Fact sheet on planar S-systems

Balázs Boros, Josef Hofbauer, Stefan Müller, and Georg Regensburger

November 28, 2018

Abstract

This document is a collection of results on planar S-systems. Without explanations. It is based on the paper “Planar S-systems: Global stability and the center problem” by Balázs Boros, Josef Hofbauer, Stefan Müller, and Georg Regensburger in *Discrete and Continuous Dynamical Systems - Series A*, 39(2):707-727, 2019.

1 ODE

$$\boxed{\begin{aligned}\dot{x}_1 &= \alpha_1\, x_1^{g_{11}} x_2^{g_{12}} - \beta_1\, x_1^{h_{11}} x_2^{h_{12}} \\ \dot{x}_2 &= \alpha_2\, x_1^{g_{21}} x_2^{g_{22}} - \beta_2\, x_1^{h_{21}} x_2^{h_{22}}\end{aligned}}$$

$$x = \frac{x_1}{x_1^*} \qquad \gamma_1 = \frac{\alpha_1 (x_1^*)^{g_{11}} (x_2^*)^{g_{12}}}{x_1^*} = \frac{\beta_1 (x_1^*)^{h_{11}} (x_2^*)^{h_{12}}}{x_1^*}$$

$$y = \frac{x_2}{x_2^*} \qquad \gamma_2 = \frac{\alpha_2 (x_1^*)^{g_{21}} (x_2^*)^{g_{22}}}{x_2^*} = \frac{\beta_2 (x_1^*)^{h_{21}} (x_2^*)^{h_{22}}}{x_2^*}$$

$$\boxed{\begin{aligned}\dot{x} &= \gamma_1 \left(x^{g_{11}} y^{g_{12}} - x^{h_{11}} y^{h_{12}}\right) \\ \dot{y} &= \gamma_2 \left(x^{g_{21}} y^{g_{22}} - x^{h_{21}} y^{h_{22}}\right)\end{aligned}}$$

$$u = \frac{1}{\gamma_1} \log x \qquad v = \frac{1}{\gamma_2} \log y$$

$$\begin{aligned}
a_1 &= \gamma_1 (g_{11} - 1) & b_1 &= \gamma_2 g_{12} \\
a_2 &= \gamma_1 (h_{11} - 1) & b_2 &= \gamma_2 h_{12} \\
a_3 &= \gamma_1 g_{21} & b_3 &= \gamma_2 (g_{22} - 1) \\
a_4 &= \gamma_1 h_{21} & b_4 &= \gamma_2 (h_{22} - 1)
\end{aligned}$$

$$\boxed{\begin{aligned}\dot{u} &= \mathrm{e}^{a_1 u + b_1 v} - \mathrm{e}^{a_2 u + b_2 v} \\ \dot{v} &= \mathrm{e}^{a_3 u + b_3 v} - \mathrm{e}^{a_4 u + b_4 v}\end{aligned}}$$

2 Matrices

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

$$H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$$

$$J = \begin{pmatrix} a_1 - a_2 & b_1 - b_2 \\ a_3 - a_4 & b_3 - b_4 \end{pmatrix}$$

$$\Gamma = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}$$

3 Number of equilibria

ODE	state space	number of equilibria
$\dot{x}_1 = \alpha_1\, x_1^{g_{11}} x_2^{g_{12}} - \beta_1\, x_1^{h_{11}} x_2^{h_{12}}$ $\dot{x}_2 = \alpha_2\, x_1^{g_{21}} x_2^{g_{22}} - \beta_2\, x_1^{h_{21}} x_2^{h_{22}}$	$\mathbb{R}^2_+$	$\begin{cases} 1, & \text{if } \det(G-H) \neq 0 \\ 0 \text{ or } \infty, & \text{if } \det(G-H) = 0 \end{cases}$
$\dot{x} = \gamma_1 \left(x^{g_{11}} y^{g_{12}} - x^{h_{11}} y^{h_{12}}\right)$ $\dot{y} = \gamma_2 \left(x^{g_{21}} y^{g_{22}} - x^{h_{21}} y^{h_{22}}\right)$	$\mathbb{R}^2_+$	$\begin{cases} 1, & \text{if } \det(G-H) \neq 0 \\ \infty, & \text{if } \det(G-H) = 0 \end{cases}$
$\dot{u} = \mathrm{e}^{a_1 u + b_1 v} - \mathrm{e}^{a_2 u + b_2 v}$ $\dot{v} = \mathrm{e}^{a_3 u + b_3 v} - \mathrm{e}^{a_4 u + b_4 v}$	$\mathbb{R}^2$	$\begin{cases} 1, & \text{if } \det J \neq 0 \\ \infty, & \text{if } \det J = 0 \end{cases}$

Assume for the rest that

$$\det(G-H) \neq 0$$

or, equivalently,

$$\det J \neq 0.$$

The equivalence is a consequence of the equality

$$J = (G-H) \cdot \Gamma.$$

4 The unique equilibrium

ODE	unique equilibrium
$\dot{x}_1 = \alpha_1\, x_1^{g_{11}} x_2^{g_{12}} - \beta_1\, x_1^{h_{11}} x_2^{h_{12}}$ $\dot{x}_2 = \alpha_2\, x_1^{g_{21}} x_2^{g_{22}} - \beta_2\, x_1^{h_{21}} x_2^{h_{22}}$	$(x_1^*, x_2^*) = \left(\left(\frac{\beta_1}{\alpha_1}\right)^{\frac{g_{22}-h_{22}}{\det(G-H)}} \left(\frac{\beta_2}{\alpha_2}\right)^{-\frac{g_{12}-h_{12}}{\det(G-H)}}, \left(\frac{\beta_1}{\alpha_1}\right)^{-\frac{g_{21}-h_{21}}{\det(G-H)}} \left(\frac{\beta_2}{\alpha_2}\right)^{\frac{g_{11}-h_{11}}{\det(G-H)}} \right)$
$\dot{x} = \gamma_1 \left(x^{g_{11}} y^{g_{12}} - x^{h_{11}} y^{h_{12}}\right)$ $\dot{y} = \gamma_2 \left(x^{g_{21}} y^{g_{22}} - x^{h_{21}} y^{h_{22}}\right)$	$(x^*, y^*) = (1, 1)$
$\dot{u} = \mathrm{e}^{a_1 u + b_1 v} - \mathrm{e}^{a_2 u + b_2 v}$ $\dot{v} = \mathrm{e}^{a_3 u + b_3 v} - \mathrm{e}^{a_4 u + b_4 v}$	$(u^*, v^*) = (0, 0)$

