Mathematical Programming 42 (1988) 489-563
North-Holland

GENERALIZED POLYMATROIDS AND SUBMODULAR FLOWS

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Received 13 November 1986
Revised manuscript received 18 April 1988

Polyhedra related to matroids and sub- or supermodular functions play a central role in combinatorial optimization. The purpose of this paper is to present a unified treatment of the subject. The structure of generalized polymatroids and submodular flow systems is discussed in detail along with their close interrelation. In addition to providing several applications, we summarize many known results within this general framework.

Contents

Supported by a grant from the Alexander von Humboldt Stiftung and by the Institut für Ökonometrie und Operations Research of the University of Bonn, Bonn, Nassestr. 2, Federal Republic of Germany.

CHAPTER I. PRELIMINARIES

1. Introduction

The first relationship between matroid theory and what is now called combinatorial optimization was a theorem by Rado (1942) on the existence of independent transversals of a family of sets. As a next significant contribution Rado (1957) proved that the greedy algorithm works correctly not only on graphs but on matroids as well.

In the middle of the sixties investigations by J. Edmonds (1965a) and (1965b) further emphasized the role of matroids in combinatorial optimization. Edmonds's matroid intersection and matroid partition theorems along with Rado's theorem became prototypes of matroid min–max theorems. These three results are somehow on the same level in the sense that they can be derived from each other by elementary constructions. The weighted matroid intersection problem of J. Edmonds, seems to be on a higher level. Edmonds also developed polynomial-time algorithms for the matroid partition and for the weighted matroid intersection problems.

A second fundamental idea is the use of linear programming in combinatorial optimization. The idea goes back to works of Dantzig, Ford, Fulkerson and Hoffman, who applied linear programming to derive combinatorial results concerning networks. Later, Edmonds realized that linear programming can also be used in cases when the constraint matrix corresponding to the combinatorial problem is not necessarily totally unimodular.

The principle of using linear programming is nowadays rather well-known: Associate points in R^n with combinatorial objects to be investigated, determine the linear inequalities describing the convex hull P of these points, and apply the linear programming duality theorem in order to obtain a min–max result for the optimal object.

Matroid polyhedra were amongst the first polyhedra defined this way by Edmonds (1971). The independent sets of a matroid are the combinatorial objects to be investigated. The convex hull of their incidence vectors defines the matroid polyhedron. Edmonds (1971) showed that the matroid polyhedron is described by $\{x \in R_+^S: x(A) \leq r(A) \text{ for every } A \subseteq S\}$, where r is the rank function and S the ground set of the matroid. A much deeper result of Edmonds (1970) establishes the polyhedron of common independent sets of two matroids.

As a natural generalization of matroid polyhedra Edmonds (1970) introduced polymatroids. A fundamental feature of polymatroids is that the optimum of a linear objective function over a polymatroid can be calculated by a greedy algorithm. Furthermore, the defining linear system is totally dual integral (TDI). Edmonds also established the polymatroid intersection theorem (1970) stating, roughly, that the defining linear system of the intersection of two polymatroids is also TDI. Further problems on polymatroids have been investigated in a thesis by R. Giles (1975) (written under the supervision of J. Edmonds).

A drawback of the concept of polymatroids is that the role of sub- and supermodular functions is asymmetric and, also that only bounded and non-negative submodular functions are considered. This is why other models, similar to polymatroids, have been introduced: contrapolymatroids by Shapley (1971), base polyhedra by Fujishige (1984c), submodular systems by Fujishige (1984d). Some other disadvantages of polymatroids are: a face and a translate of a polymatroid are not polymatroids (although they are "nice" integral polyhedra), the intersection of a polymatroid and a box is a "polymatroid-like" polyhedron but it is not a polymatroid.

In order to overcome these difficulties and to unify the above-mentioned models the concept of generalized polymatroids or g-polymatroids has been introduced by Frank (1984c). The two most important features of polymatroids—the validity of the greedy algorithm and the intersection theorem—also hold for g-polymatroids.

In this paper we discuss properties of g-polymatroids in detail. We also reveal an interrelation between g-polymatroids and a more sophisticated model called submodular flows. This concept was introduced by Edmonds and Giles (1977) in order to give a general framework for network flows, polymatroid intersections and a theorem on covering of directed cuts by Lucchesi and Younger (1978). Other interesting models were defined and investigated by Hoffman (1982) and his co-workers. Since these pioneering works many other models concerning submodular functions and graphs have been introduced. Among them are polymatroidal network flows by Hassin (1982) and Lawler and Martel (1982), independent flows by Fujishige (1978), kernel systems by Frank (1979) and a very general model by Schrijver (1984b). An excellent survey on these models and their relationship can be found in Schrijver (1984a).

The general purpose of the present paper is to analyse generalized polymatroids, submodular flows, their relationship and various applications. We strive to summarize known results, as well.

The paper is divided into six chapters. In this first introductory chapter we present the required terminology and notation, and outline some known fundamental results on polymatroids and submodular functions. Chapter II introduces the concepts of truncation and bi-truncation of submodular functions. The concept of g-polymatroids along with basic features are also presented here. Chapter III offers constructions, characterizations and examples of g-polymatroids. In the fourth chapter we briefly review the greedy algorithm for g-polymatroids and exhibit some new applications of it. (Except in this chapter we are not concerned with algorithmic aspects of submodular functions. It should however be remarked that in the recent years many important papers appeared on this topic.) Submodular flows are investigated in Chapter V. We show that submodular flow polyhedra are exactly the projections of intersections of two g-polymatroids. Furthermore, various constructions and applications of submodular flows will be provided. The final sixth chapter includes polyhedral results concerning g-polymatroids and submodular flow polyhedra. In particular, we characterize adjacent vertices of the intersection of two matroid polyhedra.

Finally, let us draw attention to a paper by Lovász (1983) which surveys the basic theory of submodular functions. That paper also exhibits various important constructions of submodular functions, the knowledge of which is very useful in studying the present theory. (Here we do not repeat those constructions.)

2. Notation, preliminaries

The elements of polyhedral theory can be found in Pulleyblank (1983). For a detailed theory see Schrijver (1986). Here we shall need the following concepts and results. Let A be an $m \times n$ matrix and b an m-vector. An inequality $cx \leq g$ ($c \in \mathbb{R}^n$, $g \in \mathbb{R}$) is a *consequence* of $Ax \leq b$ if c is a non-negative combination $c = yA$ ($y \geq 0$, $y \in \mathbb{R}^m$) of the rows of A for which $yb \leq g$. If y can be chosen integer valued we say that $cx \leq g$ is an *integral consequence* of $Ax \leq b$.

Farkas' Lemma 2.1. *An inequality $cs \leq g$ is a consequence of $Ax \leq b$ if and only if every x satisfying $Ax \leq b$ satisfies $cx \leq g$.*

If both $cx \leq g$ and $cx \geq g$ are consequences of $Ax \leq b$ we say that $cx \leq g$ is an *implicit* equality (with respect to $Ax \leq b$).

Let $Q = \{x \in \mathbb{R}^n: Ax \leq b\}$ be a polyhedron. A *face* of Q is a polyhedron $Q_F = \{x \in Q, A_F x = b_F\}$ where A_F is an $m_F \times n$ submatrix of A and b_F is the corresponding "subvector" of b. We shall also consider the empty set to be a face. A maximal face which is not Q is called a *facet*. If $\{v\}$ is a face for $v \in Q$, then v is called a *vertex*. Two vertices u, v are *adjacent* if the segment spanned by u and v is a face. The *dimension* dim Q of Q is the maximum number of affinely independent points of Q minus 1. The *co-dimension* co-dim Q of Q is the maximum number of linearly independent implicit equalities (with respect to $Ax \leq b$). This quantity depends only on Q and dim Q + co-dim $Q = n$. Every facet has the same dimension, namely, dim $Q - 1$. Similarly, every minimal face has the same dimension. Let a be a row of A. We say that an inequality $ax \leq b_a$ is *facet-inducing* if $Q_a = \{x \in Q: ax = b_a\}$ is a facet of Q where b_a is the component of b corresponding to a. A polyhedron is called *integral* if each of its faces contains an integral point. (In particular, its vertices are integral.)

We say that a linear system $Ax \leq b$ is *totally dual integral* as introduced by Edmonds and Giles (1977) or TDI if, for any integral $d \in \mathbb{R}^n$, the dual of the linear program (max dx: $Ax \leq b$) has an integer optimal solution vector whenever it has an optimal solution. The following fundamental result was proved by Hoffman (1974) when the polyhedron $\{x: Ax \leq b\}$ is bounded and by Edmonds and Giles (1977) in general:

Theorem 2.2. *If the entries of A and b are integers, a TDI system $Ax \leq b$ defines an integral polyhedron.*

We call a linear system $Ax \leq b$ *box TDI* if for every $f, g \in \mathbb{R}^n$ the system $\{f \leq x \leq g, Ax \leq b\}$ is TDI.

Suppose that Q is a full dimensional rational polyhedron. Then there is a uniquely determined matrix A for which $Q = \{x: Ax \leq b\}$ for some b, the rows of A are in a one-to-one correspondence with the facets of Q, and each row of A is an integral vector so that the greatest common divisor of its components is one. Schrijver (1981) proved that Q has a unique minimal TDI system describing Q. If the minimal TDI describing system is the above minimal describing system $Ax \leq b$, we say that Q is *facet-TDI*.

The following simple observation will prove useful.

