for second order linear differential equations. Stability concepts. Stability of a system of linear differential equations: stable, unstable and central subspace, linearization near a stationary point. Ljapunov's method. Asymptotic behavior, boundary sets, attractor set, Poincaré–Bendixson theory. Stability of a periodic solution, Poincaré map.

3.4. Dynamical systems

Topological classification of dynamical systems. Flow-box theorem. Topological classification of linear systems. Hartman–Grobman theorem. Classification of nonlinear systems using the Poincaré normal form. Stable, unstable, center manifold. Local phase portrait in a neighborhood of a periodic orbit. The stable, unstable and center manifold of a periodic orbit.

3.5. Bifurcation theory, chaos

Bifurcations of dynamical systems. Saddle-node and Andronov–Hopf bifurcations. Two co-dimensional bifurcations. Structural stability. Morse–Smale systems. Discrete dynamical systems. Stability of periodic orbits. Definition of chaotic orbit, symbolic dynamics, examples. Chaos in the Lorenz differential equation. Types of attractors, chaotic attractor.

3.6. Operator semigroups and their applications to differential equations

Infinite dimensional dynamical systems. Operator semigroups, generators. Abstract Cauchy problem, theorems of existence. Stability, asymptotic properties of operator semigroups. Delayed differential equations, applications. Reaction-diffusion equations, the existence and stability of traveling waves.

3.7. Elements of distribution theory

Fourier transform of tempered distributions in L_1 and L_2. Application for the construction of fundamental solutions. Sobolev spaces. Extension, equivalent norms in $H_0^1(\Omega)$. Trace operator. Theorems on compact embeddings. Characterization of $H^k(\mathbb{R}^n)$ and $H^k(\mathbb{R}^n_+)$ via Fourier transform. Smoothness of functions in a Sobolev space. Anisotropic Sobolev spaces.

3.8. Boundary value problems for linear elliptic equations

Generalized boundary value and eigenvalue problems for symmetric equations. The variational formulation of boundary value and eigenvalue problems. Fredholm alternative for general elliptic boundary value problems.

3.9. Initial and boundary value problems for linear hyperbolic and parabolic equations

The classical and the generalized problem. The uniqueness of the solution. The proof of the existence of solutions and their computation using the Fourier and the Galerkin method.

3.10. Nonlinear equations

The basics of the theory of monotone operators. Existence theorems for monotone and pseudo-monotone operators, their applications for nonlinear elliptic equations. The solution of abstract evolution equations using monotone operators. Initial value problems with monotone and pseudo-monotone operators, application for initial value problems of nonlinear parabolic equations.

4. Functional analysis

Main subject: *Required: topics* 4.1 *to* 4.3, *and* 7 *further (arbitrary) topics.*
Secondary subject: *Required: topics* 4.1 *to* 4.3, *and* 2 *further (arbitrary) topics.*

4.1. Basics of functional analysis

Banach-spaces and duality. Operators between Banach-spaces. Hahn–Banach theorem. Baire category theorem. Banach–Steinhaus theorem. Banach's open mapping and closed graph theorems. Riesz–Schauder theory of compact operators. Fredholm alternative.

4.2. Function spaces

Approximation and integral representation. Urysohn's theorem and Dieudonné's decomposition theorem for locally compact spaces. Stone–Weierstrass theorem. Daniel–Stone theorem. Riesz's theorems. Sobolev spaces. Equivalent norms. Embedding theorems. Trace of H^1 functions.

4.3. Geometry of Hilbert spaces and operators between Hilbert spaces

Riesz's decomposition and representation theorems. Hilbert–Schmidt theorem for compact normal (self-adjoint) operators. Dunford's classical spectral theorem for continuous normal (self-adjoint) operators.

4.4. Operator extensions

Semi-bounded self-adjoint operators. Neumann, Friedrich and Krein extension theorems. Integration with respect to projection-valued measures. Operator calculus. The classical Stone theorem.

4.5. Topological vector spaces

Constructions of topological vector spaces. Families of seminorms and locally convex spaces. Inductively constructed locally convex spaces. Strict inductive limits. Geometric form of the Hahn–Banach theorem. Separation theorems of Hahn and Banach.

4.6. Metrizable topological vector spaces

Criterium of metrization. Generalization of Banach's open mapping theorem to complete metrizable topological vector spaces. Closed graph theorem for complete metrizable topological spaces. Generalizatons of the open mapping theorem.

4.7. Spaces of continuous linear operators

Boundedness on topological vector spaces and characterization of seminormablity. Bounded sets in strict inductive limits. G-topologies in operator spaces. Ascoli theorems. Alaoglu–Bourbaki and Banach–Alaoglu theorems. Banach's equicontinuity theorem. General form of the Banach–Steinhaus theorem. Operator topologies in the space of continuous linear Hilbert space operators. Neumann-algebras.

4.8. Duality theory in topological vector spaces

Dual pairs. Polar sets and polar topologies. Topologies compatible with duality and their characterization (Mackey–Arens theorem). Mackey spaces. Barreled, infrabarreled, bornological and ultrabornological spaces. Semi-reflexive and reflexive spaces. Montel spaces.

4.9. Compact convex sets

Carathéodory–Minkowski theorem. Krein–Milman theorem. Probability Radon measures over compact spaces and construction of concentration. Barycenter of probability Radon measures over compact convex sets. Barycentric decomposition of a point of a metrizable compact convex set (abstract Choquet theorem). Choquet ordering. Barycentric decomposition of non-metrizable compact convex sets.

4.10. Abstract integral theory

Vector nets and elementary integrals. Upper integrals and integrals. L^p and F^p spaces respect to integration. Riesz–Fischer theorem. Beppo–Levi theorem. Lebesgue theorem. Multiplication of upper integrals. Lebesgue–Fubini theorem. Integral of vector-valued functions. Vector-valued measures and integral with respect to vector-valued measures. Gelfand–Dunford theorem. Topological integral and integral theory of ring of sets.

4.11. Banach-algebras

Properties of the spectrum and the resolvent. Character and Gelfand representation of a Banach algebra. Bochner's theorem. Holomorphic functional calculus in unital Banach algebras. Regular Banach algebras. Enveloping C^*-algebras and the abstract Stone theorem.