5 The Jacobian matrix at the unique equilibrium

ODE	Jacobian matrix
$\dot{x}_1 = \alpha_1\, x_1^{g_{11}} x_2^{g_{12}} - \beta_1\, x_1^{h_{11}} x_2^{h_{12}}$ $\dot{x}_2 = \alpha_2\, x_1^{g_{21}} x_2^{g_{22}} - \beta_2\, x_1^{h_{21}} x_2^{h_{22}}$	$\begin{pmatrix} \alpha_1\, (x_1^*)^{g_{11}} (x_2^*)^{g_{12}} & 0 \\ 0 & \alpha_2\, (x_1^*)^{g_{21}} (x_2^*)^{g_{22}} \end{pmatrix} \cdot (G-H) \cdot \begin{pmatrix} \frac{1}{x_1^*} & 0 \\ 0 & \frac{1}{x_2^*} \end{pmatrix}$
$\dot{x} = \gamma_1 \left(x^{g_{11}} y^{g_{12}} - x^{h_{11}} y^{h_{12}}\right)$ $\dot{y} = \gamma_2 \left(x^{g_{21}} y^{g_{22}} - x^{h_{21}} y^{h_{22}}\right)$	$\Gamma \cdot (G-H)$
$\dot{u} = \mathrm{e}^{a_1 u + b_1 v} - \mathrm{e}^{a_2 u + b_2 v}$ $\dot{v} = \mathrm{e}^{a_3 u + b_3 v} - \mathrm{e}^{a_4 u + b_4 v}$	J

6 The dihedral group D_4 acts on our family of ODEs

$\mathbf{r}_0$	id	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}a_1 & a_2 & a_3 & a_4\\ b_1 & b_2 & b_3 & b_4\end{pmatrix}$
$\mathbf{r}_1$	$+90°$	$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}-b_4 & -b_3 & -b_1 & -b_2\\ a_4 & a_3 & a_1 & a_2\end{pmatrix}$
$\mathbf{r}_2$	$+180°$	$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$	$\begin{pmatrix}-a_2 & -a_1 & -a_4 & -a_3\\ -b_2 & -b_1 & -b_4 & -b_3\end{pmatrix}$
$\mathbf{r}_3$	$+270°$	$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$	$\begin{pmatrix}b_3 & b_4 & b_2 & b_1\\ -a_3 & -a_4 & -a_2 & -a_1\end{pmatrix}$
$\mathbf{s}_0$	x-axis	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$	$\begin{pmatrix}a_1 & a_2 & a_4 & a_3\\ -b_1 & -b_2 & -b_4 & -b_3\end{pmatrix}$
$\mathbf{s}_1$	$x = y$ line	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}b_3 & b_4 & b_1 & b_2\\ a_3 & a_4 & a_1 & a_2\end{pmatrix}$
$\mathbf{s}_2$	y-axis	$\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}-a_2 & -a_1 & -a_3 & -a_4\\ b_2 & b_1 & b_3 & b_4\end{pmatrix}$
$\mathbf{s}_3$	$y = -x$ line	$\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$	$\begin{pmatrix}-b_4 & -b_3 & -b_2 & -b_1\\ -a_4 & -a_3 & -a_2 & -a_1\end{pmatrix}$

$\mathbf{r}_i \cdot \mathbf{r}_j = \mathbf{r}_{i+j}$

	$\mathbf{r}_0$	$\mathbf{r}_1$	$\mathbf{r}_2$	$\mathbf{r}_3$
$\mathbf{r}_0$	$\mathbf{r}_0$	$\mathbf{r}_1$	$\mathbf{r}_2$	$\mathbf{r}_3$
$\mathbf{r}_1$	$\mathbf{r}_1$	$\mathbf{r}_2$	$\mathbf{r}_3$	$\mathbf{r}_0$
$\mathbf{r}_2$	$\mathbf{r}_2$	$\mathbf{r}_3$	$\mathbf{r}_0$	$\mathbf{r}_1$
$\mathbf{r}_3$	$\mathbf{r}_3$	$\mathbf{r}_0$	$\mathbf{r}_1$	$\mathbf{r}_2$

$\mathbf{r}_i \cdot \mathbf{s}_j = \mathbf{s}_{i+j}$

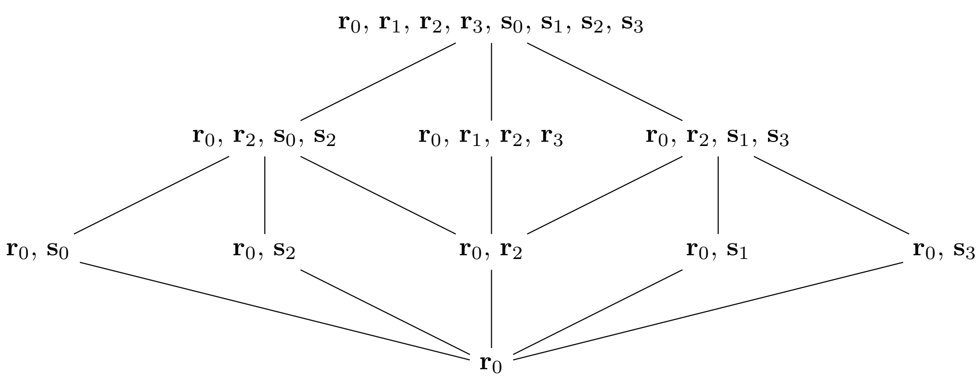
	$\mathbf{s}_0$	$\mathbf{s}_1$	$\mathbf{s}_2$	$\mathbf{s}_3$
$\mathbf{r}_0$	$\mathbf{s}_0$	$\mathbf{s}_1$	$\mathbf{s}_2$	$\mathbf{s}_3$
$\mathbf{r}_1$	$\mathbf{s}_1$	$\mathbf{s}_2$	$\mathbf{s}_3$	$\mathbf{s}_0$
$\mathbf{r}_2$	$\mathbf{s}_2$	$\mathbf{s}_3$	$\mathbf{s}_0$	$\mathbf{s}_1$
$\mathbf{r}_3$	$\mathbf{s}_3$	$\mathbf{s}_0$	$\mathbf{s}_1$	$\mathbf{s}_2$

$\mathbf{s}_i \cdot \mathbf{r}_j = \mathbf{s}_{i-j}$

	$\mathbf{r}_0$	$\mathbf{r}_1$	$\mathbf{r}_2$	$\mathbf{r}_3$
$\mathbf{s}_0$	$\mathbf{s}_0$	$\mathbf{s}_3$	$\mathbf{s}_2$	$\mathbf{s}_1$
$\mathbf{s}_1$	$\mathbf{s}_1$	$\mathbf{s}_0$	$\mathbf{s}_3$	$\mathbf{s}_2$
$\mathbf{s}_2$	$\mathbf{s}_2$	$\mathbf{s}_1$	$\mathbf{s}_0$	$\mathbf{s}_3$
$\mathbf{s}_3$	$\mathbf{s}_3$	$\mathbf{s}_2$	$\mathbf{s}_1$	$\mathbf{s}_0$

