Annals of Discrete Mathematics 41 (1989) 179–194

On Connectivity Properties of Eulerian Digraphs

A. Frank*
Department of Computer Science
Eötvös University
Budapest, Hungary

Dedicated to the memory of G. A. Dirac

Directed counterparts of theorems of Rothschild and Whinston and of Lovász concerning Eulerian graphs are proved. As a consequence, a polynomial time algorithm is presented to solve the integral two-commodity flow problem in directed graphs in the case where all capacities are 'Eulerian'.

1 Introduction

Connectivity results in graph theory start with Menger's theorem which has (at least) four versions, according to whether we are interested in the maximum number of edge-disjoint or node-disjoint paths from s to t in a directed or an undirected graph. The max-flow-min-cut (MFMC) theorem can be considered as an extension of the directed edge-version of Menger's theorem. T. C. Hu [4] proved a max-flow-min-cut theorem for two-commodity flows in undirected graphs. Unfortunately, in this case the maximum flow is not necessarily integer-valued. Rothschild and Whinston [14] found an integer-valued version of Hu's theorem for Eulerian graphs. Two other interesting connectivity results concerning Eulerian graphs are due to L. Lovász [7].

The purpose of the present note is to prove directed counterparts of these results. We also describe a polynomial time algorithm for integral two-commodity flows in directed graphs when the capacities are Eulerian (that is, at every node the sums of out-going and in-going capacities are equal).

*Supported by a grant from the Alexander von Humboldt Stiftung and by the Institute of Econometrics and Operations Research of the University of Bonn.

The proofs make use of the operation of splitting off incident edges at a node. This powerful technique was used in [14] and in [7]. It also played a basic role in connectivity results of Mader ([8] and [9]) concerning non-Eulerian graphs.

In an undirected graph $G = (V, E)$, *splitting off* two edges vu and ut means replacing vu and ut by a new edge vt. Similarly, in a directed graph, *splitting off* two edges vu and ut is an operation that replaces vu and ut by a new edge vt. If $v = t$, we leave out the resulting loop vt.

Throughout, we work with connected loopless graphs (directed or undirected) on node set V. For $X \subseteq V$, put $\overline{X} := V - X$. Note that $A \subset B$ means that $A \subseteq B$, but $A \neq B$. For $s, t \in V$, a subset $X \subset V$ is called an $s\bar{t}$-set if $s \in X$ and $t \notin X$. We do not distinguish between an element v and the one-element set $\{v\}$.

For a graph $G = (V, E)$, $d_G(X, Y)$ denotes the number of edges with one end in $X - Y$ and one end in $Y - X$. (When it is not ambiguous, we leave out the subscript G). We use $d_G(X)$ for $d_G(X, \overline{X})$. For two graphs $G = (V, E)$ and $H = (V, F)$, $G + H$ denotes a graph with node set V and edge set $E \cup F$. In a directed graph G, $\varrho_G(X)$ (respectively $\delta_G(X)$) denotes the number of edges entering (leaving) X. For a function $c: E \to R$,

$$\varrho_c(X) := \sum (c(e) : e \text{ enters } X).$$

We will use the following two equalities.

$$\varrho_G(X) + \varrho_G(Y) = \varrho_G(X \cap Y) + \varrho_G(X \cup Y) + d_G(X, Y) \qquad (1)$$

$$d_G(X) + d_G(Y) = d_G(X \cap Y) + d_G(X \cup Y) + 2d_G(X, Y) \qquad (2)$$

The *connectivity* $c(X, Y)$ between two disjoint subsets X and Y of nodes of an undirected graph is the maximum number of edge-disjoint paths connecting X and Y. In a digraph, the *di-connectivity* $dc(X, Y)$ from X to Y is the maximum number of edge-disjoint directed paths from X to Y.

An undirected graph $G = (V, E)$ is *Eulerian* if $d_G(v)$ is even for every $v \in V$. A digraph $G = (V, E)$ is *Eulerian* if $\varrho_G(v) = \delta_G(v)$ for every $v \in V$. A graph G is called *acyclic* if it does not contain directed circuits.

2 Edge-disjoint paths

The *edge-disjoint paths problem* is as follows: Given an undirected graph $G = (V, E)$ and k pairs of nodes, find k edge-disjoint paths in G connecting the corresponding terminals. In general, the problem is $\mathcal{NP}$-complete [6], but for fixed k it is polynomially solvable [13]. For arbitrary k, some important special cases are well characterized ([3], [12], [10], [11], and [18]). See also Schrijver's survey paper [15].