4.12. C^*-algebras

Uniqueness of the C^*-norm. Characterization of commutative C^*-algebras (first Gelfand–Naimark theorem). Spectral theorem for normal elements of a C^*-algebra and continuous functional calculus. Baer-C^*-algebras and the topological Schur-lemma. Projector-nets (Hilbert-nets and Neumann-nets) and the basics of non-classical probability theory. MSC-algebras and abstract polar decomposition. Ultraspectral C^*-algebras and abstract spectral theorem.

4.13. Representations of Banach $*$-algebras

Decomposition of non-degenerate $*$-representations of $*$-algebras into cyclic $*$-representations. Representable positive functionals over $*$-algebras. GNS-construction and Sebestyén's theorem of representability. Abstract Gelfand–Raikov theorem. Decomposition of cyclic $*$-representation of Banach-$*$-algebras into irreducible ones (abstract Choquet theorem). Kadison representation of C^*-algebras. Existence of faithful $*$-representations of C^*-algebras (second Gelfand–Naimark theorem).

4.14. Topological groups

Characterization and projective construction of topological groups. Semi-metrizability. Uniformly continuous maps between topological groups. Transitive continuous representations. Wigner–Neumann theorem. Dual of a topological group. Construction of invariant Radon measures and continuous unitary representations. Existence and uniqueness of the Haar measure.

4.15. Continuous unitary representation of locally compact groups

Convolution and measure algebra of locally compact groups. Characterization of measure algebras with unit elements and commutative measure algebras. The fundamental theorem of harmonic analysis. Gelfand–Raikov theorem. Choquet thorem in harmonic analysis.

4.16. Continuous unitary representations of commutative locally compact groups

Dual of commutative locally compact groups. Abstract Fourier transform. Decomposition of continuous unitary representations of commutative locally compact groups into irreducible ones (Stone theorem). Spectrum of continuous unitary representations.

4.17. Induced unitary representations

Factorization of Radon measures over locally compact groups. Bruhat-type intersection function. Construction of the Mackey-type scalar product and its properties. Induced unitary representations and imprimitivity systems. The Mackey criterium of inducibility (imprimitivity theorem). Glimm's condition and the Mackey representation theorem.

4.18. Nonlinear functional analysis

Strict differentiability and inverse function theorem. Topological algebraic complementers. Immersions and Banach submanifolds. Submersions and normal equations. Submersions and linearizability of functions. Theorem of constant rank.

4.19. Surface measures

Riemann submanifolds in finite dimensional real Euclidean spaces. Theorem of collective parametrization. General theorem of integration by substitution and surface measures. Elementary form of the Gauss–Ostrogradsky theorem. Green formulas and their applications in the theory of partial differential equations. Nonelementary forms of the Gauss–Ostrogradsky theorem.

5. Differential geometry

Main subject: *Choose* 4 *out of the following* 5 *topics.*
Secondary subject: *Choose any* 1 *of the following* 5 *topics.*

5.1. Classical differential geometry

Differential geometry of curves in the n-dimensional euclidean space. The length of curves, Crofton's formula, Cauchy formula, Barbier's theorem, natural parametrization. Osculating subspaces, Frenet–Serret frame, generalized curvatures, Frenet–Serret formulas, the fundamental theorem of curves. Osculating circle and osculating spheres. Evolute, involute and parallel curves of plane curves.

Global differential geometry of curves. Total curvature, Hopf's Umlaufsatz, Fenchel's theorem, Fáry–Milnor theorem. Characterization of convex plane curves, the four-vertex theorem.

Hypersurfaces in $\mathbb{R}^n$. Tangent space of a regularly parametrized hypersurface, curvature of a curve on a surface, normal section and normal curvature, Meusnier's theorem. Weingarten map, principal directions and principal curvatures. Euler's theorem, Rodrigues' curvature formula. Gaussian and Minkowski curvature, the Gauss–Codazzi–Maniardi equations. The intrinsic geometry of surfaces, Theorema Egregium. The fundamental theorem of hypersurfaces (Bonnet's theorem). Covariant derivative on a hypersurface. Integration on a hypersurface.

Lines of curvature. Curvature lines of surfaces of revolution. Dupin's theorem, the curvature lines of quadratic surfaces. Ruled surfaces, directrix, the structure of developable surfaces. Asymptotic lines on surfaces with negative curvature, Beltrami–Enneper theorem. Surfaces with constant negative curvature, Chebyshev nets, Hilbert's theorem. Characterization of convex surfaces, rigidity theorems. The Gauss–Bonnet theorem (local and global variants). Minimal surfaces.

Textbooks:

[1] do Carmo, M. P.: *Differential geometry of curves and surfaces*
[2] Stoker, J. J.: *Differential geometry*
[3] Csikós B.: *Differential geometry* (lecture notes on the homepage of Balázs Csikós)

5.2. The theory of differentiable manifolds

Differentiable manifolds, tangent spaces, differentiable maps, the tangent map. Immersions, embeddings, submersions. Submanifolds. Vector fields, Lie bracket. Lie groups. The Lie algebra of a Lie group. Integral curves of a vector field, complete vector fields. Integrable distributions, Frobenius' theorem. Tensor fields. Lie derivative. The ring $\Omega^*(M)$ of differential forms. Derivations and antiderivations in $\Omega^*(M)$, interior product with vector fields, the exterior derivative. Cartan's formula for the Lie derivative of differential forms. Integration of differential forms on smooth singular chains and regular domains. The volume form of a Riemannian manifold. The general Stokes theorem and its classical special cases. The de Rham cohomology ring, homomorphisms induced by smooth maps, homotopic invariance. Poincaré lemma. Covariant derivation, its torsion and curvature tensors. Parallel transport of a tangent vector along a curve. Extending the covariant derivative to tensor fields. Bianchi identities.

Textbooks:

[1] Warner, F.: *Foundations of differentiable manifolds and Lie groups*
[2] Auslander, L.; MacKenzie, R. E.: *Introduction to differentiable manifolds*

5.3. Semi-Riemannian geometry

Semi-Riemannian manifolds. The Levi–Civita covariant derivative, parallel transport. Identities of the curvature tensor, intersection curvature. Manifolds with constant curvature, Schur's theorem. Ricci tensor, Einstein manifolds, Schouten–Struik theorem, Singer–Thorpe theorem. Algebraic curvature tensors, Kulkarni–Nomizu product and the decomposition of the curvature tensor, the Weyl tensor. Conformal invariance of the Weyl tensor. Geodesics. Exponential map. Normal neighborhoods. Geodesic completeness, the completeness of Riemannian manifolds, the Hopf–Rinow theorem. Completeness of Lorentz manifolds.