$\mathbf{s}_i \cdot \mathbf{s}_j = \mathbf{r}_{i-j}$

	$\mathbf{s}_0$	$\mathbf{s}_1$	$\mathbf{s}_2$	$\mathbf{s}_3$
$\mathbf{s}_0$	$\mathbf{r}_0$	$\mathbf{r}_3$	$\mathbf{r}_2$	$\mathbf{r}_1$
$\mathbf{s}_1$	$\mathbf{r}_1$	$\mathbf{r}_0$	$\mathbf{r}_3$	$\mathbf{r}_2$
$\mathbf{s}_2$	$\mathbf{r}_2$	$\mathbf{r}_1$	$\mathbf{r}_0$	$\mathbf{r}_3$
$\mathbf{s}_3$	$\mathbf{r}_3$	$\mathbf{r}_2$	$\mathbf{r}_1$	$\mathbf{r}_0$



7 Local asymptotic stability for all rate constants

In the rest, we consider only

$$\begin{aligned}\dot{u} &= \mathrm{e}^{a_1 u + b_1 v} - \mathrm{e}^{a_2 u + b_2 v}\\ \dot{v} &= \mathrm{e}^{a_3 u + b_3 v} - \mathrm{e}^{a_4 u + b_4 v}\,.\end{aligned}$$

However, in Theorems 1 and 4, we think of a_1, a_2, a_3, a_4 and b_1, b_2, b_3, b_4 as

$$\begin{aligned}a_1 &= \gamma_1(g_{11} - 1), & b_1 &= \gamma_2 g_{12},\\ a_2 &= \gamma_1(h_{11} - 1), & b_2 &= \gamma_2 h_{12},\\ a_3 &= \gamma_1 g_{21}, & b_3 &= \gamma_2(g_{22} - 1),\\ a_4 &= \gamma_1 h_{21}, & b_4 &= \gamma_2(h_{22} - 1).\end{aligned}$$

Recall that

$$J = \begin{pmatrix} a_1 - a_2 & b_1 - b_2 \\ a_3 - a_4 & b_3 - b_4 \end{pmatrix}$$

and assume throughout that $\det J > 0$. (Observe that both $\det J$ and $\operatorname{tr} J$ are invariant under the symmetries $\mathbf{r}_0$, $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, $\mathbf{s}_0$, $\mathbf{s}_1$, $\mathbf{s}_2$, $\mathbf{s}_3$.)

Theorem 1. *The following are equivalent.*

1. *Local asymptotic stability for all rate constants, i.e., for all* $\gamma_1, \gamma_2 > 0$.
2. *Either*
   (a) $J = \begin{pmatrix} - & * \\ * & - \end{pmatrix}$,
   (b) $J = \begin{pmatrix} 0 & + \\ - & - \end{pmatrix}$,
   (c) $J = \begin{pmatrix} 0 & - \\ + & - \end{pmatrix}$,
   (d) $J = \begin{pmatrix} - & - \\ + & 0 \end{pmatrix}$, *or*
   (e) $J = \begin{pmatrix} - & + \\ - & 0 \end{pmatrix}$.

8 Non-existence of periodic solutions

Lemma 2. *Assume that* $a_1 \le a_2$ *and* $b_3 \le b_4$ *hold with* $(a_1 - a_2, b_3 - b_4) \neq (0,0)$. *Then there is no periodic solution.*

9 Boundedness of all the solutions (in positive time)

Lemma 3. *(a) If* $J = \begin{pmatrix} - & * \\ * & - \end{pmatrix}$ *then boundedness holds.*

(b1) If $J = \begin{pmatrix} + & + \\ - & - \end{pmatrix}$ *then boundedness implies* $a_3 \le a_2 < a_1 \le a_4$.

(b2) If $J = \begin{pmatrix} 0 & + \\ - & - \end{pmatrix}$ *then boundedness is equivalent to* $a_3 \le a_2 = a_1 \le a_4$.

(c1) If $J = \begin{pmatrix} + & - \\ + & - \end{pmatrix}$ *then boundedness implies* $a_4 \le a_2 < a_1 \le a_3$.

(c2) If $J = \begin{pmatrix} 0 & - \\ + & - \end{pmatrix}$ *then boundedness is equivalent to* $a_4 \le a_2 = a_1 \le a_3$.

(d1) If $J = \begin{pmatrix} - & - \\ + & + \end{pmatrix}$ *then boundedness implies* $b_1 \le b_4 < b_3 \le b_2$.

(d2) If $J = \begin{pmatrix} - & - \\ + & 0 \end{pmatrix}$ *then boundedness is equivalent to* $b_1 \le b_4 = b_3 \le b_2$.

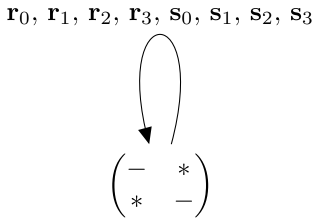
(e1) If $J = \begin{pmatrix} - & + \\ - & + \end{pmatrix}$ *then boundedness implies* $b_2 \le b_4 < b_3 \le b_1$.

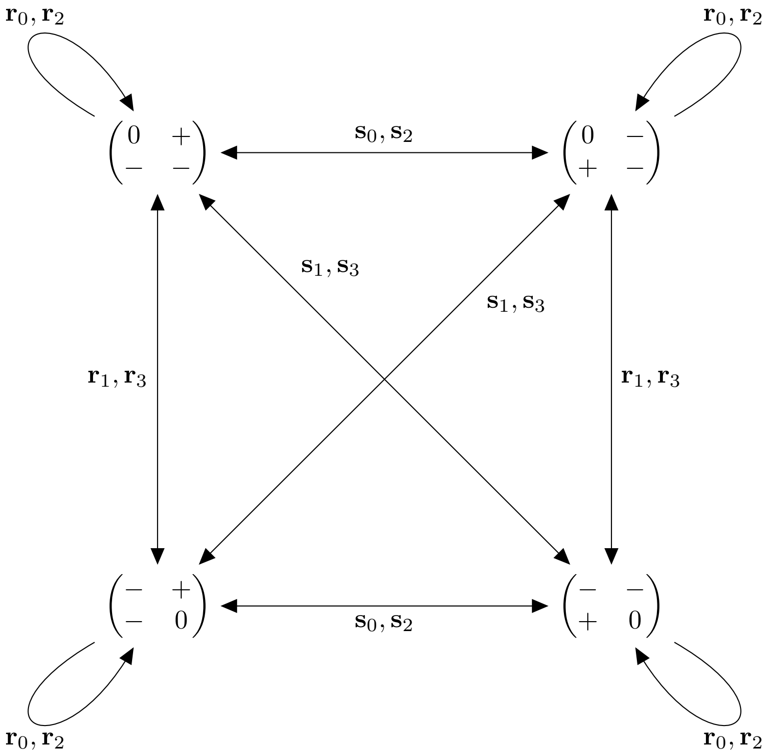
(e2) If $J = \begin{pmatrix} - & + \\ - & 0 \end{pmatrix}$ *then boundedness is equivalent to* $b_2 \le b_4 = b_3 \le b_1$.