Proposition 2.3. *If a linear system, that arises from $Ax \leq b$ by adjoining some integral consequences, is TDI, then so is $Ax \leq b$.*

For two polyhedra P_1 and P_2 in $\mathbb{R}^S$ the polyhedron $P = \{x: x = x_1 + x_2$ for some $x_1 \in P_1$, $x_2 \in P_2\}$ is called the *sum* of P_1 and P_2 and denoted by $P_1 + P_2$. Let $\{S_1, S_2, \ldots, S_k\}$ be a partition of S ($S_i \neq \emptyset$). To every vector x in $\mathbb{R}^S$ we assign a vector $z = \varphi(x) \in \mathbb{R}^k$, called the *homomorphic image*, by the definition $z(i) = x(S_i)$. The *homomorphic image* $\varphi(P)$ of a polyhedron $P \subseteq R^S$ is defined by $\varphi(P) := \{\varphi(x): x \in P\}$.

Throughout the paper we use a finite ground set S. We do not distinguish between a subset $X \subseteq S$ and its characteristic vector $\chi_X \in \mathbb{R}^S$ (defined by $\chi_X(s) = 1$ if $s \in X$ and $=0$ if $s \notin X$). $\bar{X}$ denotes the complement $S - X$ of X. A *singleton* $\{v\}$ is a one-element set. We shall denote $\{v\}$ by v. "$X \subseteq S$" means that X is a subset of S. "$X \subset S$" means that $X \subseteq S$ but $X \neq S$. For two elements $u, v \in S$ a set X is called a $u\bar{v}$-set if $u \in X$, $v \notin X$. Two subsets $X, Y \subseteq S$ are said to be

co-disjoint if $X \cup Y = S$,

intersecting if none of $X - Y$, $Y - X$, $X \cap Y$ is empty,

crossing if they are intersecting and $X \cup Y \neq S$.

A family $\mathcal{F}$ of subsets of S is called a *ring-family* if $X, Y \in \mathcal{F}$ implies $X \cap Y$, $X \cup Y \in \mathcal{F}$. $\mathcal{F}$ is an *intersecting* (*crossing*) family if this implication is required only for intersecting (crossing) X, Y.

A family $\mathcal{F}$ is a *chain* or *chain family* if $X, Y \in \mathcal{F}$ implies that $X \subseteq Y$ or $Y \subseteq X$. $\mathcal{F}$ is a *laminar* (*cross-free*) family if it does not include intersecting (crossing) subsets.

With every family $\mathcal{F}$ of subsets of S we associate a directed graph $G(\mathcal{F}) = (S, E)$ where $E = \{uv$: there is no $u\bar{v}$-set in $\mathcal{F}\}$. We call G the *digraph* of $\mathcal{F}$.

It is easy and well known that, if $\mathcal{F}$ is a ring family with $S, \emptyset \in \mathcal{F}$, $G(\mathcal{F})$ uniquely determines $\mathcal{F}$. Namely, $\mathcal{F} = \{X \subseteq S$; no edge of G leaves $X\}$.

Let $\mathcal{F}'$ be an intersecting family with $\emptyset \in \mathcal{F}'$. Define $\mathcal{F} := \{X: X = \bigcup X_i$ for some sets $X_i \in \mathcal{F}'\}$. The proof of the following statement is straightforward and so we omit it.

Proposition 2.4. *$\mathcal{F}$ is the smallest ring family including $\mathcal{F}'$. $\mathcal{F}$ consists of those sets which are unions of pairwise disjoint members of $\mathcal{F}'$. Furthermore, $G(\mathcal{F}) = G(\mathcal{F}')$.*

Let $\mathcal{F}''$ be a crossing family with $\emptyset$, $S \in \mathcal{F}''$. Define $\mathcal{F}' := \{X: X = \bigcap X_i$ for some sets $X_i \in \mathcal{F}''\}$.

One can see that $\mathcal{F}'$ is an intersecting family and $G(\mathcal{F}'') = G(\mathcal{F}')$. Applying Proposition 2.4 to $\mathcal{F}'$ we have:

Proposition 2.5. *Let $\mathcal{F}''$ be a crossing family with $\emptyset$, $S \in \mathcal{F}''$. The smallest ring family $\mathcal{F}$ including $\mathcal{F}''$ is $\mathcal{F} = \{X: X$ is the union of some disjoint subsets X_i, $i = 1, 2, \ldots,$ where each X_i is the intersection of pairwise co-disjoint members of $\mathcal{F}''\}$. Moreover, $G(\mathcal{F}) = G(\mathcal{F}'')$.*

For a vector $x \in \mathbb{R}^S$ and a subset $A \subseteq S$ we use the notation $x(A) := \sum (x(v): v \in A)$. Let $b: 2^S \to \mathbb{R} \cup \{+\infty\}$ be a set function. We shall suppose throughout that $b(\emptyset) = 0$. Let us define $\mathcal{F}(b) = \{X: b(X)$ is finite$\}$. b is called *finite* if $\mathcal{F}(b) = 2^S$. For $T \subset S$ the *restriction* $b|_T: 2^T \to \mathbb{R} \cup \{\infty\}$ of b to the subsets of T is $b|_T(X) = b(X)$ for $X \subseteq T$.

We associate three kinds of polyhedra with b:
$S(b) := \{x \in \mathbb{R}^S, x(A) \leq b(A)$ for every $A \subseteq S\}$,
$B(b) := \{x \in \mathbb{R}^S, x(S) = b(S), x(A) \leq b(A)$ for every $A \subseteq S\}$,
$P(b) := \{x \in \mathbb{R}^S, x \geq 0, x(A) \leq b(A)$ for every $A \subseteq S\}$.

A set-function b is called *fully submodular* (or *submodular*) if the *submodular inequality*

$$b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y) \tag{2.1}$$

holds for every $X, Y \subseteq S$.

b is *intersecting* (*crossing*) submodular if (2.1) is required only for intersecting (crossing) X, Y.

Let b be a submodular function. It is easy to check that $\mathcal{F}(b)$ is a ring-, intersecting-, crossing family, respectively, if b is a fully-, intersecting-, crossing submodular function. Consequently, it is equivalent to speak about a fully submodular function defined on 2^S and about a finite fully submodular function defined on a ring family. An analogous statement holds for intersecting and crossing submodular functions. We find it more convenient to work with functions defined on 2^S.

We shall use the notation b'', b', b for crossing, intersecting, fully submodular functions, respectively. In applications the following constructions of crossing submodular functions will prove useful.

Let b' be an intersecting (in particular, a fully) submodular function. Let b_2' be a function obtained from b' by reducing $b'(X)$ by a positive constant on every $X \subseteq S$ except $X = \emptyset$. Let b_1' be a function obtained from b' by reducing $b'(X)$ on singletons (by possibly different non-negative values). Similarly, let b'' be a crossing (in particular, intersecting or fully) submodular function. Define b_2'' by reducing $b''(X)$ by a positive constant on every X except $X = \emptyset$ and $X = S$ and define $b_1''(X)$ by reducing $b''(X)$ on singletons and on complements of singletons by non-negative values.

Proposition 2.6. *b_1' and b_2' are intersecting submodular functions. b_1'' and b_2'' are crossing submodular functions.*

If b is a fully submodular function, $S(b)$ is called a *submodular* polyhedron (Fujishige (1984d)) and $B(b)$ is called a *base-polyhedron* (Fujishige (1984c)). If $b(S) = 0$, $B(b)$ is a *0-base polyhedron*. If b is fully submodular, non-negative, monotone increasing (i.e., $b(X) \geq b(Y)$ if $X \supseteq Y$), and finite, then b is called a *polymatroid function* and the polyhedron $P(b)$ a *polymatroid* (Edmonds (1970)). The *base polyhedron of a polymatroid* $P(b)$ is $B(b)$. Edmonds introduced (the concept of) polymatroids as compact subsets of $\mathbb{R}_+^S$ with certain properties and proved that the two definitions of polymatroids are equivalent. Since submodular functions play a central role in applications we found it more appropriate to use them in the definition of polymatroids.

The rank function r of a matroid M is a polymatroid function and $P(r)$ is called the *matroid polyhedron*. An integer-valued polymatroid function b with $b(X) \leq |X|$ ($X \subseteq S$) is a matroid function. Using this fact and the next theorem it follows that an integral polymatroid in the unit cube is a matroid polyhedron. Edmonds (1971) proved that the vertices of $P(r)$ are exactly the (characteristic vectors of) independent sets. He also showed that the vertices of $B(r)$ are exactly the bases of M. We call $B(r)$ a *matroid base polyhedron*.

A set function p is (fully, intersecting, crossing) *supermodular* if $-p$ is (fully, intersecting, crossing) submodular. A *contra-polymatroid* is a polyhedron $\{x \in \mathbb{R}^S: x(A) \geq p(A)$ for every $A \subseteq S\}$ associated with a supermodular function p. See Shapley (1971).

A set function m is *modular* if (2.1) holds with equality everywhere. It is trivial that finite modular set functions m (with $m(\emptyset) = 0$) and vectors are essentially the same and we do not distinguish between them.

Let $p: 2^S \to \mathbb{R} \cup \{-\infty\}$, $b: 2^S \to \mathbb{R} \cup \{\infty\}$ be set functions. We say that p and b are *compliant* if they satisfy the following *cross inequality*

$$b(X) - p(Y) \geq b(X - Y) - p(Y - X) \tag{2.2}$$

for every subset X, Y of S. If (2.2) is required only for intersecting X, Y, we say that p and b are *weakly compliant*.

With the pair of set-function (p, b) we associate a polyhedron $Q(p, b)$, as follows: $Q(p, b) = \{x \in \mathbb{R}^S; p(A) \leq x(A) \leq b(A)$ for every $A \subseteq S\}$.

A pair (p, b) is called a *strong pair* if $-p$ and b are fully submodular and p and b are compliant. For a strong pair (p, b) the polyhedron $Q(p, b)$ is called a *generalized polymatroid*. A pair (p, b) is called a *weak pair* if $-p$ and b are intersecting submodular functions and p and b are weakly compliant.