First and second variation of the arc length. Jacobi fields along a geodesic. Conjugate points. Index form of a geodesic segment, the Morse index theorem. The Rauch comparison theorem. Cartan–Hadamard theorem.

Geometry of submanifolds. Second fundamental tensor. The shape operator. Normal connection. The Gauss, Codazzi and Ricci equations on curvature tensors. Focal points of a submanifold. Totally geodesic submanifolds.

Fundamentals of the theory of general relativity. Geodesics of Lorentz manifolds, completeness. Time orientation, observation field, static spacetimes. Einstein equation, the Schwarzschild solution, and an extension by Kruskal. The causal structure of time-oriented Lorentz manifolds. Hawking's singularity theorem.

Textbooks:

[1] Szenthe J.: *A Riemann-geometria elemei* (in Hungarian)
[2] do Carmo, M. P.: *Riemannian geometry*
[3] O'Neill, B.: *Semi-Riemannian geometry*
[4] Hawking, S. W.; Ellis, G. F. R.: *The large scale structure of space-time*

5.4. Homogeneous and symmetric Riemann spaces

Isometries on a Riemannian manifold, Steenrod–Myers theorem. Smooth group action on a differentiable manifold, the classification of orbits. The quotient G/H of Lie groups as a differentiable manifold. Criterion for the existence of an invariant Riemannian metric on the homogeneous space G/H. Reductive homogeneous spaces. The characterization of locally symmetric Riemannian manifolds.

Compact Lie groups with a bi-invariant Riemannian metrics as symmetric spaces. The general construction of symmetric Riemann spaces. Description of the Levi–Civita connection and its curvature tensor on a symmetric space using the Lie algebra of the isometry group. Totally geodesic submanifolds, the rank of a symmetric space. Decomposition of simply connected symmetric Riemann spaces. Symmetric spaces of compact and noncompact type, the sign of the intersection curvatures, duality. Classification of irreducible symmetric spaces.

Holonomy groups of Riemannian manifolds, the Berger–Simmons theorem and its connection with symmetric spaces.

Textbooks:

[1] Berger, M.: *A panoramic view of Riemannian geometry*
[2] Cheeger, J.; Ebin, D. G.: *Comparison theorems in Riemannian geometry*
[3] Helgason, S.: *Differential geometry, Lie groups and symmetric spaces*
[4] Warner, F. W.: *Foundations of differentiable manifolds and Lie groups*

5.5. Lie groups

Lie groups, the correspondence of local Lie groups and Lie algebras. Lie functors. One-parameter subgroups and the exponential map. The universal enveloping algebra, Poincaré–Birkhoff–Witt theorem, Hopf algebras. The Baker–Campbell–Hausdorff formula, and its form due to Dynkin. Construction of a local Lie group from its Lie algebra, the globalization of local Lie groups. Cartan's theorem.

Lie algebras and their representations. Nilpotent and solvable Lie algebras, Engel's theorem, semisimple and reductive Lie algebras. The trace form of a representation, the Killing form. Cartan's criteria for the solvability and semisimplicity of Lie algebras. Casimir operators. Cohomology of Lie algebras, Whitehead lemmas and applications, Levi–Maltsev theorem.

Semisimple Lie algebras. Cartan subalgebra, weight eigenspaces, root space decomposition. Complex semisimple Lie algebras and their irreducible representations. Compact Lie algebras, their correspondence with complex semisimple Lie algebras.

TEXTBOOKS:

[1] Postnikov, M. M.: *Lie groups and Lie algebras; Lectures in geometry V.*
[2] Goto, M.; Grosshans, F. D.: *Semisimple Lie algebras*
[3] Csikós B.: *Lie-csoportok és Lie-algebrák* (notes on the homepage of Balázs Csikós; in Hungarian)

6. Topology

Main subject: *Prepare for all of the topics listed below.*
Secondary subject: *Required topics: 6.1.1, 6.1.2, 6.2.1 (without Gromov's theorem), 6.2.2.*

6.1. Algebraic topology

6.1.1 Homotopy theory. Homotopy groups. Long homotopy exact sequences of pairs and triplets of spaces, and fibrations. Freudenthal's theorem on homotopy groups of spheres. The homotopical Whitehead theorem. Homotopy excision. The Freudenthal suspension theorem. Pontryagin construction. The 0th, 1st and 2nd stable homotopy groups of spheres. Classification of vector bundles and principal G-bundles.

6.1.2 Homology theory. The definition of simplicial, singular, and cellular homologies and their isomorphism. The homological version of Whitehead's theorem. Lefschetz's theorem. Poincaré duality. Poincaré–Hopf theorem. Cup product. The Künneth formula on singular homology. The dual of the cup product is the intersection. De Rham cohomology. Gradient, curl and divergence in $\mathbb{R}^3$. The universal coefficient theorem.

6.1.3 The Leray–Hirsch spectral sequence. Serre's killing space method. Mod C theorems.

6.1.4 The definition of characteristic classes, and applications: immersions, embeddings. Exotic spheres. Cobordisms and characteristic numbers.

6.2. Differential topology

6.2.1 Embeddings and immersions. Whitney's theorems. Whitney–Graustein theorem. The theorems of Smale, Hirsch and Gromov. Compression theorem. Immersions of surfaces.

6.2.2 Morse theory, its application to prove Poincaré duality.

6.2.3 The h-cobordism theorem. The generalized Poincaré conjecture.

6.3. Cobordisms

Thom construction. The Pontryagin–Thom theorem on characteristic numbers, and its proof due to Buoncristiano. Computing $\Omega_*\otimes\mathbb{Q}$. Dold and Wall manifolds. Rohlin, Wall and Atiyah exact sequences. The 2-torsion of Ω_*. Ω_* has no odd torsion.

Textbooks:

[1] Hatcher, A. E.: *Algebraic topology*
[2] Hirsch, G.: *Differential topology*
[3] Milnor, J. W.: *Characteristic classes*
[4] Milnor, J. W.: *Morse theory*
[5] Milnor, J. W.: *h-cobordism theorem*
[6] Husemöller, D.: *Vector bundles*
[7] Switzer, R. M.: *Algebraic topology*
[8] Lecture notes by András Juhász at the homepage of András Szűcs (in Hungarian)

7. Discrete, combinatorial, finite and convex geometry

Main subject: *Choose* 4 *out of the following* 5 *topics.*
Secondary subject: *Choose any* 1, *except the first of the following* 5 *topics.*

7.1. Euclidean and affine geometry

Synthetic geometry of the euclidean space. Axiomatic foundations and development of euclidean geometry.