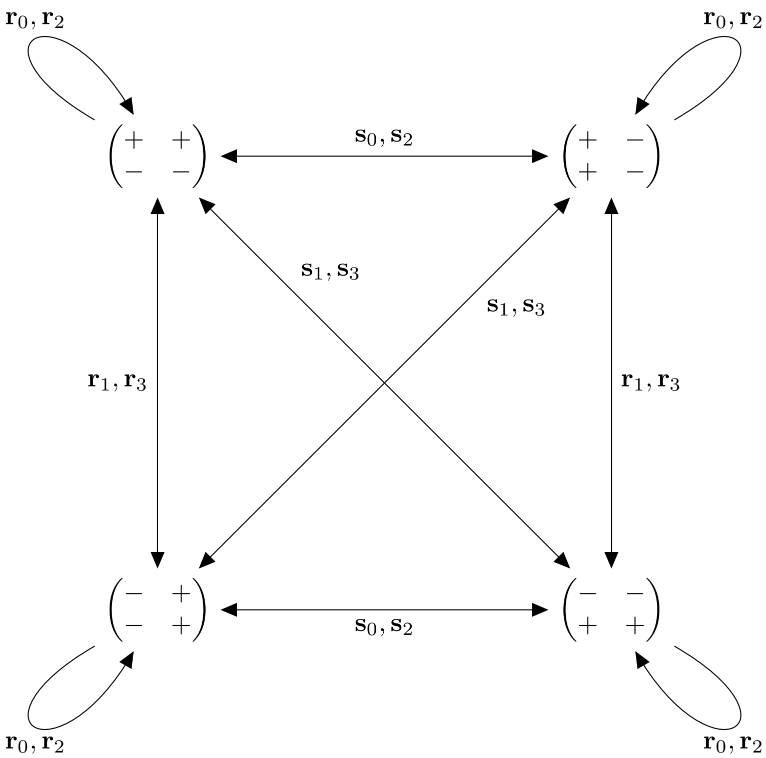
10 Global asymptotic stability for all rate constants

Theorem 4. *The following are equivalent.*

1. *Global asymptotic stability for all rate constants, i.e., for all* $\gamma_1, \gamma_2 > 0$.
2. *Either*
   (a) $J = \begin{pmatrix} - & * \\ * & - \end{pmatrix}$,
   (b) $J = \begin{pmatrix} 0 & + \\ - & - \end{pmatrix}$ *and* $a_3 \leq a_1 = a_2 \leq a_4$,
   (c) $J = \begin{pmatrix} 0 & - \\ + & - \end{pmatrix}$ *and* $a_4 \leq a_1 = a_2 \leq a_3$,
   (d) $J = \begin{pmatrix} - & + \\ - & 0 \end{pmatrix}$ *and* $b_2 \leq b_3 = b_4 \leq b_1$, *or*
   (e) $J = \begin{pmatrix} - & - \\ + & 0 \end{pmatrix}$ *and* $b_1 \leq b_3 = b_4 \leq b_2$.


$\mathbf{r}_0$, $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, $\mathbf{s}_0$, $\mathbf{s}_1$, $\mathbf{s}_2$, $\mathbf{s}_3$
$\begin{pmatrix} - & * \\ * & - \end{pmatrix}$