Sometimes the following reformulation is useful. Mark each terminal pair by an edge. The graph $H = (V, F)$ of marker edges is called a *demand graph*, while we call the original graph G a *supply graph*. The edge-disjoint paths problem is: In $G + H$, find k edge-disjoint circuits such that each of them contains one demand edge.

A natural necessary condition for the solvability is the

Cut Criterion. $\quad d_G(X) \geq d_H(X) \quad$ for every $X \subset V$.

If H consists of k parallel edges, then the cut criterion is sufficient (: undirected-edge Menger). The following result of Rothschild and Whinston is a strengthening of Hu's two-commodity flow theorem [4].

Theorem 2.1 ([14]). *If H consists of two sets of parallel edges and $G + H$ is Eulerian, then the cut criterion is necessary and sufficient for the solvability of the edge-disjoint paths problem.*

The *directed edge-disjoint paths problem* can be defined analogously. Here we formulate only the corresponding second version: Let $G = (V, E)$ and $H = (V, F)$ be directed graphs. Find $|F|$ edge-disjoint directed circuits in $G + H$, so that every circuit contains one edge of H. Such a circuit will be called a *good circuit*.

Again, the following criterion is obviously necessary.

Directed Cut Criterion. $\quad \varrho_G(X) \geq \delta_H(X) \quad$ for every $X \subset V$.

If H consists of parallel edges of the same direction, the cut criterion is sufficient; this is exactly the directed-edge version of Menger's theorem. If H consists of two oppositely directed edges (i.e., if H is a two-edge circuit), the directed edge-disjoint paths problem is $\mathcal{NP}$-complete [2], and the directed cut criterion is not sufficient. Even if $G + H$ is Eulerian, the directed counterpart of theorem 2.1 does not hold in general. This can be seen from the graph of Figure 1. Here the solid lines denote the edges of H; $G + H$ is Eulerian, and the directed cut criterion holds, but there is no solution. However, we have the following theorem.

Theorem 2.2. *Suppose that $G + H$ is an Eulerian digraph, H consists of two sets of parallel edges, and H is acyclic. Then the directed cut criterion is necessary and sufficient for the solvability of the directed edge-disjoint paths problem.*

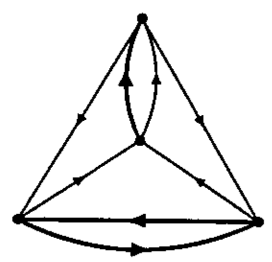

Figure 1.

Proof. Let $G+H$ be a minimal counter-example. Assume that H consists of k_1 edges from t_1 to s_1 and k_2 edges from t_2 to s_2. Let $T = \{s_1, s_2, t_1, t_2\}$ be the set of terminal nodes. By Menger's theorem, $k_1 > 0$ and $k_2 > 0$.

Call a pair $\{vu, ut\}$ of edges *feasible* if splitting them off does not destroy the directed cut criterion. Since $G+H$ is a minimal counter-example, there cannot be feasible pairs $\{vu, ut\}$ for $u \in V - T$.

Call a set X *tight* if $\varrho_G(X) = \delta_H(X)$.

CLAIM 1. *If X is tight, then so is $\overline{X}$.*

PROOF. Since $G + H$ is Eulerian,

$$\varrho_G(X) + \varrho_H(X) = \delta_G(X) + \delta_H(X),$$

whence the claim follows.

CLAIM 2. *If X and Y are tight sets, then $d_H(X,Y) \geq d_G(X,Y)$, and if equality holds, then both $X \cap Y$ and $X \cup Y$ are tight.*

PROOF. Using (1) and the directed cut criterion, we immediately get

$$\begin{aligned}\varrho_G(X \cap Y) - \varrho_H(X \cap Y) + \varrho_G(X \cup Y) - \varrho_H(X \cup Y) + d_G(X,Y) - d_H(X,Y)\\ = \varrho_G(X) - \varrho_H(X) + \varrho_G(Y) - \varrho_H(Y)\\ = 0 + 0,\end{aligned}$$

from which the claim follows.

CLAIM 3. *If X and Y are tight, then $d_H(X,\overline{Y}) \geq d_G(X,\overline{Y})$, and if equality holds, then both $X - Y$ and $Y - X$ are tight.*

PROOF. Immediate from Claims 1 and 2.

CLAIM 4. *For $i = 1, 2$, $s_i t_i$ is not an edge in G.*

PROOF. Otherwise, a demand edge $t_i s_i$ and $s_i t_i$ form a good circuit C. After deleting the two edges of C, the hypotheses of the theorem continues to hold, so the resulting digraph contains $|F|-1$ edge-disjoint good circuits. These circuits, together with C, constitute $|F|$ edge-disjoint good circuits in $G + H$, a contradiction.