Elements of affine geometry. An affine space over a field. Affine subspaces. Affine independence, affine basis, affine coordinate systems. Equations of subspaces. Affine maps. Projections and symmetries of subspaces. The fundamental theorem of affine geometry. Centroid of a mass distribution, ratio of lengths, barycentric coordinates.

Geometry of euclidean vector spaces. Orientation. Gram–Schmidt orthogonalization. Application of Grassmann algebras in the geometry of subspaces. Angle and principal angles of subspaces. Distance between subspaces. Volume. Gram determinants, the Cayley–Menger determinant.

Geometry of spheres. Power of a point respect to a sphere, inversion, pencil of spheres. Spherical trigonometry.

Orthogonal groups and their structures in different dimensions. The canonical form of isometries and infinitesimal isometries. Classification of isometries in the plane and in the space. Generation of the orthogonal group by reflections, Cartan's theorem. Classification of n-dimensional regular polyhedra, their symmetry groups. Classification of discrete subgroups of $Iso(\mathbb{R}^2)$ and finite subgroups of $Iso(\mathbb{R}^3)$. Similarities.

Textbooks:

[1] Hajós Gy.: *Bevezetés a geometriába* (in Hungarian)
[2] Berger, M.: *Geometry I, II*

7.2. Non-euclidean Geometry

Elements of projective geometry. Projective extension of an affine space. Projective space derived from a vector space. The dual space. Projective coordinates. Projective subspaces. Cross-ratio, harmonic quadruples. Pappus's theorem and Desargues's theorem. Collineations, the fundamental theorem of projective geometry. Projective correlations.

Geometry of quadratic forms. Artin spaces. Orthogonality. Group of a quadratic form. Theorem of Witt. Projective quadrics. Classification of projective and affine quadrics. Polarity respect to a non-degenerate quadric; the center. Conic sections on the projective plane. The diameter and the asymptote. Conic sections as projective formations. Pascal's theorem and Brianchon's theorem. Pencil of conic sections.

Elements of hyperbolic geometry. The foundation of hyperbolic geometry according to the Appendix. Absolute parallelism and correspondence, related theorems. Parallel angle, geometry of cycles, Hilbert end-calculus. Models of hyperbolic geometry and their connection: the Beltrami–Cayley–Klein model, the models of Poincaré and the hyperboloid model. Hyperbolic trigonometry. Area and volume, Schläfli folmula.

Textbooks:

[1] Szenthe J.; Juhász R.: *A geometria alapjai* (in Hungarian)
[2] Strohmajer J.: *A geometria alapjai* (in Hungarian)
[3] Berger, M.: *Geometry I, II*

7.3. Discrete and combinatorial geometry

Lattices, quadratic forms. Theorems of Blichtfeld, Pick and Minkowski, applications in number theory. Regular and semiregular tessellations in spaces of constant curvatures.

Packings and coverings on the plane and in the space. Density, overlap size, gap size, TÉRIGÉNY. Saturated packings, solidity, separability. Dirichlet–Voronoi cells, Delaunay triangulation. Lattice packings, theorem of Fáry, Minkowski–Hlawka theorem, Rogers–Shephard inequality. Theorems of Dowker. Moment lemma.

Helly-type theorems. Colorful Carathéodory theorem, fractional Helly theorem, Tverberg's theorem, Erdős–Szekeres theorems (Horton set). Needle problems, Borsuk problem, Hadwiger problem, illumination problem.

Unit distances on the plane and in the space, geometrical graphs, theorem of Koebe.

Geometric transversality, Vapnik–Červonenkis dimension, ε-net theorem.

Textbooks:

[1] Matoušek, J.: *Lectures on discrete geometry*
[2] Fejes-Tóth L.: *Lagerungen in der Ebene, auf der Kugel und im Raum* (in German)
[3] Fejes-Tóth L.: *Regular figures*
[4] Pach J.: *Combinatorial geometry*

7.4. Convex geometry

Convex subsets of the real affine space. Convex hull. Topology of convex sets. Theorem of Carathéodory. Extremal points, Krein–Milman theorem. Exposed points, theorem of Straszewicz. Separating theorems.

Convex polyhedra and polytopes, face lattice. Polar polytope. Euler's theorem. Dehn–Sommerville equations, the theorem of lower and upper bound, Gale transform. Applications of polytopes: the weak perfect graph conjecture.

Measure concentration on the sphere, Dvoretzky's theorem, Milman's QS theorem (about projections of subspaces), Uryson inequality, inverse Blaschke–Santaló inequality.

Distance function, support function, mixed volume, Steiner theorem, quermass integrals of a convex body, fundamental theorems of Hadwiger about functionals, zonoids, symmetrizations of Minkowski, Steiner and Schwarz, Brunn–Minkowski inequality.

Textbooks:

[1] Grünbaum, B.: *Convex polytopes*
[2] Szabó L.: *Konvex geometria* (in Hungarian)
[3] Giannopolus, A.A.; Milman, V.D.: *Euclidean structure in finite-dimensional normed spaces* (in W.B. Jonson and J. Lindenstrauss ed., Handbook of the geometry of Banach spaces)
[4] Pisier, G.: *The volume of convex bodies and Banach space geometry*
[5] Schneider, R.: *Convex bodies: the Brunn–Minkowski theory*

7.5. Finite geometry

Existence of finite planes, Bruck–Ryser theorem. Coordinates on projective planes, connection of the configuration theorems (Desargues, Pappus) with the algebraic properties of the coordinate structure. Arcs, ovals, Segre's theorem, hyperovals. Covering sets, applications of the Rédei polynomial. Higher dimensional projective spaces. Description of collineations and polarities. Quadrics, Hermite varieties, circle geometries, generalized quadrilaterals. Some applications of finite geometries (graph theory, code theory).