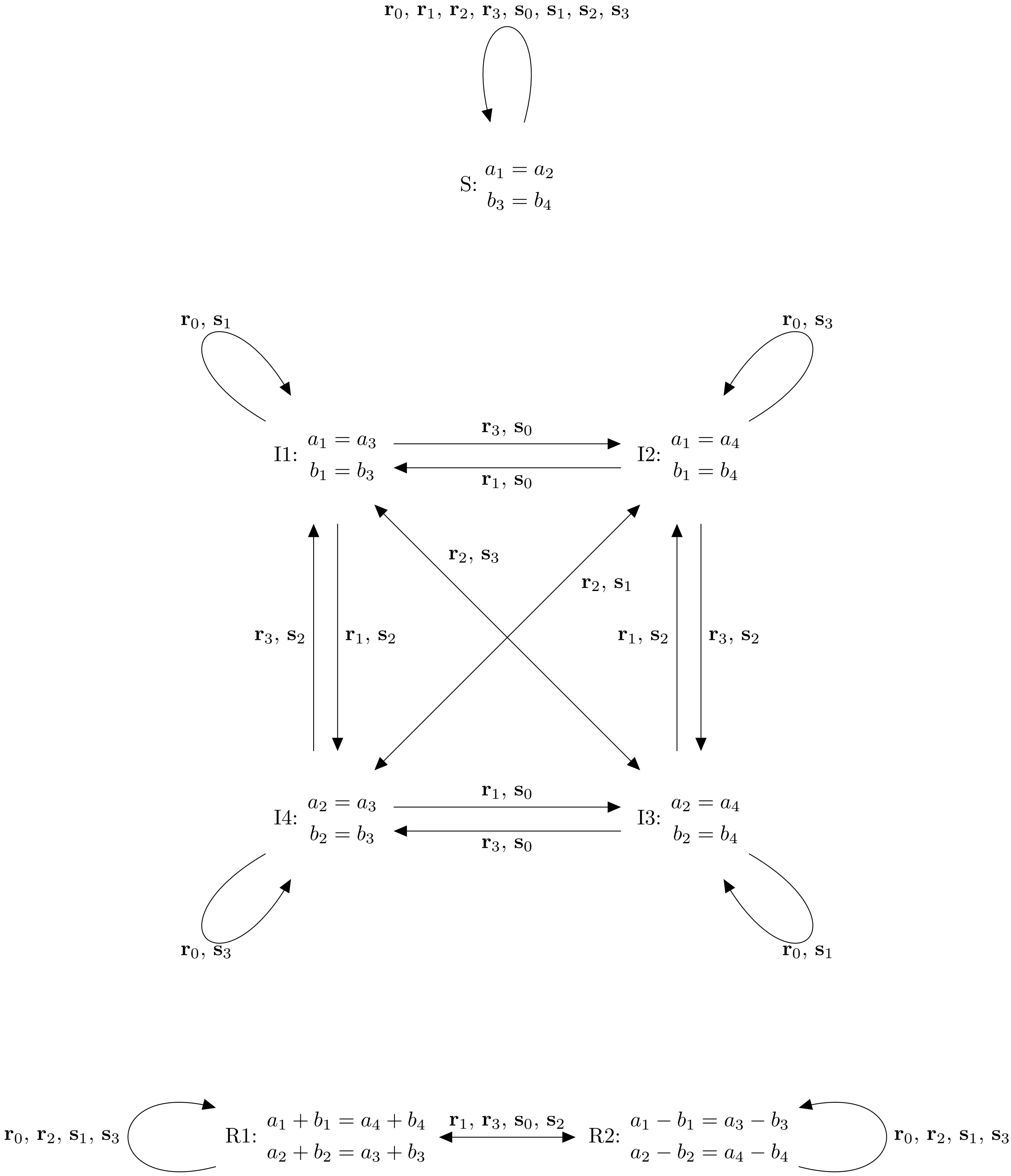

$\mathbf{r}_0, \mathbf{r}_2$
$\begin{pmatrix} 0 & + \\ - & - \end{pmatrix}$
$\mathbf{s}_0, \mathbf{s}_2$
$\begin{pmatrix} 0 & - \\ + & - \end{pmatrix}$
$\mathbf{s}_1, \mathbf{s}_3$
$\mathbf{r}_1, \mathbf{r}_3$
$\begin{pmatrix} - & + \\ - & 0 \end{pmatrix}$
$\begin{pmatrix} - & - \\ + & 0 \end{pmatrix}$


$\mathbf{r}_0, \mathbf{r}_2$
$\begin{pmatrix} + & + \\ - & - \end{pmatrix}$
$\mathbf{s}_0, \mathbf{s}_2$
$\begin{pmatrix} + & - \\ + & - \end{pmatrix}$
$\mathbf{s}_1, \mathbf{s}_3$
$\mathbf{r}_1, \mathbf{r}_3$
$\begin{pmatrix} - & + \\ - & + \end{pmatrix}$
$\begin{pmatrix} - & - \\ + & + \end{pmatrix}$

11 Local center problem

case	condition	constant of motion	Dulac
S	$a_1 = a_2$ $b_3 = b_4$	$\left[\frac{e^{(a_3-a_1)u}}{a_3-a_1} - \frac{e^{(a_4-a_1)u}}{a_4-a_1}\right] - \left[\frac{e^{(b_1-b_4)v}}{b_1-b_4} - \frac{e^{(b_2-b_4)v}}{b_2-b_4}\right]$	$e^{-a_1u-b_4v}$
I1	$a_1 = a_3$ $b_1 = b_3$	$+\frac{e^{p(u-v)}}{p} + \frac{e^{qu}}{q} - \frac{e^{rv}}{r}$, where $\begin{cases} p = a_1 - a_2 \\ q = a_4 - a_2 \\ r = b_2 - b_4 \end{cases}$	$e^{-a_2u-b_4v}$
I2	$a_1 = a_4$ $b_1 = b_4$	$-\frac{e^{p(u+v)}}{p} + \frac{e^{qu}}{q} + \frac{e^{rv}}{r}$, where $\begin{cases} p = a_1 - a_2 \\ q = a_3 - a_2 \\ r = b_2 - b_3 \end{cases}$	$e^{-a_2u-b_3v}$
I3	$a_2 = a_4$ $b_2 = b_4$	$+\frac{e^{p(-u+v)}}{p} - \frac{e^{qu}}{q} + \frac{e^{rv}}{r}$, where $\begin{cases} p = a_1 - a_2 \\ q = a_3 - a_1 \\ r = b_1 - b_3 \end{cases}$	$e^{-a_1u-b_3v}$
I4	$a_2 = a_3$ $b_2 = b_3$	$-\frac{e^{p(-u-v)}}{p} - \frac{e^{qu}}{q} - \frac{e^{rv}}{r}$, where $\begin{cases} p = a_1 - a_2 \\ q = a_4 - a_1 \\ r = b_1 - b_4 \end{cases}$	$e^{-a_1u-b_4v}$
R1	$a_1 + b_1 = a_4 + b_4$ $a_2 + b_2 = a_3 + b_3$	?	?
R2	$a_1 - b_1 = a_3 - b_3$ $a_2 - b_2 = a_4 - b_4$	?	?
R1 ∩ R2	$a_1 + b_1 = a_4 + b_4$ $a_2 + b_2 = a_3 + b_3$ $a_1 - b_1 = a_3 - b_3$ $a_2 - b_2 = a_4 - b_4$	$\left[1 + e^{r(u+v)}\right]\left[e^{qu} + e^{qv}\right]^{-\frac{r}{q}}$, where $\begin{cases} q = a_4 - a_1 = a_2 - a_3 = b_3 - b_2 = b_1 - b_4 \\ r = a_3 - a_1 = a_2 - a_4 = b_2 - b_4 = b_3 - b_1 \end{cases}$	$e^{-a_1u-b_4v}(e^{qu} + e^{qv})^{-\frac{q+r}{q}}$, where $\begin{cases} q = a_4 - a_1 = a_2 - a_3 = b_3 - b_2 = b_1 - b_4 \\ r = a_3 - a_1 = a_2 - a_4 = b_2 - b_4 = b_3 - b_1 \end{cases}$

In the above table, $\frac{e^{\gamma z}}{\gamma}$ is understood as z whenever $\gamma = 0$.



$\mathbf{r}_0$, $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, $\mathbf{s}_0$, $\mathbf{s}_1$, $\mathbf{s}_2$, $\mathbf{s}_3$
S: $a_1 = a_2$, $b_3 = b_4$
$\mathbf{r}_0$, $\mathbf{s}_1$
$\mathbf{r}_0$, $\mathbf{s}_3$
I1: $a_1 = a_3$, $b_1 = b_3$
$\mathbf{r}_3$, $\mathbf{s}_0$
$\mathbf{r}_1$, $\mathbf{s}_0$
I2: $a_1 = a_4$, $b_1 = b_4$
$\mathbf{r}_2$, $\mathbf{s}_3$
$\mathbf{r}_2$, $\mathbf{s}_1$
$\mathbf{r}_3$, $\mathbf{s}_2$
$\mathbf{r}_1$, $\mathbf{s}_2$
$\mathbf{r}_1$, $\mathbf{s}_2$
$\mathbf{r}_3$, $\mathbf{s}_2$
I4: $a_2 = a_3$, $b_2 = b_3$
$\mathbf{r}_1$, $\mathbf{s}_0$
$\mathbf{r}_3$, $\mathbf{s}_0$
I3: $a_2 = a_4$, $b_2 = b_4$
$\mathbf{r}_0$, $\mathbf{s}_3$
$\mathbf{r}_0$, $\mathbf{s}_1$
$\mathbf{r}_0$, $\mathbf{r}_2$, $\mathbf{s}_1$, $\mathbf{s}_3$
R1: $a_1 + b_1 = a_4 + b_4$, $a_2 + b_2 = a_3 + b_3$
$\mathbf{r}_1$, $\mathbf{r}_3$, $\mathbf{s}_0$, $\mathbf{s}_2$
R2: $a_1 - b_1 = a_3 - b_3$, $a_2 - b_2 = a_4 - b_4$
$\mathbf{r}_0$, $\mathbf{r}_2$, $\mathbf{s}_1$, $\mathbf{s}_3$