CLAIM 5. $T \subset V$.

PROOF. Let $T = V$. Since $k_1 > 0$ and $s_1 t_1$ is not in G, there are two edges $s_1 x$ and xy in G such that $s_1 \neq x \neq y \neq s_1$. Since every pair is infeasible, there is a tight set A for which $x \in A$ and $s_1, y \notin A$. Since $\delta_G(A) > 0$ and A is tight, $\varrho_H(A) > 0$. Similarly, $\delta_H(A) > 0$. Hence, $t_1 s_1$ leaves A, and $t_2 s_2$ enters A. By Claim 4, $t_1 \neq x$. So we must have $y = t_2$ and $x = s_2$, contradicting Claim 4.

Since G is connected, there is a node $u \in V - T$ such that $us \in E$ for some $s \in T$. The next lemma immediately implies the Theorem. □

Lemma. *There exists an edge vu in E such that $\{vu, us\}$ is feasible.*

Proof. Assume that $s = t_1$ (the other possibilities are analogous). If there is no tight $s\overline{u}$-set, then any vu edge will do. Assume first that there is exactly one maximal tight $s\overline{u}$-set, denoted by X. We claim that there is an edge $vu \in E$ such that $v \notin X$. Otherwise, we have

$$\varrho_G(X + u) < \varrho_G(X) = \delta_H(X) = \delta_H(X + u),$$

contradicting the directed cut criterion. Note that the pair $\{vu, us\}$ is feasible.

Next, suppose that there are two maximal tight $s\overline{u}$-sets, denoted by X and Y. Since their union is not tight, by Claims 1 and 2, $d_H(X,Y) > 0$ and $d_H(X,\overline{Y}) > 0$. Since $t_1 \in X \cap Y$, among the two edges of H only $t_2 s_2$ can contribute to $d_H(X,Y)$. Therefore, we cannot have three maximal tight $s\overline{u}$-sets, X_1, X_2, X_3, since otherwise $s_2 t_2$ would connect $X_i - X_j$ and $X_j - X_i$, $1 \leq i < j \leq 3$, which is impossible. Assume that $t_2 \in X - Y$ and $s_2 \in Y - X$. We claim that there is an edge vu in E such that $v \notin X \cup Y$. Otherwise, $\delta_G(X) = 0$ (since $\overline{X}$ is tight), and

$$\varrho_G(Y + u) < \varrho_G(Y) = \delta_H(Y) = \delta_H(Y + u),$$

contradicting the directed cut criterion. Now the edge pair $\{vu, us\}$ is feasible. □

Notice that Theorem 2.2 easily implies Theorem 2.1. Indeed, a theorem by Ford and Fulkerson [1] says that the undirected edges of a mixed graph can be oriented in such a way that the resulting digraph is Eulerian if and only if every node has an even number of incident edges (directed or undirected), and every cut contains at least as many undirected edges as is the difference between the numbers of entering and leaving directed edges in this cut. Apply this result to the mixed graph obtained from $G+H$ by directing the edges of H from t_1 to s_1 and from t_2 to s_2. By the undirected cut criterion, the necessary and sufficient condition above is satisfied, so there is an Eulerian orientation of $G+H$. The undirected cut criterion also implies the directed cut criterion for this orientation. Thus, Theorem 2.1 follows from Theorem 2.2.

By a *star* we mean a digraph in which either all the edges enter the same node or all the edges leave the same node.

Theorem 2.3. *If $G+H$ is an Eulerian digraph, and H is the union of two stars, then the directed cut criterion is necessary and sufficient for the solvability of the directed edge-disjoint paths problem.*

Proof. We derive this result from Theorem 2.2 by using an elementary construction. Assume that H is the union of two stars, H_1 and H_2. If every edge of H_i leaves t_i, adjoin a new node s_i to V. For every edge t_ix of H_i, add a new edge s_ix to G, and in H_i, replace t_ix by t_is_i. Analogously, if, in H_i, every edge enters s_i, adjoin a new node t_i to V. For every edge xs_i of H_i, add a new edge xt_i to G, and in H_i, replace xs_i by t_is_i. In the resulting problem, the directed cut criterion continues to hold, so we can apply Theorem 2.2. □

A kind of converse to Theorem 2.3 is also true. A digraph $H=(V,F)$ is called *suitable* if, in $G+H$, the directed cut criterion is sufficient for the directed edge-disjoint paths problem for every digraph G for which $G+H$ is Eulerian. Theorem 2.3 says that the union of two stars is suitable.