Textbooks:

[1] Kiss Gy.; Szőnyi T.: *Véges geometriák* (in Hungarian)
[2] Hughes, D. R.; Piper, F. C.: *Projective planes*
[3] Hirschfeld, J. W. P.: *Projective geometries over finite fields*
[4] Hirschfeld, J. W. P.; Thas, J. A.: *General Galois geometries*

8. Probability theory

9. Stochastic processes

Main subject: *Choose 3 out of the following 4 main topics.*
Secondary subject: *Choose any 1 out of the 4 secondary topics (not from among the topics in the main subject).*

9.A Main subject

9.1. Markov chains and Markov processes

Markov property. Transition probabilities. Chapman–Kolmogorov equations in discrete and continuous parameter case. Classification of the states. Recurrence. Positive states. Ergod theorems (for transition probabilities and for the mean of functions). Sationary distribution.
Description of finite state-space Markov chains with positive matrices. Perron–Frobenius theorems. Birth-death processes.
Infinitesimal generator of Markov processes with continuous trajectories. Properties of the transition-probability function, continuity, differentiability, Chapman–Kolmogorov equation. Feller processes. Potentials. Erdős–Kac theorem.

9.2. Stationary processes

Strong and weak stationary processes. Case of the Gaussian processes.
Random orthogonal stochastic measure. Bochner–Khinchin theorem, Herglotz theorem. Spectral representation of stationary processes. The spectral measure. The Gaussian case.
The consequencies of the L^2 isometry. Kotelnikov–Shannon theorem about the density of the sampling. Ergodicity.
Classification of the weak stationary processes based on the linear filtration. Wold decomposition. Spectral density function of absolutely regular processes.
Strong stationary processes, mixing. Birkhoff's ergodic theorem about strong stationary processes. The correlogram and the periodogram. Consistent estimation of the spectral function.

9.3. Convergence of measures in function spaces

Processes with continuous trajectories, modulus of continuity. Functions without discontinuity of the second kind. Spaces $C[0,1]$ and $D[0,1]$, tightness in these spaces.
The Wiener measure. Different constructions and the properties of the Wiener process. The behaviour of the trajectories.
Weak invariance principle about the summation process (Donsker theorem). The convergence of the empirical cumulative distribution function as a stochastic process to the Brownian bridge.
Random orthogonal measure. Wiener integral and its applications.

9.4. Processes with independent increments

Characterization of the charactersitic function of the processes with independent and stationary increments, the Lévy–Khinchin theorem.
Representation of the processes with independent increments as an integral with respect to a random point measure.
Decomposition of the trajectories of the processes with independent increments as a sum of a continuous and a jump process. Gaussian processes with independent increments.
Differential equation about the expected value of the functionals of the processes with independent increments.

Textbooks:

[1] S. Karlin, H. M. Taylor: *A First Course in Stochastic processes.* Academic Press, New York, San Francisco, London, 1975.
[2] M. H. A. Davis: *Linear Estimation and Stochastic Control.* Chapman and Hall, London, 1977.
[3] M. B. Priestley: *Spectral Analysis and Time Series.* Academic Press, New York, 1981.
[4] Yu. Rozanov: *Stationary time series.* Holden Day, San Francisco, 1967.
[5] K. L. Chung, K. L.: *Markov Chains with Stationary Transition Probabilities.* Springer, Berlin, 1967.
[6] D. L. Isaacson, R. W. Madsen: *Markov Chains: Theory and Applications.* Wiley, New York, 1976.

9.B Secondary subject: Markov chains, Markov processes

Stochastic processes: Markov property, strong Markov property, homogeneity. Discrete-time Markov chains: definition, transition-probability matrix, Chapman–Kolmogorov equations. Classification of the states. Period, recurrence. Convergence of the transition-probabilities. Stationary distribution. Law of large numbers and central limit theorem for functional of irreducible, positive recurrent Markov chain.
Transition probabilities with taboo states. Regular measure, Doeblin quotient theorem. Time-reversed Markov chain. Absorbing probabilities. Perron–Frobenius theorems.
Birth-death processes.
Infinitesimal generator of Markov processes with continuous trajectories. Properties of the transition-probability function, continuity, differentiability, Chapman–Kolmogorov equation. Feller processes.
Potentials. Erdős–Kac theorem.

Textbooks:

[1] S. Karlin, H. M. Taylor: *A First Course in Stochastic processes.* Academic Press, New York, San Francisco, London, 1975.
[2] K. L. Chung, K. L.: *Markov Chains with Stationary Transition Probabilities.* Springer, Berlin, 1967.
[3] D. L. Isaacson, R. W. Madsen: *Markov Chains: Theory and Applications.* Wiley, New York, 1976.
[4] G. Kemeny, J. L. Snell: *Finite Markov Chains.* Van Nostrand, Princeton, 1960.

9.C Secondary subject: Stationary processes

Strong and weak stationary processes. Case of the Gaussian processes.
Random orthogonal stochastic measure. Bochner–Khinchin theorem, Herglotz theorem. Spectral representation of stationary processes. The spectral measure.
The consequencies of the L^2 isometry. The fundamental theorem of the sampling. Ergodicity.
Classification of the weak stationary processes based on the linear filtration. Wold decomposition and its relation to the Lebesgue decomposition of the spectral measure. Spectral density function of absolutely regular processes.
ARMA processes. Weak stationary processes with rational spectral density function. Representation of the ARMA processes with state equation.
Strong stationary processes. Birkhoff's ergodicity theorem.
Different types of mixing.

Textbooks:

[1] S. Karlin, H. M. Taylor: *A First Course in Stochastic processes.* Academic Press, New York, San Francisco, London, 1975.
[2] T. W. Anderson: *The Statistical Analysis of Time Series.* Wiley, New York 1971.
[3] M. B. Priestley: *Spectral Analysis and Time Series.* Academic Press, New York, 1981.
[4] Yu. Rozanov: *Stationary time series.* Holden Day, San Francisco, 1967.

9.D Secondary subject: Processes with independent increments

Characterization of the charactersitic function of the processes with independent and stationary increments. The characteristic function of the infinitely divisible distributions, the Lévy–Khinchin formula. Poisson point process and integral. Characterization of the properties of the distribution (nonnegativity, finite variance) using the characteristic function. The characteristic function of stable distributions. Generating random variables from stable distributions. The order of the tail.probabilities of stable distributions.
Representation of the point processes with independent increments as an integral with respect to a random point measure.
Decomposition of the trajectories of the processes with independent increments as a sum of a continuous and a jump process. Gaussian processes with independent increments.
Functionals of processes with independent increments. Differential equation about the expected value.

Textbooks:

[1] I. I. Gikhman, A. V. Skorokhod: *Introduction to the Theory of Random Processes.* W. B. Saunders, Philadelphia, London, Toronto, 1969.
[2] O. Kallenberg: *Random Measures.* Academic Verlag, Berlin, 1976.
[3] V. V. Petrov: *Sums of Independent Random Variables.* Springer, Berlin, 1972.