12 Intersections of the seven center manifolds

case	ODE	J	$\det J > 0$
S ∩ I1	$\dot u = 1 - \mathrm{e}^{pv}$ $\dot v = 1 - \mathrm{e}^{qu}$	$\begin{pmatrix} 0 & -p \\ -q & 0 \end{pmatrix}$	$pq < 0$
S ∩ I2	$\dot u = 1 - \mathrm{e}^{pv}$ $\dot v = \mathrm{e}^{qu} - 1$	$\begin{pmatrix} 0 & -p \\ q & 0 \end{pmatrix}$	$pq > 0$
S ∩ I3	$\dot u = \mathrm{e}^{pv} - 1$ $\dot v = \mathrm{e}^{qu} - 1$	$\begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix}$	$pq < 0$
S ∩ I4	$\dot u = \mathrm{e}^{pv} - 1$ $\dot v = 1 - \mathrm{e}^{qu}$	$\begin{pmatrix} 0 & p \\ -q & 0 \end{pmatrix}$	$pq > 0$
S ∩ R1	$\dot u = \mathrm{e}^{pv} - \mathrm{e}^{qv}$ $\dot v = \mathrm{e}^{qu} - \mathrm{e}^{pu}$	$\begin{pmatrix} 0 & p-q \\ q-p & 0 \end{pmatrix}$	$p \neq q$
S ∩ R2	$\dot u = \mathrm{e}^{pv} - \mathrm{e}^{qv}$ $\dot v = \mathrm{e}^{-pu} - \mathrm{e}^{-qu}$	$\begin{pmatrix} 0 & p-q \\ q-p & 0 \end{pmatrix}$	$p \neq q$
S ∩ I1 ∩ R2	$\dot u = 1 - \mathrm{e}^{pv}$ $\dot v = 1 - \mathrm{e}^{-pu}$	$\begin{pmatrix} 0 & -p \\ p & 0 \end{pmatrix}$	$p \neq 0$
S ∩ I2 ∩ R1	$\dot u = 1 - \mathrm{e}^{pv}$ $\dot v = \mathrm{e}^{pu} - 1$	$\begin{pmatrix} 0 & -p \\ p & 0 \end{pmatrix}$	$p \neq 0$
S ∩ I3 ∩ R2	$\dot u = \mathrm{e}^{pv} - 1$ $\dot v = \mathrm{e}^{-pu} - 1$	$\begin{pmatrix} 0 & p \\ -p & 0 \end{pmatrix}$	$p \neq 0$
S ∩ I4 ∩ R1	$\dot u = \mathrm{e}^{pv} - 1$ $\dot v = 1 - \mathrm{e}^{pu}$	$\begin{pmatrix} 0 & p \\ -p & 0 \end{pmatrix}$	$p \neq 0$
S ∩ R1 ∩ R2	$\dot u = \mathrm{e}^{pv} - \mathrm{e}^{-pv}$ $\dot v = \mathrm{e}^{-pu} - \mathrm{e}^{pu}$	$\begin{pmatrix} 0 & 2p \\ -2p & 0 \end{pmatrix}$	$p \neq 0$
R1 ∩ R2	$\dot u = \mathrm{e}^{pv} - \mathrm{e}^{(p+q)u+qv}$ $\dot v = \mathrm{e}^{qu+(p+q)v} - \mathrm{e}^{pu}$	$\begin{pmatrix} -p-q & p-q \\ q-p & p+q \end{pmatrix}$	$pq < 0$
I1 ∩ R2	$\dot u = 1 - \mathrm{e}^{pu+qv}$ $\dot v = 1 - \mathrm{e}^{-qu-pv}$	$\begin{pmatrix} -p & -q \\ q & p \end{pmatrix}$	$\lvert p\rvert < \lvert q\rvert$
I2 ∩ R1	$\dot u = 1 - \mathrm{e}^{pu+qv}$ $\dot v = \mathrm{e}^{qu+pv} - 1$	$\begin{pmatrix} -p & -q \\ q & p \end{pmatrix}$	$\lvert p\rvert < \lvert q\rvert$
I3 ∩ R2	$\dot u = \mathrm{e}^{pu+qv} - 1$ $\dot v = \mathrm{e}^{-qu-pv} - 1$	$\begin{pmatrix} p & q \\ -q & -p \end{pmatrix}$	$\lvert p\rvert < \lvert q\rvert$
I4 ∩ R1	$\dot u = \mathrm{e}^{pu+qv} - 1$ $\dot v = 1 - \mathrm{e}^{qu+pv}$	$\begin{pmatrix} p & q \\ -q & -p \end{pmatrix}$	$\lvert p\rvert < \lvert q\rvert$