Let b be a set function. An *evaluation oracle* for b provides the value $b(X)$ for any subset X and tells $G(\mathcal{F}(b))$. A *minimizing oracle for* b solves the problem $\min(b(X)-m(X): A\subseteq X\subseteq B)$ where $\emptyset\subseteq A\subset B\subseteq S$ are given subsets and m is a (finite) vector. A minimizing oracle for b, when applied to $A=\{u\}$, $B=S-v$, $m=0$, can be used to decide whether an edge uv belongs to $G(\mathcal{F}(b))$. We note that Grötschel, Lovász and Schrijver (1981), relying on the ellipsoid method, constructed a polynomial-time minimizing oracle for any submodular function b (given by an evaluation oracle).

Let $G=(V,E)$ be a directed graph with node set V and edge set E. We say that an edge uv *enters* $A\subset V$ if A is a $v\bar{u}$-set. An edge *leaves* A if it enters $\bar{A}$. The number of edges entering (leaving) A is denoted by $\rho(A)$ $(\delta(A))$. For a vector $x\in\mathbb{R}^E$, $\rho_x(A):=\sum(x(e): e\in E,\ e$ enters $A)$. For a subset $F\subseteq E_{\rho_F}(A)$ is the number of elements of F entering A. $\delta_x(A)$ and $\delta_F(A)$ are defined analogously. We denote the difference $\rho_x(A)-\delta_x(A)$ by $\lambda_x(A)$. For $A,B\subseteq V$ and $x:E\to\mathbb{R}\cup\{\infty\}$ let $d_x(A,B):=\sum(x(e): e\in E,\ e$ enters one of A and B and leaves the other). If $x=\chi_E$, we use $d(A,B)$ for $d_x(A,B)$. $\boldsymbol{\rho}_A\in\mathbb{R}^E$ $(\boldsymbol{\delta}_A\in\mathbb{R}^E)$ is a $(0,1)$ vector for which $\boldsymbol{\rho}_A(e)=1$ $(\boldsymbol{\delta}_A(e)=1)$ if e enters (leaves) A. Set $\boldsymbol{\lambda}_A:=\boldsymbol{\rho}_A-\boldsymbol{\delta}_A$. We do not distinguish between the set of edges entering A and its characteristic vector $\boldsymbol{\rho}_A$. For $X\subseteq V$, $E(X)$ denotes the set of edges with both ends in X.

Proposition 2.7. *For* $x\in\mathbb{R}^E$, $\lambda_x:2^V\to\mathbb{R}$ *is a finite modular function. For* $f:E\to\mathbb{R}\cup\{-\infty\}$, $g:E\to\mathbb{R}\cup\{\infty\}$ *and* $f\leqslant g$, *the set function* $b(A)=\rho_g(A)-\delta_f(A)$ *satisfies*:

$$b(A)+b(B)=b(A\cap B)+b(A\cup B)+d_{g-f}(A,B). \tag{2.3}$$

In particular, b, ρ_x, δ_x *(where* $x\geqslant 0$*) are fully submodular.*

This is proved by showing that each edge has the same contribution to both sides of (2.3).

We call a directed graph $G=(V,E)$ *h-strongly edge-connected* ($h\geqslant 0$ is an integer) if $\rho(X)\geqslant h$ for every X, $\emptyset\subset X\subset V$. (Equivalently, by Menger's theorem, if there are h edge-disjoint directed paths from u to v for every pair u, v of nodes.)

Here we list some basic results on polymatroids and submodular functions.

Proposition 2.8 (Edmonds (1970)). *For every polymatroid* P *there is a unique polymatroid function* b *for which* $P=P(b)$, *namely*, $b(A)=\max(x(A): x\in P)$.

Theorem 2.9 (Edmonds (1970)). *Let* $b'\geqslant 0$ *be an intersecting submodular function* (b' *need not be monotone or finite*). *Then* $P(b')$ *is a polymatroid. Its unique defining polymatroid function* b_1 *is*

$$b_1(X)=\min(\textstyle\sum b(X_i): \bigcup X_i\supseteq X,\ X_i\text{'s are disjoint}).$$

Let $g:S\to\mathbb{R}_+\cup\{\infty\}$ be a vector and b a polymatroid function. Proposition 2.6 and Theorem 2.9 imply

Theorem 2.10 (Edmonds (1970)). $P(b)\cap\{x\in\mathbb{R}^S: x\leqslant g\}$ *is a polymatroid. Its unique defining function* b_1 *is*

$$b_1(X)=\min(b(Y)+g(X-Y): Y\subseteq X).$$

In particular (choosing $g\equiv 1$*) a family* $\mathcal{F}=\{F\subseteq S: |X|\leqslant b(X)$ *for every* $X\subseteq F\}$ *is the family of independent sets of a matroid whose rank function is*

$$r(A)=\min(b(X)+|A-X|: X\subseteq A).$$

Perhaps the most important result for polymatroids is the Polymatroid Intersection Theorem of Edmonds (1970). See Chapter V.

The following useful theorem concerning crossing submodular functions is due fo S. Fujishige (1984a).

Theorem 2.11. *For a crossing submodular function* b'' *the polyhedron* $B(b'')$ *is non-empty if and only if*

$$\text{(a) } \textstyle\sum b''(Z_i)\geqslant b''(S) \quad \textit{and} \qquad \text{(b) } \sum b''(\bar{Z}_i)\geqslant (k-1)b''(S) \tag{2.4}$$

for every partition $\{Z_1,Z_2,\dots,Z_k\}$ of S.

An interesting relationship between sub- and supermodular functions is the

Discrete Separation Theorem 2.12 (Frank (1982)). *Let* p *and* b *be fully super- and submodular functions, respectively. There is a finite modular function* m *for which* $p\leqslant m\leqslant b$ *if and only if* $p\leqslant b$. *If* p *and* b *are integer-valued,* m *can be chosen integer-valued, too.*

In Section V.3 we give a new proof. In Section IV.4 we shall present a new algorithmic proof.

We call a non-negative function $y:2^S\to R_+$ *a weighted chain* if the family $\mathcal{F}=\{X: y(X)>0\}$ is a chain. With every weighted chain y we associate a non-negative vector $\pi=\sum_A y(A)\chi_A$ $(\in\mathbb{R}_+^S)$, called the *depth vector* of y.

This is a one-to-one correspondence: for a non-negative vector $\pi\in\mathbb{R}_+^S$ let $0\leqslant\pi_1<\cdots<\pi_k$ be the distinct values of π and let $X_i=\{s: \pi(s)\geqslant\pi_i\}$. Define $y(X)=\pi_i-\pi_{i-1}$ if $X=X_i$ $(i=1,\dots,k)$ (where π_0 is 0). Obviously y is a weighted chain and its depth vector is π. We call this y the *weighted chain* of π.

Let b be a fully submodular function. There is a natural way to extend b to all non-negative vectors. Namely, let $\pi\in\mathbb{R}_+^S$ be a non-negative vector and y its weighted chain. Define $\hat{b}(\pi):=\sum(y(X)\cdot b(X): y(X)>0)$.

Obviously $\hat{b}$ is (positively) homogeneous, i.e., $\hat{b}(\mu\pi)=\mu\hat{b}(\pi)$ for every positive μ.

Theorem 2.13 (Lovász (1983)). *The extended $\hat{b}$ is convex, i.e., $\hat{b}(\alpha)+\hat{b}(\beta) \geq 2\hat{b}((\alpha+\beta)/2)$. Equivalently (by the homogeneity), $\hat{b}$ is subadditive, i.e., $\hat{b}(\alpha)+\hat{b}(\beta) \geq \hat{b}(\alpha+\beta)$.*

The extension $\hat{b}$ is strongly related to the greedy algorithm. Call an ordering $s_1, s_2, \ldots, s_n$ of the elements of S *compatible* (with π and b) if $\pi(s_i) \geq \pi(s_{i-1})$ $(i=2,3,\ldots,n)$ and $b(S_i)$ is finite for $S_i=\{s_1, s_2, \ldots, s_i\}$ $(i=1,2,\ldots,n)$. Suppose $s_1, s_2, \ldots, s_n$ is a compatible ordering.

Define a vector $x_0 \in \mathbb{R}^S$ by $x_0(s_i)=b(S_i)-b(S_{i-1})$ $(S_0 := \emptyset)$.

Greedy Algorithm Theorem 2.14. (a) (Edmonds (1971)). *If b is a polymatroid function, then $x_0 \in P(b)$ and $\hat{b}(\pi)=\max(\pi x\colon x \in P(b))=\pi x_0$.*

(b) (Fujishige and Tomizawa (1983)). *If b is an arbitrary fully submodular function, then $x_0 \in B(b)$ and $\hat{b}(\pi)=\max(\pi x\colon x \in B(b))=\pi x_0$.*

Corollary 2.15. *The linear systems $\{x \geq 0,\ x(A) \leq b(A)$ for every $A \subseteq S\}$ and $\{x(S)=b(S),\ x(A) \leq b(A)$ for every $A \subseteq S\}$ defining $P(b)$ and $B(b)$, respectively, are TDI.*

Let S_1 and S_2 be disjoint sets and $b_i : 2^{S_i} \to \mathbb{R}$ $(i=1,2)$ arbitrary set functions. Define $b : 2^{S_1 \cup S_2} \to \mathbb{R}$ by $b(X)=b_1(X \cap S_1)+b_2(X \cap S_2)$. We call b the *direct sum* of b_1 and b_2. Obviously, if b_1 and b_2 are fully submodular then so is b.