9.E Secondary subject: Stochastic analysis

Local martingale, semimartingale. Integral with respect to a semimartingale. The properties of the integral. Quadratic variation, BDG inequality, isometry theorem. Itô formula, Lévy characterization, Girsanov theorem, Kazamaki's and Novikov's condition. Itô integral.
Stochastic differential equations, strong and weak solutions, uniqueness in law and pathwise-uniqueness of the solutions, relation between them. Weak solution with change of measure and time. Fubini's theorem, local time. Occupation times formula. The case of Hölder-continuous coefficients in one dimension. Tsirelson's example. Ordering theorem.

Textbooks:

[1] D. Revuz, M. Yor: *Continuous Martingales and Brownian Motion.* 3rd ed. Springer, Berlin, 1999.
[2] P. E. Protter: *Stochastic Integration and Differential Equations.* 2nd ed. Springer, Berlin, 1990.

10. Statistics

11. Algebra

Main subject: *Required topics: the first two, mandatory topics, and* 3 *further (arbitrary) topics from among the* 9 *elective ones.*
Secondary subject: *Required: the first two, mandatory topics: Group theory and Ring theory.*

Mandatory topics

11.1. Group theory

- Free groups. The Nielsen–Schreier theorem. Groups given by generators and relations. Varieties of groups, Birkhoff's theorem.
- Group action. Permutation groups, transitive, primitive, multiply transitive groups.
- The Sylow theorems. Nilpotent groups, central series. p-groups, the Frattini subgroup, Burnside's basis theorem.
- Solvable groups, theorems of Hall.
- Abelian groups. Fundamental theorem of finitely generated Abelian groups.
- Linear groups, the simplicity of $\mathrm{PSL}(n, F)$
- Linear representations of finite groups over the field of complex numbers. Irreducible, completely reducible representations, Maschke's theorem, Schur's lemma. Characters. The equivalence of representations. Character table, orthogonality relations.
- Burnside's p-q-theorem: groups of order $p^\alpha q^\beta$ are solvable.
- Induced representations, Frobenius reciprocity, Frobenius groups.

11.2. Ring theory

- Associative algebras. Structure theory: radical, semisimplicity. Chain conditions, Hilbert's basis theorem.
- Categories, functors, natural transformations. Basic properties of the Hom and the tensor functors. Exactness of functors, projective and injective modules.
- Chain complexes, homology modules, long exact sequence of homology modules.
- Commutative rings. Prime and primary ideals. Decompositions of ideals. The Hilbert Nullstellensatz.
- Basic notions about Lie algebras. Solvable and nilpotent Lie algebras. Root systems, quadratic forms. Dynkin diagrams, classification of complex semisimple Lie algebras.

Elective topics

11.3. Finite groups

- Group extensions. Semidirect product, the Schur–Zassenhaus theorem.
- The transfer and its applications. Burnside's normal p-complement theorem.
- Finite p-groups. Extraspecial, regular, powerful p-groups, p-groups of maximal class.
- Commutator calculus.
- Subgroup lattices.

11.4. Simple groups

- The classification of finite simple groups.
- Classical simple groups.
- Simple groups of Lie type.

- Sporadic simple groups.

11.5. Permutation groups

- Description of multiply transitive groups.
- Primitive permutation groups, the O'Nan–Scott theorem.
- Wreath product.
- Permutation groups of prime degree.
- Frobenius groups, Zassenhaus groups.

11.6. Group representations

- Clifford's theorems.
- Character degrees.
- Characterizations using the centralizers of involutions.
- Projective representations, the Schur multiplier.

11.7. Commutative algebra and algebraic geometry

- Localization.
- Discrete valuation rings, other famous classes of rings.
- Krull's principal ideal theorem.
- Elements of algebraic geometry: affine and projective algebraic varieties.
- Birational mappings.
- The Riemann–Roch theorem.

11.8. Non-commutative rings

- Artinian and Noetherian rings.
- The Noether–Skolem theorem.
- Generalizations of Artinian rings, some other special classes of rings.
- Goldie's theory.
- Rings with polynomial identities.
- Central simple algebras, the Brauer group.
- Field theory, skew fields.

11.9. Homological algebra

- The structure of projective and injective modules.
- Derived functors, the Ext and the Tor functors.
- Equivalence classes of exact sequences of modules and the Ext functor, the Yoneda product.
- Homological dimensions.
- Applications in representation theory.
- Some famous conjectures and theorems in homological algebra.

11.10. Universal algebra

- Varieties, free algebras, theorems of Birkhoff.
- Congruence identities, Maltsev type theorems.
- Clones.
- Completeness results, discriminator varieties.
- Commutator theory. Foundations of tame congruence theory and some applications (e.g. the free spectrum, residually small varieties, decidability, algebraic aspects of the constraint satisfaction problem).

11.11. Lattice theory

- Distributive and modular lattices, Boole algebras, the duality between distributive lattices and posets.
- Free lattices.
- Projective geometries and complemented modular lattices.
- Desarguesian lattices, coordinatization.
- Representation of lattices in partition lattices.
- Algebraic lattices.

12. Number theory

Main subject: *Choose* 4 *of the following topics.*
Secondary subject: *Choose* 2 *of the following topics.*

12.1. Combinatorial number theory

The Brun sieve and its applications. Schnirelmann's theorems, prime numbers form a basis. Further addition theorems. Additive and multiplicative Sidon sequences. Divisibility in sequences, primitive sequences. The "larger sieve". Hilbert cube in dense sequences, applications. Theorems of van der Waerden and Szemerédi about arithmetic progressions. Schur's theorem.

12.2. Exponential sums. Applications of Fourier analysis in number theory

Additive and multiplicative characters, their connection, applications. Gauss sums. The Pólya–Vinogradov inequality. Estimates for the smallest quadratic nonresidue. Kloosterman sums. Analytic, arithmetic and character form of the large sieve, applications. Irregularities of distribution in arithmetic progressions, lower bound for the character sums. Uniform distribution, Weyl's criterion. Discrepancy, the Erdős–Turán inequality, irregularities of distributions. Van der Corput's method, applications.

12.3. Additive number theory

The Erdős–Fuchs theorem. Estimation of Weyl sums. The Hardy–Littlewood method, Waring's problem. The Goldbach conjecture, Vinogradov's theorem. 3-long arithmetic progressions in dense sequences. Structure and multiplicative properties of sum sets and difference sets. Partitions.