13 Global center problem

case	clockwise version(s)		anticlockwise version(s)	
S	$a_3 \le a_1 = a_2 \le a_4$ $b_2 \le b_3 = b_4 \le b_1$	$J = \begin{pmatrix} 0 & + \\ - & 0 \end{pmatrix}$	$a_4 \le a_1 = a_2 \le a_3$ $b_1 \le b_3 = b_4 \le b_2$	$J = \begin{pmatrix} 0 & - \\ + & 0 \end{pmatrix}$
I1	$a_3 = a_1 \le a_2 \le a_4$ $b_2 \le b_4 \le b_3 = b_1$	$J = \begin{pmatrix} - & + \\ - & + \end{pmatrix}$	$a_4 \le a_2 \le a_1 = a_3$ $b_1 = b_3 \le b_4 \le b_2$	$J = \begin{pmatrix} + & - \\ + & - \end{pmatrix}$
I2	$a_3 \le a_2 \le a_1 = a_4$ $b_2 \le b_3 \le b_4 = b_1$	$J = \begin{pmatrix} + & + \\ - & - \end{pmatrix}$	$a_4 = a_1 \le a_2 \le a_3$ $b_1 = b_4 \le b_3 \le b_2$	$J = \begin{pmatrix} - & - \\ + & + \end{pmatrix}$
I3	$a_3 \le a_1 \le a_2 = a_4$ $b_2 = b_4 \le b_3 \le b_1$	$J = \begin{pmatrix} - & + \\ - & + \end{pmatrix}$	$a_4 = a_2 \le a_1 \le a_3$ $b_1 \le b_3 \le b_4 = b_2$	$J = \begin{pmatrix} + & - \\ + & - \end{pmatrix}$
I4	$a_3 = a_2 \le a_1 \le a_4$ $b_2 = b_3 \le b_4 \le b_1$	$J = \begin{pmatrix} + & + \\ - & - \end{pmatrix}$	$a_4 \le a_1 \le a_2 = a_3$ $b_1 \le b_4 \le b_3 = b_2$	$J = \begin{pmatrix} - & - \\ + & + \end{pmatrix}$
R1	$a_3 \le a_2 \le a_1 \le a_4$ $b_2 \le b_3 \le b_4 \le b_1$ $a_3 \le a_1 \le a_2 \le a_4$ $b_2 \le b_4 \le b_3 \le b_1$	$J = \begin{pmatrix} + & + \\ - & - \end{pmatrix}$ $J = \begin{pmatrix} - & + \\ - & + \end{pmatrix}$	$a_4 \le a_1 \le a_2 \le a_3$ $b_1 \le b_4 \le b_3 \le b_2$ $a_4 \le a_2 \le a_1 \le a_3$ $b_1 \le b_3 \le b_4 \le b_2$	$J = \begin{pmatrix} - & - \\ + & + \end{pmatrix}$ $J = \begin{pmatrix} + & - \\ + & - \end{pmatrix}$
R2	$a_3 \le a_2 \le a_1 \le a_4$ $b_2 \le b_3 \le b_4 \le b_1$ $a_3 \le a_1 \le a_2 \le a_4$ $b_2 \le b_4 \le b_3 \le b_1$	$J = \begin{pmatrix} + & + \\ - & - \end{pmatrix}$ $J = \begin{pmatrix} - & + \\ - & + \end{pmatrix}$	$a_4 \le a_1 \le a_2 \le a_3$ $b_1 \le b_4 \le b_3 \le b_2$ $a_4 \le a_2 \le a_1 \le a_3$ $b_1 \le b_3 \le b_4 \le b_2$	$J = \begin{pmatrix} - & - \\ + & + \end{pmatrix}$ $J = \begin{pmatrix} + & - \\ + & - \end{pmatrix}$

Theorem 5. *A local center is*

1. *a global center if and only if*

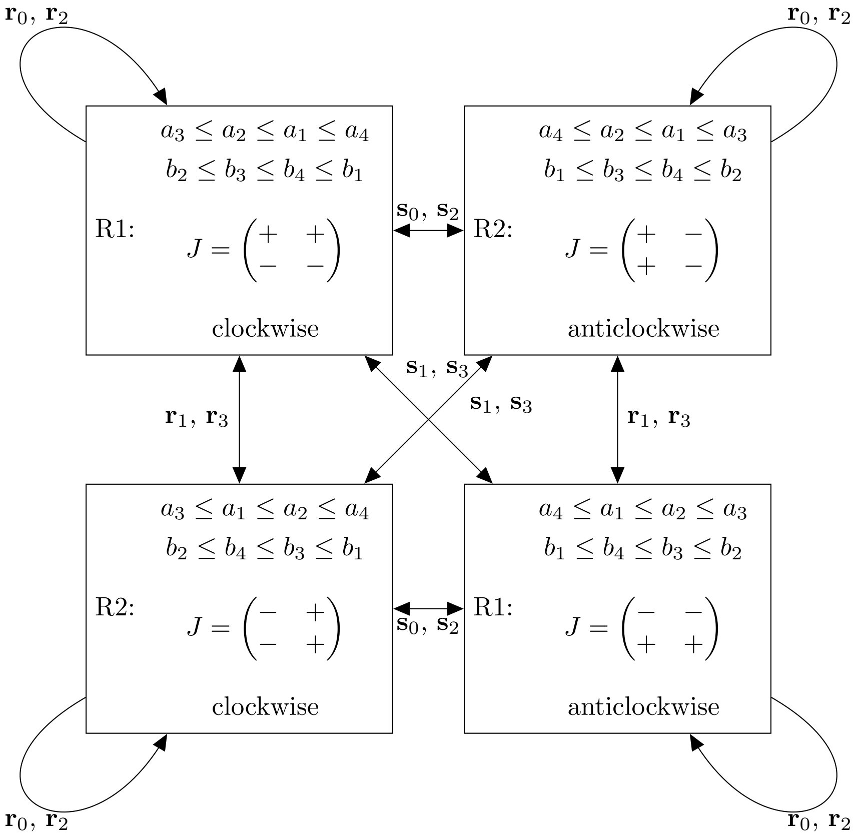
$$\min(a_3, a_4) \le \min(a_1, a_2) \le \max(a_1, a_2) \le \max(a_3, a_4) \text{ and}$$
$$\min(b_1, b_2) \le \min(b_3, b_4) \le \max(b_3, b_4) \le \max(b_1, b_2),$$

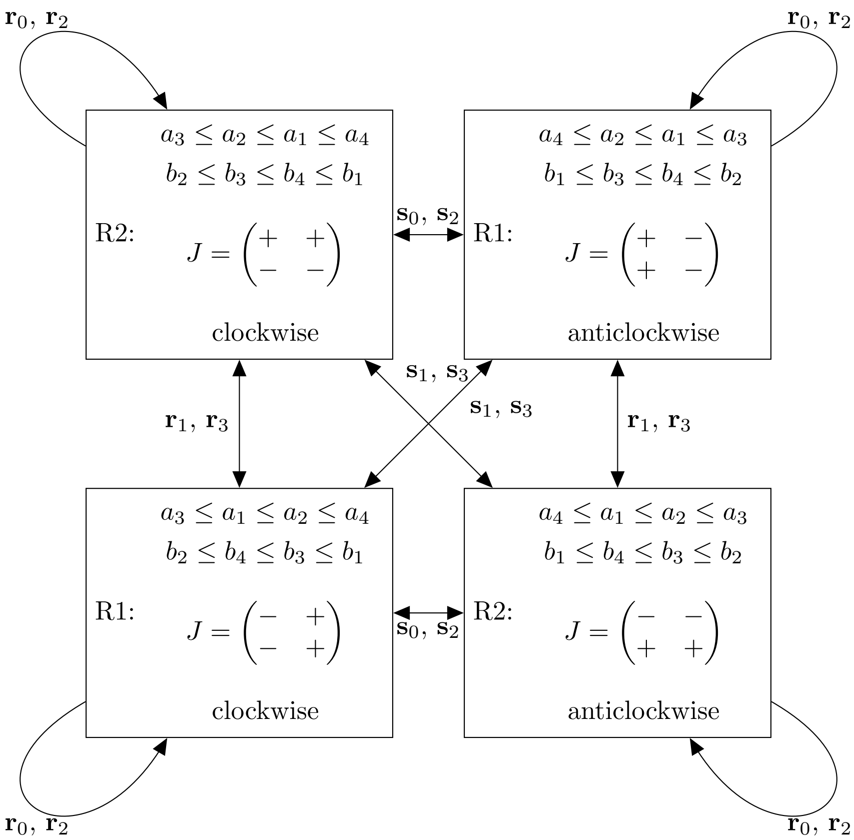
2. *a clockwise global center if and only if*