Proposition 2.16. *For a non-negative integer vector $\pi \in \mathbb{Z}_+^S$ and a fully submodular function b the set function b_π defined by $b_\pi(X) := \hat{b}(\pi+\chi_X)-\hat{b}(\pi)$ is fully submodular. Furthermore, b_π depends only on the level sets of π.*

Proof. Let $0 \leq \pi_0 < \pi_1 < \ldots < \pi_k$ denote the distinct values of π. We can suppose that $\pi_0=0$, for otherwise reduce every component of π by π_0 and observe that for the resulting π' one has $b_\pi = b_{\pi'}$. If $\pi \equiv 0$, then $b_\pi(X)=b(X)$ so we can suppose that $k>0$. Let $S_i=\{v\colon \pi(v) \geq \pi_i\}$ $(i=0,1,\ldots,k)$ be the level sets. Assume first that $\pi_i-\pi_{i-1} \geq 2$ for some $i=1,2,\ldots,k$. Then S_i is a level set of $\pi+\chi_X$ so for $\pi'=\pi-\chi_{S_i}$ we have $b_\pi = b_{\pi'}$ and the second part of the theorem follows. Now we can suppose that $\pi_i = i$ $(i=0,1,2,\ldots,k)$.

Define $b_i : 2^{S_{i+1}-S_i} \to \mathbb{Z}_+$ $(i=0,1,\ldots,k)$ by $b_i(X)=b(X \cup S_i)-b(S_i)$ (where $S_{k+1}=\emptyset$). Obviously each b_i is fully submodular and b_π is the direct sum of b_i's. □

Chapter II. Generalized polymatroids

1. Truncation, bi-truncation

Let $b' : 2^S \to \mathbb{R} \cup \{\infty\}$ be an arbitrary set function. Define $b : 2^S \to \mathbb{R} \cup \{\infty\}$ as follows:

$$b(X)=\min(\textstyle\sum b'(X_i)\colon \{X_i\} \text{ a partition of } X). \tag{1.1}$$

We call b the *lower truncation* of b'. (Sometimes it is called Dilworth truncation). If in (1.1) min is replaced by max, we call b the *upper truncation* of b'.

Truncation Theorem 1.1 (Lovász (1977)). *The lower truncation b of an intersecting submodular function b' is fully submodular. Moreover, $S(b)=S(b')$.*

(The theorem can be stated analogously for supermodular functions and upper truncation.)

Proof. The second statement is straightforward since obviously $b \leq b'$ and hence $S(b) \subseteq S(b')$. On the other hand any inequality $x(A) \leq b(A)$ is an (integral) consequence of certain inequalities $x(A_i) \leq b'(A_i)$ where $\{A_i\}$ partitions A and $b(A)=\sum b'(A_i)$. Consequently $S(b)=S(b')$.

To prove the submodularity of b let $A, B \subseteq S$. Let $b(A)=\sum b'(A_i)$ for a certain partition $\{A_1, \ldots, A_k\}$ of A and $b(B)=\sum b'(B_i)$ for a certain partition $\{B_1, \ldots, B_h\}$ of B.

Let $\mathcal{F}=\{A_1, \ldots, A_k, B_1, \ldots, B_h\}$. Then $\mathcal{F}$ satisfies the following:

every $v \in A \cap B$ is covered twice, every $v \in (A-B) \cup (B-A)$ is covered once by $\mathcal{F}$. (1.2)

Denote $b'(\mathcal{F}) := \sum\{b'(X)\colon X \in \mathcal{F}\}$. If there are two intersecting sets A_i, B_j in $\mathcal{F}$, revise $\mathcal{F}$ by replacing A_i and B_j by $A_i \cap B_j$ and $A_i \cup B_j$. The new family $\mathcal{F}_1$ satisfies (1.2) and since b' is intersecting submodular $b'(\mathcal{F}_1) \leq b'(\mathcal{F})$.

Apply this uncrossing operation as long as there are intersecting sets. Since in every step $\sum(|X|^2\colon X \in \mathcal{F})$ strictly increases (check!) after a finite number of steps we obtain an $\mathcal{F}_0$ satisfying (1.2) for which $b'(\mathcal{F}_0) \leq b'(\mathcal{F})$ and $\mathcal{F}_0$ is laminar. Then $\mathcal{F}_0=\mathcal{P}_1 \cup \mathcal{P}_2$ where $\mathcal{P}_1$ and $\mathcal{P}_2$ are disjoint, $\mathcal{P}_1$ is a partition of $A \cap B$ and $\mathcal{P}_2$ is a partition of $A \cup B$.

By definition $b(A \cap B) \leq b'(\mathcal{P}_1)$ and $b(A \cup B) \leq b'(\mathcal{P}_2)$ so we have $b(A)+b(B)=b'(\mathcal{F}) \geq b'(\mathcal{F}_0)=b'(\mathcal{P}_1)+b'(\mathcal{P}_2) \geq b(A \cap B)+b(A \cup B)$, as required. □

The Truncation Theorem easily implies Theorem I.2.9.

Proof of Theorem I.2.9. Let b be the truncation of b'. Then $b \geq 0$, b is fully submodular and since $S(b) = S(b')$ also $P(b) = P(b')$. This b may not be monotone. Set $b_1(A) := \min(b(X): X \supseteq A)$. It is easily seen that b_1 is a polymatroid function and $P(b) = P(b_1)$. □

Remark. In the proof b_1 was constructed in two steps. One was truncation, the second was monotonization. In Section 2 we shall slightly extend the concept of truncation and the extension will involve monotonization as well.

A combination of Theorems I.2.12 and 1.1 is

Theorem 1.2 (Frank (1982)). *Let p' and b' be intersecting super- and submodular functions, respectively $(-p', b': 2^S \to \mathbb{R} \cup \{\infty\})$. There is a finite modular function m for which $p' \leq m \leq b'$ if and only if $\sum p'(F_i) \leq \sum b'(G_j)$ holds whenever both families $\{F_i\}$ and $\{G_i\}$ consist of disjoint subsets of S and $\bigcup F_i = \bigcup G_j$. If p', b' are integer-valued, m also can be chosen integer-valued.* □

A third application of Theorem 1.1 is "bi-truncation".

The following theorem was proved by Fujishige (1984a) and implicit in Frank (1982).

Bi-truncation Theorem 1.3. *Let $b'': 2^S \to \mathbb{R} \cup \{\infty\}$ be a crossing submodular function and $Q = B(b'')$. If Q is non-empty, there is a fully submodular function b (called the bi-truncation of b'') for which $Q = B(b)$. If b'' is integer-valued, so is b.*

Proof. We are going to construct b from b'' by using truncation twice. (This justifies the name bi-truncation). Let $k = b''(S)$ and define p'' by $p''(X) := k - b''(S - X)$. Then p'' is a crossing supermodular function and obviously $Q = Q'' := \{x \in \mathbb{R}^S;\ x(S) = k, x(A) \geq p''(A)$ for every $A \subseteq S\}$. Let p' denote the upper truncation of p''. Clearly $Q'' = Q' =: \{x \in R^S: x(S) = k\ x(A) \geq p'(A)$ for every $A = \subseteq S\}$. Applying the truncation theorem to $p' | X \cup Y$ we see that p' is supermodular on X and Y whenever X and Y are not co-disjoint. Furthermore, we claim that $p'(S) = p''(S) = k$. Indeed, for upper truncation $p'(S) \geq p''(S)$ holds in general, but $Q = Q'$ is not empty, so we cannot have $p'(S) > k$.

Let us define b' by $b'(X) = k - p'(S - X)$. b' is an intersecting submodular function ($b'(\emptyset) = 0$!) and $B(b') = Q' = Q$. Let b be the lower truncation of b'. From a second application of the Truncation Theorem we see that b is fully submodular and $B(b) = B(b') = Q$. □

Remark. The bi-truncation p of a crossing supermodular function p'' can be introduced analogously. (p is the negative of the bi-truncation of $-p''$).

Proposition 1.4. *The bi-truncation b of a crossing submodular function b'' with $b''(S) = k$ is*

$$b(X) = \min(\textstyle\sum b''(X_{ij}) + (n - m)k) \tag{1.3}$$

where m is the number of sets X_{ij}, the sets $X_i = \bigcap_j X_{ij}$, $i = 1, 2, \ldots, n$, form a partition of X, and, for fixed i, the sets $\bar{X}_{i1}, \bar{X}_{i2} \ldots$ form a partition of $\bar{X}_i$. *($\bar{Z}$ denotes $S - Z$.)*

Proof. From the proof of Theorem 1.1 $b(X) = \min \sum(b'(X_i): \{X_i\}$ a partition of $X)$. Furthermore, $b'(X_i) = k - p'(S - X_i)$ and $p'(Z_i) = \sum \max(p''(Z_{ij}): \{Z_{ij}\}$ a partition of $Z_i)$ where $Z_i = S - X_i$. Since $p''(Z_{ij}) = k - b''(S - Z_{ij})$ the statement follows. □

Remark. (1.3) becomes simpler if $k = 0$.

Remark. Observe that Proposition I.2.5 is a special case of the Bi-truncation Theorem. Namely, define $b''(X) = 0$ if $X \in \mathcal{F}''$ and $= \infty$ otherwise.

Remark 1.5. Let $\mathcal{F} = \{X_{ij}\}$ be a family where the minimum in (1.3) is attained. Then $b(Z) = b''(Z)$ holds for $Z \in \mathcal{F}$.

From the proof of Theorem 1.3 we have:

Corollary 1.6. *If b is the bi-truncation of b'', an inequality $x(A) \leq b(A)$ is an integral consequence of the inequalities $x(X) \leq b''(X)$ $(X \subseteq S)$ and $x(S) = b''(S)$. (That is, there are integers y_X associated with subsets $X \subseteq S$ which are non-negative if $X \neq S$ such that $\sum y_X \chi_X = \chi_A$ and $\sum y_x b''(X) = b(A)$).*

2. Generalized polymatroids

Let (p, b) be a strong pair. The polyhedron $Q = Q(p, b)$ is called a *generalized polymatroid* or briefly a *g-polymatroid*. If p and b are integer-valued, Q is called an *integral g*-polymatroid.

For convenience we consider the empty set as a g-polymatroid. The concept of g-polymatroids was introduced by Frank (1984c). See also Hassin (1982). In this section we present some basic features of g-polymatroids.