Theorem 2.4. *If H is suitable, then H is the union of two stars.*

Proof. First, observe that a subgraph $H'=(V,F')$ of a suitable H is also suitable. Indeed, assume that $G'+H'$ is Eulerian for a certain G', and the directed cut criterion holds, but there are not $|F'|$ edge-disjoint good circuits in $G'+H'$. Then the same statement is true for $G+H$, where G arises from G' by adding the edges in $F-F'$ oppositely directed. In Figure 2, we list some digraphs (along with G), which are not suitable. The edges of H are drawn by solid lines.

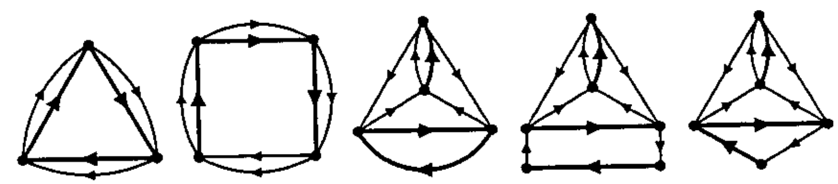

Figure 2.

It is an easy exercise to see that if a digraph H does not contain any of these graphs as a subgraph, then H is the disjoint union of two stars. □

Theorems 2.3 and 2.4 can be considered as a counterpart to a theorem by Papernov [12], saying that in the undirected case, a graph H is suitable if and only if it is the union of two (undirected) stars, or it arises from K (complete 4-gon) or C (circuit of five edges) by adding parallel edges. See also [17].

On the other hand, we cannot be overjoyed with our characterization since there is another natural necessary condition for the edge-disjoint paths problem:

Covering Criterion. For any subset $F' \subseteq F$ of the demand edges, the good circuits, using an element of F', cannot be covered by less than $|F'|$ edges of G.

This criterion immediately implies the cut criterion, but not the other way around. The graph $G+H$ in Figure 1 satisfies the cut criterion, but not the covering criterion. Note that in the undirected case, the (corresponding) covering criterion is equivalent to the cut criterion [18].

It is tempting to try to find classes for the directed case, where the covering criterion is sufficient, while the cut criterion is not. I conjectured that the covering criterion is sufficient if the digraph $G+H$ is Eulerian and planar. (By a theorem of Seymour [18], if $G+H$ is undirected, Eulerian and planar, then the cut criterion is sufficient.) However, C. Hurkens (Tilburg University) found the counter-example of Figure 3.

(Relying on the above theorem of Seymour, one can easily show that if every face of $G+H$ is a directed circuit, then the directed cut criterion is sufficient.)

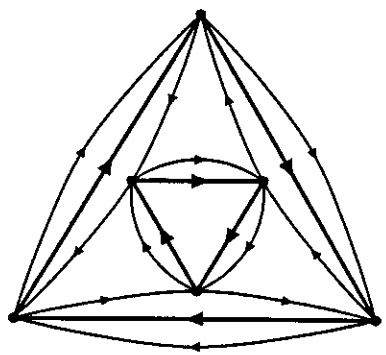

Figure 3.

3 Two-commodity flows

The proof of Theorem 2.2 suggests an algorithm to find the required good circuits. Actually, the theorem itself gives rise to an algorithm, since it implies that if the directed cut criterion holds, then either the number of edges in G from s_i to t_i is at least k_i ($i = 1, 2$), or there is a feasible pair of edges in G. In order to find feasible pairs, we have to be able to test a pair of edges for feasibility. This can be done if the directed cut criterion can be checked efficiently. By Menger's theorem, the directed cut criterion is satisfied (for G and H in Theorem 2.2) if and only if, in G, there are k_1 edge-disjoint paths from s_1 to t_1, and there are k_2 edge-disjoint paths from s_2 to t_2, and there are $k_1 + k_2$ edge-disjoint paths from $\{s_1, s_2\}$ to $\{t_1, t_2\}$. Therefore, the directed cut criterion can be checked by three max-flow-min-cut computations.

Begin the algorithm with checking the directed cut criterion. Second, repeat the following procedure as long as there is a node $u \in V$ for which $\varrho_G(u) > 0$ and $\delta_G(u) > 0$: Choose an edge $us \in E$. One by one, check every pair $\{vu, us\}$, $vu \in E$, whether it is feasible. By Theorem 2.2, we can be sure that at least one of these pairs is feasible. Once a feasible pair is found, split it off.

When the splitting phase terminates, the resulting digraph contains k_i edges from s_i to t_i ($i = 1, 2$). Each of these edges corresponds to a path of G from which the edge has arisen during the splitting phase.

The third part of the algorithm consists of reconstructing these $k_1 + k_2$ edge-disjoint paths.