$$a_3 \le \min(a_1, a_2) \le \max(a_1, a_2) \le a_4 \text{ and}$$
$$b_2 \le \min(b_3, b_4) \le \max(b_3, b_4) \le b_1,$$

3. *an anticlockwise global center if and only if*

$$a_4 \le \min(a_1, a_2) \le \max(a_1, a_2) \le a_3 \text{ and}$$
$$b_1 \le \min(b_3, b_4) \le \max(b_3, b_4) \le b_2.$$





14 Ultimately monotonic unbounded orbits

$J = \begin{pmatrix} + & + \\ - & - \end{pmatrix}$ clockwise	$J = \begin{pmatrix} - & - \\ + & + \end{pmatrix}$ anticlockwise	$J = \begin{pmatrix} + & - \\ + & - \end{pmatrix}$ anticlockwise	$J = \begin{pmatrix} - & + \\ - & + \end{pmatrix}$ clockwise
$-\frac{a_3-a_4}{b_3-b_4} < -\frac{a_1-a_2}{b_1-b_2} < \gamma < 0$ $a_1 + \gamma b_1 > a_2 + \gamma b_2$ $a_3 + \gamma b_3 < a_4 + \gamma b_4$ $v = \gamma u$	$\gamma < -\frac{a_3-a_4}{b_3-b_4} < -\frac{a_1-a_2}{b_1-b_2} < 0$ $a_1 + \gamma b_1 > a_2 + \gamma b_2$ $a_3 + \gamma b_3 < a_4 + \gamma b_4$ $v = \gamma u$	$0 < \gamma < -\frac{a_1-a_2}{b_1-b_2} < -\frac{a_3-a_4}{b_3-b_4}$ $a_1 + \gamma b_1 > a_2 + \gamma b_2$ $a_3 + \gamma b_3 > a_4 + \gamma b_4$ $v = \gamma u$	$0 < -\frac{a_1-a_2}{b_1-b_2} < -\frac{a_3-a_4}{b_3-b_4} < \gamma$ $a_1 + \gamma b_1 > a_2 + \gamma b_2$ $a_3 + \gamma b_3 > a_4 + \gamma b_4$ $v = \gamma u$
$u \to \infty$ $\frac{dv}{du} \approx -e^{[(a_4-a_1)+\gamma(b_4-b_1)]u} > \gamma$ if $(a_4 - a_1) + \gamma(b_4 - b_1) < 0$	$u \to \infty$ $\frac{dv}{du} \approx -e^{[(a_4-a_1)+\gamma(b_4-b_1)]u} < \gamma$ if $(a_4 - a_1) + \gamma(b_4 - b_1) > 0$	$u \to \infty$ $\frac{dv}{du} \approx e^{[(a_3-a_1)+\gamma(b_3-b_1)]u} < \gamma$ if $(a_3 - a_1) + \gamma(b_3 - b_1) < 0$	$u \to \infty$ $\frac{dv}{du} \approx e^{[(a_3-a_1)+\gamma(b_3-b_1)]u} > \gamma$ if $(a_3 - a_1) + \gamma(b_3 - b_1) > 0$
escape below the u-axis for $u \to \infty$ if $(a_4 - a_1) + \left(-\frac{a_1 - a_2}{b_1 - b_2}\right)(b_4 - b_1) < 0$ or $a_4 - a_1 < 0$	escape right to the v-axis for $v \to -\infty$ if $(a_4 - a_1) + \left(-\frac{a_3 - a_4}{b_3 - b_4}\right)(b_4 - b_1) > 0$ or $b_4 - b_1 < 0$	escape above the u-axis for $u \to \infty$ if $(a_3 - a_1) + \left(-\frac{a_1 - a_2}{b_1 - b_2}\right)(b_3 - b_1) < 0$ or $a_3 - a_1 < 0$	escape right to the v-axis for $v \to \infty$ if $(a_3 - a_1) + \left(-\frac{a_3 - a_4}{b_3 - b_4}\right)(b_3 - b_1) > 0$ or $b_3 - b_1 > 0$
$u \to -\infty$ $\frac{dv}{du} \approx -e^{[(a_3-a_2)+\gamma(b_3-b_2)]u} > \gamma$ if $(a_3 - a_2) + \gamma(b_3 - b_2) > 0$	$u \to -\infty$ $\frac{dv}{du} \approx -e^{[(a_3-a_2)+\gamma(b_3-b_2)]u} < \gamma$ if $(a_3 - a_2) + \gamma(b_3 - b_2) < 0$	$u \to -\infty$ $\frac{dv}{du} \approx e^{[(a_4-a_2)+\gamma(b_4-b_2)]u} < \gamma$ if $(a_4 - a_2) + \gamma(b_4 - b_2) > 0$	$u \to -\infty$ $\frac{dv}{du} \approx e^{[(a_4-a_2)+\gamma(b_4-b_2)]u} > \gamma$ if $(a_4 - a_2) + \gamma(b_4 - b_2) < 0$
escape above the u-axis for $u \to -\infty$ if $(a_3 - a_2) + \left(-\frac{a_1 - a_2}{b_1 - b_2}\right)(b_3 - b_2) > 0$ or $a_3 - a_2 > 0$	escape left to the v-axis for $v \to \infty$ if $(a_3 - a_2) + \left(-\frac{a_3 - a_4}{b_3 - b_4}\right)(b_3 - b_2) < 0$ or $b_3 - b_2 > 0$	escape below the u-axis for $u \to -\infty$ if $(a_4 - a_2) + \left(-\frac{a_1 - a_2}{b_1 - b_2}\right)(b_4 - b_2) > 0$ or $a_4 - a_2 > 0$	escape left to the v-axis for $v \to -\infty$ if $(a_4 - a_2) + \left(-\frac{a_3 - a_4}{b_3 - b_4}\right)(b_4 - b_2) < 0$ or $b_4 - b_2 < 0$