Proposition 2.1. *Polymatroids, contra-polymatroids, base polyhedra and submodular polyhedra are generalized polymatroids.*

Proof. From the definition one easily sees that if p is identically zero, a pair (p, b) forms a strong pair if and only if b is a polymatroid function. Then $Q(p, b)$ determines an ordinary polymatroid. If p is fully supermodular and $b \equiv \infty$, then (p, b) defines a contra-polymatroid. If b is fully submodular and p is defined by

$p(X) = b(S) - b(S-X)$, then (p, b) is a strong pair and $Q(p, b)$ is a base polyhedron. If b is fully submodular and $p \equiv -\infty$, then (p, b) is a strong pair and $Q(p, b)$ is a submodular polyhedron. □

Proposition 2.2. *Where (p, b) is a strong pair, the g-polymatroid $Q = Q(p, b)$ is non-empty. If Q is integral, it contains integer points.*

Proof. Induction on $|S|$. Let $s \in S$ and $S_1 = S - s$. For $p_1 = p|S_1$ and $b_1 = p|S_1$ (p_1, b_1) is a strong pair so, by the induction hypothesis, there is a vector $x_1 \in Q(p_1, b_1)$ $(\subseteq \mathbb{R}^{S_1})$. We claim that $m := \min(b(X) - x_1(X): s \in X \subseteq S) \geq M := \max(p(Y) - x_1(Y): s \in Y \subseteq S)$. Indeed, for $s \in X$, $Y \subseteq S$ we have $b(X) - p(Y) \geq b(X-Y) - p(Y-X) \geq x_1(X-Y) - x_1(Y-X) = x_1(X) - x_1(Y)$. Define $x \in \mathbb{R}^S$ in such a way that $x|S_1 = x_1$ and $m \geq x(s) \geq M$. Then $x \in Q(P, b)$. □

Proposition 2.3. $\max(x(A): x \in Q) = b(A)$ *and* $\min(x(A): x \in Q) = p(A)$ *for* $A \subseteq S$.

Proof. Because p and b play a symmetric role we prove only the first equality. Obviously, $\max(x(A): x \in Q) \leq b(A)$. If the maximum here is infinite, then $b(A) = \infty$ and we are done. So suppose that the maximum is finite and consider the following dual pair of linear programs.

$$\max(x(A): x(X) \geq p(X), x(X) \leq b(X) \quad \text{for } X \subseteq S) \tag{2.1}$$

$$\min\Bigg(\sum_{Y \subseteq S} y_Y b(Y) - \sum_{Z \subseteq S} z_Z p(Z): y, z \geq 0,$$

$$\sum_{Y \subseteq S} y_Y \chi_Y - \sum_{Z \subseteq S} z_Z \chi_Z = \chi_A\Bigg). \tag{2.2}$$

(To be more precise, if $p(X) = -\infty$ or if $b(X) = +\infty$ for some $X \subseteq S$, the corresponding primal constraint in (2.1) and dual variable in (2.2) is meant not to occur.)

Since the primal program has now a finite optimum, by the linear programming duality theorem, so does the dual. Making use of the well-known uncrossing technique (for an excellent survey, see Schrijver (1984a)) and the fact that (p, b) is a strong pair one can see that there is an optimal dual solution (y, z) such that both families $\{Y: y_Y > 0\}$ and $\{Z: z_Z > 0\}$ form a chain and $Y \cap Z = \emptyset$ whenever $y_Y > 0$ and $z_Z > 0$. Obviously, exactly one such dual solution exists, namely, $y_A = 1$ and all other dual variables are zero. Consequently, the common optimum of (2.1) and (2.2) is $b(A)$, as required. □

An important corollary of Proposition 2.3 is:

Proposition 2.4. *For a non-empty g-polymatroid Q the defining strong pair (p, b) is unique.* □

Specializing this result to polymatroids we have Theorem I.2.8. For g-polymatroids the generalization of Theorem I.2.9 is as follows

Proposition 2.5 (Frank (1984c)). *For a weak pair (p', b'), $Q = Q(p', b')$ is a g-polymatroid.*

Proof. Since the empty set is by definition a g-polymatroid we can assume that $Q \neq \emptyset$. Extend the ground set S by a new element s and define b_1'' on the subsets of $S' = S + s$ by letting

$$b_1''(X) = \begin{cases} b'(X) & \text{if } X \subseteq S, \\ -p(S-X) & \text{if } s \in X. \end{cases}$$

Then b_1'' is a crossing submodular function and Q is the projection of $B(b_1'')$ along s. Let b_1 be the bi-truncation of b_1'' and for any $X \subseteq S$ let $p(X) = -b_1(S' - X)$ and $b(X) = b_1(X)$. Then (p, b) is a strong pair and by the Bi-truncation Theorem we have $B(b_1'') = B(b_1)$ and hence $Q(p', b') = Q(p, b)$. □

Call the strong pair (p, b) constructed in the proof the *truncation* of (p', b'). Theorem 1.2 provides a characterization for $Q(p', b')$ to be non-empty.

Making use of the weak compliance of p' and b' this can be simplified:

Proposition 2.6. *For a weak pair (p', b') a g-polymatroid $Q = Q(p', b')$ is non-empty if and only if*

$$\text{(a) } \textstyle\sum b'(Z_i) \geq p'(\bigcup Z_i) \quad \textit{and} \quad \text{(b) } \sum p'(Z_i) \leq b'(\bigcup Z_i) \tag{2.3}$$

for every family of non-empty disjoint subsets $Z_1, \ldots, Z_t$ of S. If Q is non-empty and p', b' are integer-valued, then Q contains an integer point.

Proof. From Theorem 1.2 Q is empty if and only if there is a family $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ and a family $\mathcal{G} = \{G_1, G_2, \ldots, G_l\}$ such that

(a) both $\mathcal{F}$ and $\mathcal{G}$ consist of disjoint subsets,

(b) $\bigcup_{F \in \mathcal{F}} F = \bigcup_{G \in \mathcal{G}} G$, (2.4)

(c) $\sum_{F \in \mathcal{F}} b'(F) < \sum_{G \in \mathcal{G}} p'(G)$.

If there are two intersecting sets $F \in \mathcal{F}$ and $G \in \mathcal{G}$, then

$$b'(F) - p'(G) \geq b'(F-G) - p'(G-F)$$

and redefining

$$\mathcal{F} := (\mathcal{F} - \{F\}) \cup \{F - G\}, \quad \mathcal{G} := (\mathcal{G} - \{G\}) \cup \{G - F\}$$

statement (2.4) continues to hold. Furthermore, $\bigcup(F: F \in \mathcal{F})$ has become smaller (by $F \cap G$). Applying this uncrossing step as long as possible, finally we get an $\mathcal{F}$ and $\mathcal{G}$ satisfying (2.4) and no more intersecting sets $F \in \mathcal{F}$, $G \in \mathcal{G}$ exist.

For any maximal member X of $\mathcal{F} \cup \mathcal{G}$ the members $Z_1, Z_2, \ldots, Z_t$ of $\mathcal{F} \cup \mathcal{G}$ (properly) included in X form a partition of X. (If $X \in \mathcal{F}$, then $Z_i \in \mathcal{G}$, if $X \in \mathcal{G}$ then $Z_i \in \mathcal{G}$.) Since every set in $\mathcal{F} \cup \mathcal{G}$ is either maximal or belongs to the partition of exactly one maximal set of $\mathcal{F} \cup \mathcal{G}$, property c implies that for at least one maximal member X of $\mathcal{F} \cup \mathcal{G}$ the partition $\{Z_1, \ldots, Z_t\}$ of X violates (2.3). □

Remark. Observe that the proof is a simple polynomial-time algorithm provided that starting $\mathcal{F}$ and $\mathcal{G}$ satisfying (2.4) are available. In Section IV.4 we describe a method to construct such an $\mathcal{F}$ and $\mathcal{G}$.

One may wonder if crossing sub- and supermodular functions can define g-polymatroids. If p'' and b'' are crossing super- and submodular functions, respectively, which are compliant, then $Q(p'', b'')$ may have fractional vertices so it need not be a g-polymatroid in general. (For example, let $S = \{a, b, c\}$, $p'' \equiv -\infty$, $b''(X) = 1$ if $|X| = 2$ and $= \infty$ otherwise). However, we have

Proposition 2.7 (Frank (1984c) and Fujishige (1984b)). *If b'' is a crossing submodular function, then $B(b'')$ is a g-polymatroid (actually, a base polyhedron).*

Proof. Immediate from the Bi-truncation Theorem. □

Proposition 2.6 easily implies Fujishige's theorem (Theorem I.2.11). (The converse implication is equally simple.)

Proof of Theorem I.2.11. Let $s \in S$ and define p' on the subsets X of $S - s$ by $p'(X) = b''(S) - b''(S - X)$. Let $b'(X) = b''(X)$ for $X \subseteq S - s$. This definition implies that b'' is a crossing submodular function if and only if (p', b') is a weak pair. Moreover, Q is non empty if and only if $Q(p', b')$ is non-empty. From Proposition 2.6 the statement follows. □

Remark. Notice that if t is an arbitrary integer and b'' is a crossing submodular function, the polyhedron $\{x \in \mathbb{R}^S: x(S) = t, X(A) \leqslant b''(A) \text{ for } A \subseteq S\}$ is a g-polymatroid. Indeed, apply Proposition 2.7 to the crossing submodular function b_1'' where $b_1''(X) = b''(X)$ if $X \subset S$ and $b_1''(S) = t$.

Using Proposition 1.4 one can express the truncation (p, b) of a weak pair (p', b').