Let us consider the following directed two-commodity flow problem. We are given a digraph $G = (V, E)$ endowed with a non-negative integral capacity function $c: E \to Z$. Two (ordered) pairs of terminals s_i, t_i ($i = 1, 2$) are specified along with a prescribed flow value k_i. The problem is to find two flows x_1 and x_2 such that $x_i \geq 0$ is a flow from s_i to t_i of value k_i, and

$$x_1(e) + x_2(e) \leq c(e)$$

for every $e \in E$. We solve this problem under the assumption that

$$\varrho_c(v) = \delta_c(v) \qquad \text{if } v \neq s_i, t_i$$

and

$$\varrho_c(s_i) + k_i = \delta_c(s_i) \quad \text{and} \quad \varrho_c(t_i) = \delta_c(t_i) + k_i \qquad (i = 1, 2).$$

Theoretically, the problem goes back to the uncapacitated case by replacing each edge e by $c(e)$ parallel edges. This reduction gives rise to an algorithm that is not polynomial, since the required number of splitting operations is proportional to the maximal capacity.

This difficulty can be overcome if we split off a feasible pair $\{vu, us\}$ more than once at a time. How many times can a pair be split off, without violating the cut criterion? If M denotes this number, then

$$M = \min(c(vu), c(us), m)$$

where

$$m = \min(\varrho_c(X) - \delta_H(X) : u \notin X \text{ and } v, s \in X).$$

Like the way we tested the directed cut criterion, m can be computed by three max-flow-min-cut computations.

By a *weighted splitting* we mean an operation that, given two edges vu and us, reduces $c(vu)$ and $c(us)$ by M and introduces a new supply edge from v to s of capacity M. We say that a weighted splitting operation is *critical* if $M = m$.

Now, the weighted algorithm is the same as the unweighted one, except that each time we perform weighted splittings. That is, the algorithm begins with checking the directed cut criterion. The second part consists of iterating the following procedure for $u = v_i$, $i = 1, 2, \dots, |V|$: Perform a weighted splitting for every pair $\{vu, us\}$. The third part builds the two required flows x_1 and x_2 by iterating the next procedure: Assume that $x_1(e)$ and $x_2(e)$ have already been computed for an edge $e = vs$ having arisen by splitting off vu and us. Increase $x_i(vu)$ and $x_i(us)$ by $x_i(e)$ for $i = 1, 2$, and reduce $x_i(e)$ to 0.

Although this algorithm is a natural extension of the uncapacitated algorithm, proving that the number of weighted splittings is bounded by a polynomial in $|V|$ and $|E|$ requires some more work. First, observe that during the weighted splitting phase, once a subset of nodes becomes tight, it remains tight throughout. Consequently, once a pair of edges becomes infeasible, it never again becomes feasible.

Lemma. *In the course of the weighted splitting phase at most $O(|V|^2)$ weighted splittings are critical.*

Proof. Let $T = \{s_1, s_2, t_1, t_2\}$ be the set of terminal nodes. Let $\mathcal{F}$ denote the family of tight sets. For $Z \subset T$, let $\mathcal{F}_Z = \{ X : X \in \mathcal{F} \text{ and } X \cap T = Z \}$. By Claim 2 (in the proof of Theorem 2.2), $\mathcal{F}_Z$ is a ring family. (A family of subsets is called a *ring family* if it is closed under union and intersection.) Thus, $\mathcal{F}$ is the union of at most 16 ring families.

It is well-known that for every ring family $\mathcal{R} \subseteq 2^V$ for which $\emptyset, V \in \mathcal{R}$, there is a unique (transitive) digraph $D_1 = (V, E_1)$ such that

$$\mathcal{R} = \{ X : \delta_1(X) = 0 \text{ and } X \subseteq V \},$$

where $\delta_1(X)$ is the number of edges leaving X in D_1. (Namely, put $E_1 := \{ uv : \text{there is no } u\overline{v}\text{-set in } \mathcal{R} \}$.) Consequently, for a sequence $\mathcal{R}_1 \subset \mathcal{R}_2 \subset \cdots \subset \mathcal{R}_k$ of ring families, we have $k \le |V|^2$. Thus, if $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_l$ is a sequence of families of tight sets during the weighted splitting phase, then $l \le 16|V|^2$. Since a critical splitting operation strictly increases the family of tight sets, the number of these operations is at most $16|V|^2$. □

Since at a node u, at most

$$\varrho(u) \cdot \delta(u) \le |E|^2$$

non-critical weighted splittings can occur, the total number of weighted splittings is at most

$$|V| \cdot |E|^2 + 16|V|^2.$$

4 Connectivity in Eulerian digraphs

In this section we prove directed counterparts of two theorems of Lovász [7]. Let $G = (V, E)$ be a directed graph. Throughout this section, $\widetilde{G}$ is used to denote the underlying undirected graph of G. Let $A = \{v_1, v_2, \dots, v_k\} \subseteq V$ be a specified subset of nodes. A path connecting two distinct elements of A is called an *A-path*. Let $d_i = dc(v_i, A - v_i)$ for $i = 1, 2, \dots, k$.