12.4. Analytic and multiplicative number theory

The gamma function. The ζ function (analytic continuation, Riemann's functional equation, zero-free region, order of magnitude, density bound). The proof of the prime number theorem with error term. Prime numbers in arithmetic progressions, application of L-functions. The Siegel–Walfisz theorem. Prime numbers in short intervals, application of density bounds. The Selberg sieve. The Bombieri–Vinogradov theorem. Numbers built from small primes.

12.5. Arithmetic functions, probabilistic methods in number theory

The Turán–Kubilius inequality. The Erdős–Kac theorem. Mean value of multiplicative functions. Limit distribution of additive functions. Characterization of additive and multiplicative functions, subsets of unicity. Application of probabilistic methods, investigation of the sequence of sums. Algebraic number theory, diophantine equations. Algebraic number fields and their ring of integers. Quadratic and cyclotomic fields. Ideals, fundamental theorem. Dedekind rings, consequences. Ideal classes, class numbers, units, Dirichlet's theorem. Pell's equation, Mordell's equation, special cases of Fermat's last theorem, regular primes. Elements of diophantine approximations, continued fractions, connection to diophantine equations, the Thue–Siegel theorem. The Baker method. Elements of geometric number theory.

12.6. Computational number theory

Time complexity of basic operations and basic number theoretic tasks. For $n = pq$ finding p and q is polynomially equivalent to computing $\varphi(n)$. Computing powers in modular arithmetic. Factorization using algebraic identities, factorization of Mersenne numbers. Basic notions of cryptography, RSA, discrete logarithm, the Diffie–Hellman key exchange system. Primality testing, pseudoprimes. Factorization, factor basis algorithm, the quadratic sieve. Elliptic curves over finite fields, the analogue of the Diffie–Hellman key exchange. Pseudorandom binary and $[0, 1)$-sequences, application in cryptography and in relation to Monte Carlo methods.

Textbooks:

[1] Hardy, G. H.; Wright, E. M.: *An Introduction to the Theory of Numbers*
[2] Halberstam, H.; Roth, K. F.: *Sequences*
[3] Pomerance, C.; Sárközy, A.: *Combinatorial Number Theory* (in: *Handbook of Combinatorics*)
[4] Graham, S. W.; Kolesnik, G.: *Van der Corput's Method of Exponential Sums* pp. 1–44.
[5] Kuipers, L.; Niederreiter, H.: *Uniform Distribution of Sequences* pp. 1–163.
[6] Vaughan, R. C.: *The Hardy–Littlewood Method* Chapters 1–5, 7 and 10.
[7] Davenport, H.: *Multiplicative Number Theory*
[8] Elliott, P. D. T. A.: *Probabilistic Number Theory*, Chapters 3., 4., 5., 6., 9., 12. and 15.
[9] Pollard, H.; Diamond, H. G.: *The Theory of Algebraic Numbers*
[10] Mordell, L. J.: *Diophantine Equations*
[11] Shorey, T. N.; Tijdeman, R.: *Exponential Diophantine Equations*
[12] Koblitz, N.: *A Course in Number Theory and Cryptography*

13. Discrete mathematics

Main subject: *Choose 4 of the following topics.*
Secondary subject: *Choose 2 of the following topics or choose a monography with the examiner.*

13.1. Graph theory 1.

Paths and cycles, Eulerian path, Hamiltonian path. Ramsey-type theorems in graph theory, set theory, number theory and geometry. Estimations of Ramsey numbers. Turán-type theorems. Forbidden non-bipartite subgraphs, asymptotic structure of extremal graphs, number of copies of given subgraphs. Erdős–Stone–Simonovits theorem. Forbidden bipartite subgraphs, Turán number of paths and $K_{p,q}$. Constructions using algebra and finite geometry. Szemerédi regularity lemma and its applications. Turán–Ramsey-type theorems. Random graph models: Erdős–Rényi. Expected value and concentration of graph parameters. Martingale method, threshold function, evolution around $p = \frac{\log n}{n}$. Pseudorandom graphs.

13.2. Graph theory 2.

Matchings in bipartite and non-bipartite graphs. Tutte theorem. Stable matching problem, Galvin's theorem. Min-max theorems in graph theory. Vertex-connectivity, Menger's theorem and its variations. Vertex and edge colorings, Brooks' and Vizing's theorem. Chromatic number, chromatic index, critical graphs. Plane and surface embeddings, Kuratowski's theorem, four color theorem. Graph minors, Robertson–Seymour theorem. Chromatic polynomial and its properties. Perfect graphs. Construction of graphs with a large chromatic number containing no small cycles.

13.3. Set systems

Classical theorems: Sperner's theorem, LYM inequality, Erdős–Ko–Rado and Erdős–Rado theorems. Fischer's inequality and the theorem of De Bruijn–Erdős. L-intersecting extremal set systems and cross-intersecting set systems. Intersection theorems about linear subspaces. Forbidden substructures. Shadow of a set system, shadow-intersection, Kruskal–Katona theorem. Matching, covering, fractional matching and fractional covering of hypergraphs. Probabilistic method: first and second moment methods. Lovász local lemma and its applications. Quotient of the optimal fractional and integer solution. Vapnik–Chervonenkis dimension. Colorings. Discrepancy, Beck–Fiala theorem, Spencer's six standard deviations theorem.

13.4. Enumerative combinatorics

Generating function method. Ordinary and exponential generating functions. Partitions of sets and numbers, Bell numbers, Stirling numbers, recursions of these. Methods for showing identities: WZ pairs (Wilf and Zeilberger), Gosper's algorithm for indefinite summation and Zeilberger's algorithm. Asymptotic enumeration, Lagrange inversion, Cauchy method, saddle point method. Sieves. Möbius function of partially ordered sets, inversion formulas. Pólya enumeration theorem.

13.5. Symmetric structures, algebraic methods

Existence of block designs, necessary conditions, asymptotic results. Steiner triple systems, t-designs, Ray–Chaudhuri–Wilson inequality for t-designs. Projective planes, Hadamard matrices, biplanes. Constructions, extending block designs. Difference sets, multiplier theorem. Eigenvalues of graphs. Chromatic number and eigenvalues. Largest eigenvalue. Expanders. Strongly regular graphs, friendship theorem. Distance-regular graphs, association scheme. Codes, Hamming code. Examples. Perfect codes and block designs. Golay codes and Witt designs.