Proposition 2.8. $p(Y) = \max(\sum_i p'(Y_i) - \sum_j b'(X_{ij})$: $X_{ij} \subseteq Y_i$, $\{Y_i \setminus \bigcup_j X_{ij}: i = 1, 2, \ldots\}$ *is a partition of Y and for each i the sets X_{ij} $(j = 1, 2, \ldots)$ are disjoint*), $b(X) = \min(\sum_i b'(X_i) - \sum_j p'(Y_{ij})$: $Y_{ij} \subseteq X_i$, $\{X_i \setminus \bigcup_j Y_{ij}: i = 1, 2, \ldots\}$ *is a partition of X and for each i the sets Y_{ij} $(i = 1, 2, \ldots)$ are disjoint*).

Remark 2.9. In the special case when

$$p'(X) = \begin{cases} 0 & \text{if } |X| = 1, \\ -\infty & \text{otherwise,} \end{cases}$$

and b' is an arbitrary non-negative intersecting submodular function (in which case (p', b') is automatically a weak pair) the formula becomes simpler. Namely, $p \equiv 0$, $b(X) = \min(\sum b'(X_i)$: $X \subseteq \bigcup X_i$, the X_i's are disjoint). In this case $Q(p', b')$ is an ordinary polymatroid, b is its unique polymatroid function. If, in addition, the starting b' is fully submodular (but not necessarily monotone), the truncation of (p', b') is $(0, b)$ where $b(X) = \min(b'(Y), X \subseteq Y)$. This formula is well-known to make a submodular function b' monotone. The present approach tells us that monotonization can be considered as a special truncation.

We mention two special cases where the truncation formula in Proposition 2.8 is considerably simpler.

First, let $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ such that $p(S) \leqslant \alpha \leqslant \beta \leqslant b(S)$ and let (p, b) be a strong pair. Define

$$p_1'(X) = \begin{cases} p(X) & \text{if } X \neq S \\ \alpha & \text{if } X = S \end{cases} \quad \text{and} \quad b_1'(X) = \begin{cases} b(X) & \text{if } X \neq S \\ \beta & \text{if } X = S. \end{cases}$$

Then (p_1', b_1') is a weak pair.

Proposition 2.10. *The truncation (p_1, b_1) of the above (p_1', b_1') is given by*

$$p_1(X) = \max(p(X), \alpha - b(S - X)),$$
$$b_1(X) = \min(b(X), \beta - p(S - X)). \quad \square \tag{2.5}$$

Second, let $f: S \to \mathbb{R} \cup \{-\infty\}$ and $g: S \to \mathbb{R} \cup \{\infty\}$ be two vectors with $f \leqslant g$ and let (p, b) be a strong pair. Define

$$p_1'(X) = \begin{cases} p(X) & \text{if } |X| > 1, \\ \max(p(v), f(v)) & \text{if } X = \{v\}, \end{cases}$$

and

$$b_1'(X) = \begin{cases} b(X) & \text{if } |X| > 1, \\ \min(b(v), g(v)) & \text{if } X = \{v\}. \end{cases}$$

Then (p', b') is a weak pair. The next formula easily follows from either of Propositions 1.4 and 2.8.

Proposition 2.11. *The truncation (p_1, b_1) of the above (p_1', b_1') is*

$$p_1(X) = \max_Y (p(Y) + f(X - Y) - g(Y - X)),$$
$$b_1(X) = \min_Y (b(Y) + g(X - Y) - f(Y - X)). \tag{2.6}$$

The following can be proved with the help of the greedy algorithm (Fujishige and Tomizawa (1983), Hassin (1982)). (See also, Chapter IV.)

Proposition 2.12. *For a strong pair* (p, b) *the linear system* $\{x(A) \geq p(A), x(A) \leq b(A)$ *for every* $A \subseteq S\}$ *is TDI.*

Let (p, b) be the truncation of a weak pair (p', b') and let b_1 be the bi-truncation of a crossing submodular function b''_1. By Corollary 1.6 and the proof of Proposition 2.5 we have the following consequence of Proposition 2.12.

Corollary 2.13. *The linear systems* $\{x(A) \geq p'(A), x(A) \leq b'(A)$ *for every* $A \subseteq S\}$ *and* $\{x(S) = b''_1(S), x(A) \leq b''_1(A)$ *for every* $A \subseteq S\}$ *are TDI.*

Let us close this section by mentioning that a much deeper result, the intersection theorem, is also true for g-polymatroids. See, Section V.1.

Chapter III. CONSTRUCTIONS, CHARACTERIZATIONS, APPLICATIONS

1. Constructions and examples

In Chapter II we have seen that ordinary polymatroids, contra-polymatroids, base polyhedra, submodular polyhedra are special g-polymatroids. It was also mentioned that weak pairs and crossing submodular functions can define g-polymatroids, as well. In this section we show that the class of g-polymatroids is closed under various operations and several examples will also be mentioned. Throughout we suppose a g-polymatroid $Q = Q(p, b)$ is defined by a strong pair (p, b). All operations below when applied to an integral g-polymatroid result in an integral g-polymatroid provided that the parameters defining the operation are integer-valued.

1.1. Reflection. *Q is a g-polymatroid defined by the strong pair* $(-b, -p)$.

1.2. Translation. *For a vector* $v \in R^S$ *the translate* $Q + v$ *is a g-polymatroid defined by* (p_1, b_1) *where* $p_1(X) = p(X) + v(X)$, $b_1(X) = b(X) + v(X)$.

Notice that if Q is the base polyhedron of a matroid and $v = (1, 1, \ldots, 1) \in R^S$, then $Q' = -Q + v$ is the base polyhedron of the dual matroid.

1.3. Intersection with a plank. *Where* $\alpha, \beta \in R \cup \pm\{\infty\}$, $\alpha \leq \beta$, *the intersection* Q_1 *of* Q *and the "plank"* $P = \{x \in \mathbb{R}^S: \alpha \leq x(S) \leq \beta\}$ *is a g-polymatroid.* Q_1 *is non-empty if and only if* $\alpha \leq \beta$, $\alpha \leq b(S)$, $\beta \geq p(S)$. *If* Q_1 *is non-empty, its defining strong pair is given by* (II.2.5).

1.4. Intersection with a box. *Let* $f \in (\mathbb{R} \cup \{-\infty\})^S$, $g \in (\mathbb{R} \cup \{\infty\})^S$, $f \leq g$. *The intersection* Q_1 *of* $Q(p, b)$ *and a box* $B = \{x \in \mathbb{R}^S, f \leq x \leq g\}$ *is a g-polymatroid.* Q_1 *is non-empty if and only if* $f \leq g$, $f \leq b$, $p \leq g$. *Its defining strong pair* (p_1, b_1) *is given by* (II.2.6).

Proposition II.2.11 shows that a g-polymatroid $Q(p, b) \cap B$ is nonempty if and only if $f(X) \leq b(X)$ and $g(X) \geq p(X)$ for $X \subseteq S$. Hence the following corollary immediately follows.

Proposition 1.5. *Let* $B_1 = \{x \in \mathbb{R}^S, f \leq x\}$, $B_2 = \{x \in \mathbb{R}^S, g \geq x\}$, $f \leq g$, *and Q a g-polymatroid. Then* $Q \cap B_1 \cap B_2$ *is non-empty if and only if neither* $Q \cap B_1$ *nor* $Q \cap B_2$ *is empty.*

In Section III.3 we show some applications of this proposition.

Remark. A central algorithmic problem is the minimization of a fully submodular function b. As an application of the ellipsoid method Grötschel, Lovász and Schrijver (1981) gave a polynomial-time algorithm for this problem. Here we can derive a good characterization. Consider the intersection Q of $S(b)$ and $B_2=\{x\in\mathbb{R}^S: x\leqslant 0\}$. From (II.2.6) we see that $Q=S(b_1)$ where $b_1(X)=\min(b(Y): Y\subseteq X)$. By Proposition II.2.3 we have the following formula for the minimum:

$$\min(b(Y): Y\subseteq S)=\max(x(S): x\in S(b), x\leqslant 0).$$

Remark 1.6. A general matroid construction—due to J. Edmonds (1970)—can be viewed as a special case of this construction. Edmonds showed that, given a polymatroid function b, the family $\mathscr{F}=\{X: b(Y)\geqslant|Y| \text{ for } Y\subseteq X\}$ forms the family of independent sets of a matroid M. Now let the box B be defined by $f\equiv 0$ and $g(s)=1$ for $s\in S$. Then $Q_1=Q\cap B$ (the intersection of a polymatroid and the unit hypercube) is the matroid polyhedron M. The rank function b_1 of M comes from the above formula, namely, $b_1(X)=\min(b(Y)+|X-Y|: Y\subseteq X)$.

More generally, Edmonds showed that given an intersecting submodular (not necessarily monotone and finite) function $b'\geqslant 0$, family $\mathscr{F}=\{X: b'(Y)\geqslant|X\cap Y| \text{ for } Y\subseteq S\}$ is a family of independent sets of a matroid M. The rank-function b_1 of M is

$$b_1(X)=\min\left(\sum b'(X_i)+\left|X-\bigcup_1^k X_i\right|: X_1, X_2,\ldots,X_k\subseteq S\right).$$

In the present approach this formula immediately follows by considering the truncation of $(0, b')$. (See Remark II.2.9.)

Relying on bi-truncation a matroid construction was given by Frank and Tardos (1984a):

Proposition 1.7. *For a crossing submodular function b'' and for an integer k a family $\mathscr{B}=\{D: |D\cap X|\leqslant b''(X)$ for every $X\subseteq S$, $|D|=k\}$, if non-empty, forms the set of bases of a matroid.*

1.8. Projection. *For a subset T of S the projection $Q_T=\{x_T\in\mathbb{R}^T: (x_T, x_{S-T})\in Q$ for some $x_{S-T}\in\mathbb{R}^{S-T}\}$ of Q into $\mathbb{R}^T$ is a g-polymatroid defined by the strong pair (p_1, b_1) where $p_1=p\,|\,T$, $b_1=b\,|\,T$.*

Indeed, $Q(p_1, b_1)\supseteq Q_T$ obviously holds. The reversed containment immediately follows from the proof of Proposition II.2.2. □

Let s be a new element outside S and let $S_1=S+s$. The following proposition will be useful. Its proof is straightforward.