Theorem 4.1. *Let vx be an edge of an Eulerian digraph G such that $v \notin A$. There exists an edge yv such that splitting off yv and vx does not reduce $dc(v_i, A - v_i)$ for $i = 1, 2, \dots, k$.*

Proof. Call a set X *i-critical* for some $i = 1, 2, \dots, k$ if $X \cap A = \{v_i\}$ and $\delta(X) = dc(v_i, A - v_i)$.

CLAIM. *If X and Y are i-critical, then so are $X \cap Y$ and $X \cup Y$, and $d(X, Y) = 0$. If X is i-critical, and Z is j-critical ($i \ne j$), then $X - Z$ is i-critical, and $Z - X$ is j-critical, and $d(X, \overline{Z}) = 0$.*

PROOF. Functions d and c below concern $\widetilde{G}$, functions δ and dc concern G. Since G is Eulerian, $d(X) = 2\delta(X)$ and $c(X, Y) = 2dc(X, Y)$ for disjoint sets X and Y. Thus, it suffices to work with $\widetilde{G}$. By (2), we have

$$\begin{aligned} 2c(v_i, A - v_i) &\\ &= d(X) + d(Y) \\ &= d(X \cap Y) + d(X \cup Y) + 2d(X, Y) \\ &\ge 2c(v_i, A - v_i) + 2d(X, Y) \end{aligned}$$

whence the first statement follows. Similarly,

$$\begin{aligned} c(v_i, A - v_i) + c(v_j, A - v_j) &\\ &= d(X) + d(Z) \\ &= d(X - Z) + d(Z - X) + 2d(X, \overline{Z}) \\ &\ge c(v_i, A - v_i) + c(v_j, A - v_j) + 2d(X, \overline{Z}) \end{aligned}$$

which implies the second statement, and thus proves the claim.

The edges yv and vx can be split off without reducing d_i if and only if there is no i-critical set X, for which either $v \in X$ and $x, y \notin X$, or $v \notin X$ and $x, y \in X$. Thus, if there is no critical set M with $|\{v, x\} \cap M| = 1$, then for any edge yv, the pair $\{yv, vx\}$ can be split off. If there is such set, we have two cases to consider.

CASE 1. There exists a critical $v\overline{x}$-set M.

Let M be minimal, and suppose that M is i-critical for some $i = 1, 2, \dots, k$. There is an edge yv such that $y \in M$, for otherwise

$$dc(v_i, A - v_i) \le \delta(M - v) < \delta(M) = dc(v_i, A - v_i),$$

a contradiction. We claim that yv and vx can be split off. Indeed, there is no i-critical $x\overline{v}$-set C, since then $d(C,M) > 0$ contradicts the claim. There is no i-critical $v\overline{y}$-set C, for otherwise $C \cap M$ is i-critical, contradicting the minimality of M. Similarly, there is no j-critical $y\overline{v}$-set C, since then $M - C$ would be i-critical, contradicting the minimality of M, and there is no j-critical $v\overline{x}$-set, since $d(M,\overline{C}) > 0$.

Case 2. There is no critical $v\overline{x}$-set, but there is a critical $x\overline{v}$-set M.

Let M be maximal, and suppose that M is i-critical for some $i = 1, 2, \dots, k$. Then there exists an edge yv such that $y \notin M$. We claim that yv and vx can be split off. Indeed, by assumption there is no critical $v\overline{x}$-set. Similarly, there cannot be a critical $y\overline{v}$-set C, since C is i-critical; then $C \cup M$ is also i-critical, contradicting the maximality of M, and if C is j-critical ($j \neq i$), then $d(M,\overline{C}) > 0$, contradicting the claim. □

The following theorem is immediately implied by Theorem 4.1.

Theorem 4.2. *In an Eulerian digraph $G = (V,E)$, the maximum number t of pairwise edge-disjoint A-paths is*

$$\min \sum_{i=1}^{k} \delta(V_i), \tag{3}$$

where $V_1, V_2, \dots, V_k$ are disjoint subsets of V and $V_i \cap A = \{v_i\}$ ($i = 1, 2, \dots, k$). Furthermore, $t = \sum d_i$, and if M_i denotes the minimal set for which $M_i \cap A = \{v_i\}$ and $\delta(M_i) = d_i$, then the sets M_i ($i = 1, 2, \dots, k$) are pairwise disjoint, and they form an optimal solution to (3).