13.6. Combinatorial optimization, algorithms

Algorithms for the shortest path problem. Graph traversals. Minimum-cost flows. Polyhedral combinatorics. Polyhedrons of matchings. Polyhedrons of perfect graphs. Ear decompositions of 2-connected, strongly connected and factor-critical graphs. Data structures. Approximation algorithms. Splitting off and applications. Covering and packing with trees and arborescences. Matroids, greedy algorithm. Cycle matroid, duality. Matroid intersection, sum of matroids, applications. Basic elements of complexity theory. P and NP classes, NP-completeness

13.7. Infinitary combinatorics

Infinite graphs. Decomposition of graphs into trees. Aharoni–Berger–Menger theorem. Chromatic number, the dependence on the choice of axiom. De Bruijn–Erdős theorem. Triangle-free graphs with large chromatic number. Every graph with uncountable chromatic number has a cycle of length four. Set mappings. Fodor Géza's theorem. Almost disjoint families, for which cardinalities do they exist. Elekes–Homann theorem. Generalizations of Ramsey's theorem: Erdős–Rado theorem. Delta-system lemma.

Textbooks:

[1] Beth, T.; Jungnickel, D.; Lenz, J.: *Design theory*

[2] Bollobás B.: *Random graphs*

[3] Brouwer, A. E.; Cohen, A. M.: *Distance regular graphs*

[4] Hajnal A.; Hamburger P.: *Halmazelmélet* (in Hungarian)

[5] Graham, R., Knuth, D.; Patashnik, O.: *Konkrét matematika*, Műszaki Kiadó (in Hungarian)

[6] Frank A.: *Gráfelmélet* (lecture notes in Hungarian)

[7] Frank A.: *Matroidelmélet* (lecture notes in Hungarian)

[8] Hajnal P.: *Halmazrendszerek*, Polygon Kiadó (in Hungarian)

[9] Hajnal P.: *Gráfelmélet*, Polygon Kiadó (in Hungarian)

[10] Hajnal P.: *Összeszámlálási problémák*, Polygon Kiadó (in Hungarian)

[11] Lovász L.: *Kombinatorikai problémák és feladatok*, Typotex Kiadó (in Hungarian)

[12] van Lint, J. H.; Wilson, R.: *A course in combinatorics*, Oxford Univ. Press Handbook of Combinatorics

[13] Alon N.; Spencer, J.: *The probabilistic method*

14. Set theory and mathematical logic

Main subject: *Choose 4 out of 14.1 or 14.2*
Secondary subject: *Choose 2 (?) out of 14.1 or 14.2*

14.1. Set theory

14.1.1 Combinatorial set theory. Trees. Partition relations. Set mappings. Applications of the Todorcevic-walk. Stepping-up lemmas. Infinite graphs. Jónsson algebras.

14.1.2 Forcing. Chain conditions. Iterated forcing. Classical problems. Martin's Axiom, Proper forcing, Martin's Maximum.

14.1.3 Large cardinals. Inaccessible, weakly compact, Ramsey, measurable cardinals. Indescribability. Elementary embeddings. Extenders. Inner models. Covering theorems. Woodin, strongly compact, supercompact, huge cardinals. Large cardinals forcings. Saturated ideals.

14.1.4 PCF theory. Foundations of PCF theory. Transitive generators. Estimates for the power function. Jónsson algebras.

14.1.5 Set-theoretic topology. Inequalities between fundamental cardinal functions (e.g. weight, density, cellularity, π-weight, character, etc). Special classes of spaces. Cardinal functions on subspaces. sup = max problems, Products, S-, L-spaces.

14.1.6 Descriptive set theory. Borel and analytic sets. Proper and complete analytic sets. Uniformization, reduction, separation theorems. Determinacy of Borel sets. Axiom of Determinacy (AD).

14.2. Model theory

14.2.1 Model theory. Completeness, compactness theorem. The omitting types theorem. Interpolation theorems. Saturated, special, universal models. Two-cardinal theorems. Ultraproduct. Ax–Kochen–Jersov theory. Finite models. Stability, rank. Prime models. Morley categoricity theorem. Stability spectrum theorem. Shelah's main gap theorem. Definability theory. Beth theorem. Chang–Makkai theorem. Beth properties.

14.2.2 Recursion theory. Rice's theorem. Creative sets, simple sets. Post's problem. Word problems. Degrees. Jump operation. Turing machines. Computability. Decidability.

14.2.3 Proof theory and limitative theorems. Gödel-type incompliteness theorems. Kalmár's proof too. Tarski's theorem about the undefinability of truth. Cut-elimination. Gentzen's theorem. Theorems that are unprovable in PA.

14.2.4 Algebraic logic and boolean algebras with operators Boolean algebras and basics of universal algebra (e.g. discriminator varieties, congruence distributive varieties). Theories of relations. Relation calculuses. Algebras of multiple argument relations. Residuated Boolean algebras. Representation of the non-normal case by partial accessibility relations.

14.2.5 Cylindric, poliadic and relation algebras. Algebraization of arbitrary logic. Algebras of binary relations, algebras of relations with n arguments. Positive, negative theorems. Daigneault–Monk–Keisler theorem (poliadic algebras) and its limitations. Algebraizations of models, class of models, interpretations. Decidability of cylindric relativized algebras. Finite model property of $L_{\omega,\omega}$-s bounded initial segments.

14.2.6 Not completely classic logics. Computer science logics. Dynamic logic. Temporal logic. Linguistical logic semantics. Intuitionist logic and Kripke models. Lambek calculus and its completeness. Multi-modal and arrow logics. Connection to Tarski–Jónsson-type Boolean algebras with operators. Decidability questions.

Textbooks:

[1] P. Erdős, A. Hajnal, A. Máté, R. Rado: *Combinatorial Set Theory: Partition Relations for Cardinals*
[2] S. Shelah: *Proper and Improper Forcing*
[3] T. Jech: *Multiple forcing*
[4] A. Kanamori: *The higher innite*
[5] M. Burke, M. Magidor: *S.Shelah's pcf theory*
[6] I. Juhász: *Cardinal Functions, Ten Years Later*
[7] Y. N. Moschovakis: *Descriptive set theory*
[8] C. C. Chang, H. J. Keisler: *Model Theory,* North Holland
[9] S. Shelah: *Classication Theory,* North Holland
[10] H. Rogers: *Theory of Recursive Functions & Effective Computability,* McGraw-Hill, 1967 *Handbook of Mathematical Logic.* North Holland
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