Proposition 1.9 (Frank (1984c) and Fujishige (1984b)). *There is a one-to-one correspondence between strong pairs (p, b) (with $-p, b: 2^S\to\mathbb{R}\cup\{\infty\}$) and fully submodular functions $b_1: 2^{S_1}\to\mathbb{R}\cup\{\infty\}$ with $b_1(S_1)=0$, namely*

$$b_1(X)=b(X) \text{ if } X\subseteq S \quad\text{and}\quad b_1(X)=-p(S-X) \text{ if } s\in X\subseteq S+s. \tag{1.1}$$

Moreover, any g-polymatroid $Q(p, b)$ is the projection (along s) of a 0-base polyhedron $B(b_1)$. □

While proving statements for g-polymatroids Proposition 1.9 will often make it possible to restrict ourselves to 0-base polyhedra.

Proposition 1.10. *A g-polymatroid $Q=Q(p, b)$ is a base polyhedron if and only if $p(S)=b(S)$.*

Proof. By Proposition II.2.3 if Q is a base polyhedron, then $p(S)=b(S)$. Conversely, let $p(S)=b(S)$. We claim that $Q=B(b)$. Indeed, obviously $Q\subseteq B(b)$. On the other hand for $x\in B(b)$ and $A\subseteq S$ we have $x(A)\leqslant b(A)$ and $x(A)=b(S)-x(S-A)\geqslant b(S)-b(S-A)\geqslant p(A)$. Hence $x\in Q$, so $Q=B(b)$. □

Propositions 1.3 and 1.10 immediately imply:

Proposition 1.11. *Let us be given a strong pair (p, b) and a constant k for which $p(S)\leqslant k\leqslant b(S)$. Then $Q(p, b)\cap\{x\in\mathbb{R}^S: x(S)=k\}$ is a base polyhedron.* □

1.12. Face. *Every face of a g-polymatroid is a g-polymatroid.*

Proof. Let (p, b) and b_1 be given as in (1.1). A face of $Q(p, b)$ is the projection of a face of $B(b_1)$. Thus it suffices to prove that a face of a 0-base polyhedron, is a 0-base polyhedron. Let $b: 2^S\to\mathbb{R}\cup\{\infty\}$ be a fully submodular function with $b(S)=0$. It is enough to prove for a fixed $T\subset S$ that a face $Q_T=\{x\in\mathbb{R}^S; x\in B(b), x(T)=b(T)\}$ is a 0-base polyhedron. (By Proposition II.2.3 Q_T is non-empty.) Define

$$b_T(X)=b(X\cap T)+b(X\cup T)-b(T). \tag{1.2}$$

It is easily seen that b_T is fully submodular and $b_T(S)=0$.

Claim. $Q_T=B(b_T)$.

Proof. Let $x\in Q_T$. Then

$$x(X)=x(X\cap T)+x(X\cup T)-x(T)$$
$$\leqslant b(X\cap T)+b(X\cup T)-b(T)=b_T(X),$$

hence $Q_T\subseteq B(b_T)$. To see the other direction let $x\in B(b_T)$. Since $x(S)=0$ we have $b(T)=b_T(T)\geqslant x(T)=-x(S-T)\geqslant -b_T(S-T)=b(T)$, i.e., $x(T)=b(T)$.

Furthermore, for $X \subseteq S$,

$$x(X) \leq b_T(X) = b(X \cap T) + b(X \cup T) - b(T) \leq b(X).$$

Consequently, $x \in Q_T$ and hence $Q_T = B(b_T)$. □

Let b be a fully submodular function with $b(S) = 0$. Any face Q_1 of $B(b)$ is defined by a family $\mathcal{T}$ of subsets by $Q_1 = \{x : x \in B(b), x(T) = b(T) \text{ for } T \in \mathcal{T}\}$. Let $\pi := \sum(\chi_T : T \in \mathcal{T})$ and $b_1(X) := \hat{b}(\pi + \chi_X) - \hat{b}(\pi)$. In Chapter I we showed that b_1 is fully submodular. Repeated applications of formula (1.2) show

Proposition 1.13. *Q_1 is non-empty if and only if $\hat{b}(\pi) = \sum(b(X) : X \in \mathcal{T})$. If Q_1 is non-empty, $Q_1 = B(b_1)$.* □

Remark 1.14. For a matroid $M = (S, r)$ the matroid polyhedron of a deletion $M \backslash (S - T)$ is the projection of the matroid polyhedron Q of M into $\mathbb{R}^T$. The matroid polyhedron of a contraction $M \div (S - T)$ comes by projecting the face Q defined by $x(S - T) = r(S - T)$ into $\mathbb{R}^T$.

1.15. Homomorphic image. *Where $\gamma : S - S'$ is a surjective mapping, the homomorphic image $\gamma(Q)$ of the g-polymatroid Q is a g-polymatroid. The strong pair (p_1, b_1) defining $\gamma(Q)$ is $p_1(X) = p(\gamma^{-1}(X))$, $b_1(X) = b(\gamma^{-1}(X))$. Furthermore, for an integral vector $y \in \gamma(Q)$ there is an integral vector $x \in Q$ for which $y = \gamma(x)$ if Q is integral.*

Proof. Obviously (p_1, b_1) is a strong pair and $Q(p_1, b_1) \supseteq \gamma(Q)$. It suffices to prove the reverse containment for the special case when $S' = S - \{u, v\} + w$ $(u, v \in S, w \notin S)$ and

$$\gamma(s) = \begin{cases} s & \text{if } s \in S - \{u, v\}, \\ w & \text{if } s \in \{u, v\}. \end{cases}$$

Let $x_1 \in Q(p_1, b_1)$. Define $p_1' : S \to \mathbb{R} \cup \{-\infty\}$ and $b_1' : S \to \mathbb{R} \cup \{\infty\}$ as follows:

$$p_1'(X) = \begin{cases} x_1(S') & \text{if } X = S, \\ x_1(s) & \text{if } X = \{s\},\ s \in S - \{u, v\}, \\ p(X) & \text{otherwise;} \end{cases}$$

$$b_1'(X) = \begin{cases} x_1(S') & \text{if } X = S, \\ x_1(s) & \text{if } X = \{s\},\ s \in S - \{u, v\}, \\ b(X) & \text{otherwise.} \end{cases}$$

Then (p_1', b_1') is a weak pair. Applying Proposition II.2.6 we see that $Q(p_1', b_1')$ is non-empty so it has an integer point x. Since $x \in Q$ and $\gamma(x) = x_1$ we are done. □

1.16. Inverse homomorphic image. *Let $\gamma : S \to S'$ be the same as before and Q_1 a g-polymatroid on S' whose defining strong pair is (p_1, b_1). Then $\gamma^{-1}(Q_1)$ is a g-polymatroid. Its defining strong pair (p, b) is as follows. $p(A) = p_1(A')$ if $A = \gamma^{-1}(A')$ for some $A' \subseteq S'$ and $= -\infty$ otherwise. $b(A) = b_1(A')$ if $A = \gamma^{-1}(A')$ for some $A' \subseteq S'$ and $= \infty$ otherwise.* □

The next result immediately follows from results of Edmonds (1970) and was explicitly stated by Lovász (1977).

Proposition 1.17. *Any integral polymatroid P is the homomorphic image of a matroid polyhedron.*

Proof. To see this suppose $P = \{x \in \mathbb{R}_+^{S'}, x(A) \leq b(A) \text{ for } A \subseteq S'\}$. Set $S = \{v_i : v \in S', i = 1, 2, \ldots, b(v)\}$ and define γ by $\gamma(v_i) = v$. By the above construction $\gamma^{-1}(P)$ is a g-polymatroid. Denoting the unit cube by B, one can easily see that $B \cap \gamma^{-1}(P)$ is a matroid polyhedron and $P = \gamma(B \cap \gamma^{-1}(P))$. □

In Section III.2 this result will easily be extended to g-polymatroids.

1.18. Direct sum. *Let S_1 and S_2 be two disjoint sets and let (p_i, b_i) be a strong pair on S_i $(i = 1, 2)$. The "direct sum" $Q_1 \oplus Q_2$ of $Q_1 = Q(p_1, b_1)$ and $Q_2 = Q(p_2, b_2)$ defined by*

$$Q_1 \oplus Q_2 = \{x = (x_1, x_2) \in \mathbb{R}^{S_1 \cup S_2} : x_1 \in Q_1, x_2 \in Q_2\}$$

is a g-polymatroid. Its defining strong pair (p, b) is $p(A) = p_1(A \cap S_1) + p_2(A \cap S_2)$ and $b(A) = b_1(A \cap S_1) + b_2(A \cap S_2)$. □

1.19. Sum. *Let $Q_i = Q(p_i, b_i)$ be g-polymatroids on S defined by the strong pair (p_i, b_i) $(i = 1, 2)$. The "sum" of Q_1 and Q_2 defined by $Q_1 + Q_2 := \{x : x = x_1 + x_2 \text{ for some } x_1 \in Q_1, x_2 \in Q_2\}$ is a g-polymatroid. Its defining strong pair is $(p_1 + p_2, b_1 + b_2)$. Furthermore, for any integral vector $q \in Q_1 + Q_2$ there are integral vectors $q_1 \in Q_1$, $q_2 \in Q_2$ so that $q = q_1 + q_2$ provided that Q_1, Q_2 are integral.*

Proof. Let S_1 and S_2 be two disjoint copies of S and let γ be a map of $S_1 \cup S_2$ onto S defined by $\gamma(v_1) = \gamma(v_2) = v$ for $v \in S$. Then $Q_1 + Q_2 = \gamma(Q_1 \oplus Q_2)$ and the statement follows from the properties of the homomorphic image. □

1.19 was proved by Giles (1975) for polymatroids.