Remark. Lovász's theorem on A-paths of an undirected Eulerian graph [7] easily follows if we apply Theorem 4.2 to any Eulerian orientation of G. On the other hand, the converse derivation is not difficult either.

Let $G = (V,E)$ be an Eulerian digraph, and let $\widetilde{G}$ denote the underlying undirected Eulerian graph. We show that there are t directed edge-disjoint A-paths in G, provided that there are t edge-disjoint A-paths in $\widetilde{G}$.

For a directed A-path or circuit P, we say that an internal node v of P is *in-bad* (on P) if both edges of P incident to v enter v; v is said to be *out-bad* if the two incident edges on P leave v. Let $b(P)$ denote the number of bad nodes of P. Let $\mathcal{P}$ be a decomposition of E into undirected A-paths $P_1, P_2, \dots, P_t$ and some circuits such that

$$\sum(b(P) : P \in \mathcal{P})$$

is minimal. We claim that every path P is a directed path in G (although not necessarily a simple path). Suppose, indirectly, that a path $P \in \mathcal{P}$ contains an in-bad node v (say). Since G is Eulerian, v must be an out-bad node of another member Q of $\mathcal{P}$. At least one of the two possible switches of P and Q at v gives rise to another partition $\mathcal{P}'$ of E into t (possibly not simple) A-paths and some circuits for which $\sum(b(P) : P \in \mathcal{P}')$ is smaller, a contradiction. See Figure 4.

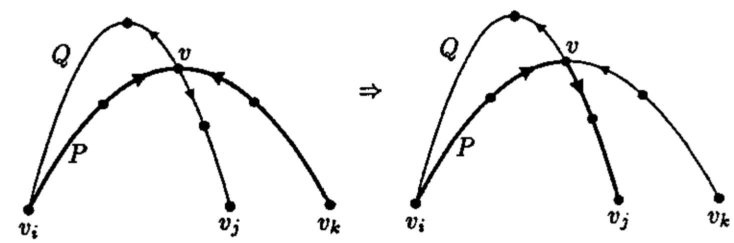

Figure 4. Switch of P and Q at v.

Remark. From an algorithmic point of view, the edge-disjoint A-paths problem and its capacitated version (when to every edge a non-negative capacity is assigned such that $\varrho_c(v) = \delta_c(v)$ for every $v \in V$) can be solved quite analogously to the two-commodity flow problem analysed in Section 3. We leave out the details.

Theorem 4.3. *Let $G = (V,E)$ be an Eulerian digraph and $vx \in E$. There exists an edge $yv \in E$ such that splitting off yv and vx does not reduce $dc(s,t)$ for any $s,t \in V - v$.*

Proof. Call a set C *critical* with respect to a pair of nodes s,t ($\neq v$) if C is a $t\overline{s}$-set and $dc(s,t) = \varrho(C)$. If C is critical for s,t, then C is critical for t,s. The set C is said to be *critical* with respect to s,t in $\widetilde{G}$ if $|\{s,t\} \cap C| = 1$ and $c(s,t) = d(C)$. Obviously, C is critical in G if and only if C is critical in $\widetilde{G}$. The next lemma concerns $\widetilde{G}$.

Lemma. *If X and Y are critical, then either*

(i) *$X \cap Y$ and $X \cup Y$ are critical and $d(X,Y) = 0$, or*

(ii) *$X - Y$ and $Y - X$ are critical and $d(X,\overline{Y}) = 0$.*

Proof of Lemma. Let X be critical for x_1, x_2, and let Y be critical for y_1, y_2. We have three possibilities to consider.

(a) $X \cap Y$ separates one of the pairs $\{x_1, x_2\}, \{y_1, y_2\}$, and $X \cup Y$ separates the other. (A set A is said to *separate* x and y if $|A \cap \{x, y\}| = 1$.) Then

$$d(X) + d(Y) = d(X \cap Y) + d(X \cup Y) + 2d(X, Y)$$
$$\geq d(X) + d(Y) + 2d(X, Y)$$

from which (i) follows.

(b) $X - Y$ separates one of the pairs $\{x_1, x_2\}, \{y_1, y_2\}$, and $Y - X$ separates the other. Applying (a) to the sets X and $\bar{Y}$, we obtain (ii).

(c) X separates $\{y_1, y_2\}$, and Y separates $\{x_1, x_2\}$, and one of the pairs $\{x_1, x_2\}, \{y_1, y_2\}$ is separated by both $X \cap Y$ and $X \cup Y$. Now

$$d(X) = c(x_1, x_2) \geq d(Y) = c(y_1, y_2) \geq d(X).$$

Suppose that the pair $\{y_1, y_2\}$ is separated by both $X \cap Y$ and $X \cup Y$. Then we have

$$d(X) = d(Y) = c(y_1, y_2)$$

and

$$d(X) + d(Y) = d(X \cap Y) + d(X \cup Y) + 2d(X, Y)$$
$$\geq c(y_1, y_2) + c(y_1, y_2) + 2d(X, Y)$$
$$= d(X) + d(Y) + 2d(X, Y)$$

from which (i) follows.

This completes the proof of the lemma. □

The lemma implies the corresponding assertion for G. Therefore, if X and Y are critical in G, and the edge vx enters both, then $X \cup Y$ is also critical. Hence, there is a unique maximal critical $x\bar{v}$-set M (if there is one at all). If no such M exists, any edge yv can be split off. If we have such an M, there is an edge yv with $y \notin M$. Indeed, if M is critical for s, and t and the required edge yv did not exist, then

$$dc(s, t) = \varrho(M) > \varrho(M + v) \geq dc(s, t),$$

a contradiction. By the construction of M, the edges yv and vx can be split off without reducing $dc(s, t)$ for any $s, t \in V - v$. □

Remark. W. Mader proved [9] that, given a not necessarily Eulerian digraph $G = (V, E)$ and a node v such that $dc(s, t) \geq k$ for every $s, t \in V - v$ and $\varrho(v) = \delta(v)$, a pair of edges yv, vx can be split off such that the connectivity from any $s \in V - v$ to any $t \in V - v$ continues to be at least k.

Remark. B. Jackson [5] also proved Theorems 4.1 and 4.3.

References

[1] L. R. Ford and D. R. Fulkerson, *Flows in Networks,* Princeton Univ. Press, Princeton, N. J. (1962).

[2] S. Fortune, J. Hopcroft and J. Wyllie, "The directed subgraph homeomorphism problem," *Theoretical Computer Science* **10** (1980), 111–121.

[3] A. Frank, "Edge-disjoint paths in planar graphs," *J. Combin. Theory (B)* **39** (1985), 164–178.

[4] T. C. Hu, "Multicommodity network flows," *Operations Res.* **11** (1963), 344–360.

[5] B. Jackson, "Some remarks on arc-connectivity, vertex splitting and orientation in digraphs." Preprint (1985).

[6] R. M. Karp, "On the computational complexity of combinatorial problems," *Networks* **5** (1975), 45–68.

[7] L. Lovász, "On some connectivity properties of Eulerian graphs," *Acta Math. Acad. Sci. Hungar.* **28** (1976), 129–138.

[8] W. Mader, "A reduction method for edge connectivity in graphs," *Annals of Discrete Math.* **3** (1978), 145–164.

[9] W. Mader, "Konstruktion aller n-fach kantenzusammenhängenden Digraphen," *Europ. J. Combinatorics* **3** (1982), 63–67.

[10] H. Okamura, "Multicommodity flows in graphs," *Discrete Applied Math.* **6** (1983), 55–62. (*Proc. London Math. Soc. (3)* **42** (1981), 178–192).

[11] H. Okamura and P. D. Seymour, "Multicommodity flows in planar graphs," *J. Combin. Theory (B)* **31** (1981), 75–81.

[12] B. A. Papernov, "Feasibility of multicommodity flows," *Studies in Discrete Optimization* (A. A. Friedman, ed.), Isdat. "Nauka," Moskow (1976), 230–261. (In Russian).

[13] N. Robertson and P. D. Seymour, "Graph minors — a survey," *Surveys in Combinatorics* (I. Anderson, ed.), *London Math. Soc. Lecture Notes* **103** (1985), 154–171.

[14] B. Rothschild and A. Whinston, "Feasibility of two-commodity network flows," *Operations Res.* **14** (1966), 1121–1129.

[15] A. Schrijver, "Min-max results in combinatorial optimization," *Mathematical Programming — The State of the Art* (A. Bachem, M. Grötschel and B. Korte, eds.), Springer Verlag (1982), 439–500.

[16] P. D. Seymour, "Disjoint paths in graphs," *Discrete Math.* **29** (1980), 293–309.

[17] P. D. Seymour, "Four-terminus flows," *Networks* **10** (1980), 79–86.

[18] P. D. Seymour, "On odd cuts and plane multicommodity flows," *Proc. London Math. Soc. (3)* **42** (1981), 178–192.