The sum of more than two g-polymatroids can be defined analogously.

Remark 1.20. It is known (Edmonds (1965a), Nash-Williams (1967)) that, given k matroids $M_1, M_2, \ldots, M_k$, the family $\mathcal{F} = \{X : X = F_1 \cup F_2 \cup \cdots \cup F_k, F_i \in M_i\}$ forms a family of independent sets of the matroid M called the *sum* of $M_1, \ldots, M_k$. By the preceding construction and Remark 1.6 the matroid polyhedron $Q(M)$ of

M is $B \cap \sum Q(M_i)$ (B is the unit cube) and its rank function is $r(X) = \min(\sum r_i(Y) + |X - Y|: Y \subseteq X)$.

1.21. Cone g-polymatroids. *A g-polymatroid Q defined by the strong pair (p, b) forms a cone if and only if $p(A)$, $b(A) \in \{-\infty, 0, +\infty\}$ for $A \subseteq S$.*

Proof. Obviously such a pair defines a cone. Conversely, $b(A) = \max(x(A): x \in Q) \geq 0$ since $0 \in Q$. If $b(A) > 0$ for some $A \subseteq S$, then $x(A) > 0$ for some $x \in Q$ and then $\lambda x \in Q$ for $\lambda > 0$. Consequently $b(A) = +\infty$. That $p(A) \in \{0, -\infty\}$ can be seen similarly. □

1.22. Dominant. *For a g-polymatroid $Q = Q(p, b)$ defined by a strong pair (p, b) the dominant $Q + \mathbb{R}_+^S$ of Q is a g-polymatroid. Its defining strong pair is (p, b_1) where $b_1 \equiv \infty$.*

Indeed, $\mathbb{R}_+^S$ is a g-polymatroid so 1.19 can be applied to $\mathbb{R}_+^S$ and Q. □

2. Characterizations

In this section we are going to provide two kinds of characterizations of g-polymatroids. The first one extends Proposition III.1.17. The second characterization extends those for polymatroids given by Edmonds (1970).

Proposition 2.1. *Every g-polymatroid Q is the sum of a bounded g-polymatroid and a cone g-polymatroid.*

Proof. It suffices to prove this statement for base polyhedra $B(b)$. The following lemma is easy to prove.

Lemma 2.2. *For every fully submodular function b with $b(S) < \infty$ there is a finite fully submodular function b_1 such that $b(X) = b_1(X)$ whenever $b(X) < \infty$, namely*

$$b_1(X) = \min(b(Y) + 2M|Y - X|: Y \supseteq X)$$

where $M = \max(|b(X)|: X \subseteq S, b(X) < \infty)$. □

Define b_2 by

$$b_2(X) = \begin{cases} 0 & \text{if } b(X) < \infty, \\ \infty & \text{if } b(X) = \infty. \end{cases}$$

Now b_2 is fully submodular, $B(b_2)$ is a cone, $B(b_1)$ is bounded and $B(b) = B(b_1) + B(b_2)$. □

Remark. It is well-known that a pointed polyhedron P (i.e., a polyhedron which does not include a straight line) is the sum of the convex hull of the vertices of P and a cone. S. Fujishige noticed that the convex hull of the vertices of a pointed g-polymatroid is not a g-polymatroid in general. His example: $S = \{1, 2\}$, $b(1) = 1$, $b(2) = \infty$, $b(S) = 2$, $p(1) = -\infty$, $p(2) = p(S) = 0$.

Proposition 2.3. *A bounded base polyhedron $B(b)$ is the translate of the base polyhedron of a polymatroid function.*

Proof. Let b be finite and fully submodular. Let $M = 2\max(|b(A)|, A \subseteq S)$ and denote v a vector each component of which is M. It is easy to see that $b_1(X) = b(X) + M|X|$ is a polymatroid function and $B(b) = B(b_1) - v$. □

Summing up these propositions we have

Theorem 2.4. *Every g-polymatroid is the sum of a bounded g-polymatroid and a cone g-polymatroid. A bounded integral g-polymatroid can be obtained from a matroid base polyhedron by taking a homomorphic image, a translation, and a projection.*

Edmonds' fundamental theorem on characterizing polymatroids is as follows.

Proposition 2.5 (Edmonds (1970)). *The following are equivalent:*
(a) *P is a (not-necessarily integral) polymatroid*
(b) *P is a compact non-empty subset of $\mathbb{R}_+^S$ such that*
(i) *for every $z \in \mathbb{R}_+^S$, $y \in P$ with $y \leq z$ the maximum of $(x(S): x \in P, y \leq x \leq z)$ is independent of the choice of y (that is for every maximal vector y in P below z the value $y(S)$ is the same).*
(ii) *$0 \leq x \leq y \in P$ implies $x \in P$.*

(Actually, Edmonds used property (b) to define polymatroids.) Notice that in b, convexity is not assumed.

For a vector $x \in \mathbb{R}^S$ and a subset $A \subseteq S$ let $x|_A$ denote a vector $y \in \mathbb{R}^A$ for which $y(s) = x(s)$ for every $s \in A$.

The corresponding characterization for g-polymatroids is:

Proposition 2.6. *The following are equivalent:*
(a) *Q is a (not-necessarily integral) g-polymatroid*
(b) *Q is a closed subset of $\mathbb{R}^S$ such that*
(i) *for every $z \in (\mathbb{R} \cup \{\pm\infty\})^S$, $A \subseteq S$ and $y \in Q$ for which $y|_A \leq z|_A$ and $y|_{S-A} \geq z|_{S-A}$ the maximum of*

$$(x(A): x \in Q, y|_A \leq x|_A \leq z|_A \text{ and } y|_{S-A} \geq x|_{S-A} \geq z|_{S-A})$$

is independent of the choice of y, and

(ii) *property* (i) *holds when* $-Q$ *is substituted for* Q.

The proof of this statement goes along a similar line to that of Proposition 2.5 (See Welsh (1976) and Giles (1975)) so we do not include it.

One may wonder whether or not there is an analogous characterization for the set of integral points of an integral polymatroid or more generally, of an integral g-polymatroid.

Proposition 2.7. *The following are equivalent:*

(a) Q *is the set of integral points of an integral g-polymatroid,*

(b) $Q \subseteq \mathbb{Z}^S$ *is such that* (i) *and* (ii) *in Proposition* 2.6 *hold when* $z \in (\mathbb{Z} \cup \{\pm\infty\})^S$ *(rather than* $z \in (\mathbb{R} \cup \{\pm\infty\})^S$.

To prove $a \Rightarrow b$ is easy. The other direction can be proved similarly to that in Proposition 2.6. One (small) difficulty to be overcome comes from the fact that in the proof of Propositions 2.5 and 2.6 a certain ε-increasing is used ($\varepsilon < 1$) (see Welsh (1976) and Giles (1975)) while here we have to work with integral vectors. □

There may be examples where the integral vectors of a (suspected) polymatroid are defined in a special way and one has to prove that the given structure is indeed an integral g-polymatroid. To do that Proposition 2.7 may be advantageous. Such a situation will be mentioned in the next section (Proposition 3.8).

3. Applications

In this section we exhibit several results in combinatorial optimization that are related to g-polymatroids.

Orientations

Let $G = (V, E)$ be an undirected graph and $m: V \to Z$ an integer-valued function.

Lemma 3.1. *There is an orientation of the edges of G such that* $\rho(v) = m(v)$ *for every* $v \in V$ *if and only if*

$$m(X) \geqslant |E(X)| \quad \textit{for every } X \subseteq V \quad \textit{and} \qquad m(V) = |E|. \tag{3.1}$$

(Here ρ denotes the in-degree function of the orientation and $E(X)$ denotes the set of edges induced by X.) This lemma appeared in Frank and Gyárfás (1978) and it is an easy consequence of Hall's theorem. We remark that $|E(X)|$ is fully supermodular and the vectors m satisfying (3.1) form a base polyhedron.

We derive the following classical result of Nash-Williams:

Theorem 3.2 (Nash-Williams (1969)). *An undirected graph has an h-strongly edge-connected orientation if and only if every cut contains at least 2h edges.*

(A digraph is *h-strongly edge-connected* if there are at least h directed edge entering any non-empty proper subset of nodes.) (Actually, Nash-Williams proved his theorem in a stronger form. We were not able to derive that version.)

Proof. The necessity is straightforward. To see the sufficiency let $p''(X) = h + |E(X)|$ if $0 \subset X \subset V$ and $p''(\emptyset) = 0$, $p''(V) = |E|$. Then p'' is an (integer-valued) crossing supermodular function. Let $Q = \{m \in \mathbb{R}^V : m(X) \geqslant p''(X)$ for every $X \subseteq V$, $m(V) = p''(V)\}$ be a base polyhedron. Q is nonempty since (the possibly fractional) vector $d/2$ is in Q by the hypothesis ($d(v)$, $v \in V$, is the degree of v in G). Indeed,

$$\sum\left(\frac{d(v)}{2} : v \in X\right) = \tfrac{1}{2}d(X, \bar{X}) + |E(X)| \geqslant h + |E(X)|$$

and

$$\sum\left(\frac{d(v)}{2} : v \in V\right) = |E|.$$

By Proposition II.2.2, Q contains an integral point m. This m satisfies (3.1) so by Lemma 3.1 there is an orientation for which $\rho(v) = m(v)$ for $v \in V$. This orientation is h-strongly edge-connnected since $\rho(X) = \sum(\rho(v) : v \in X) - |E(X)| = m(X) - |E(X)| \geqslant h$ for every $\emptyset \subset X \subset V$. □

The following more general orientation problem was investigated by Frank (1980). Let $h: 2^V \to Z_+ \cup \{\infty\}$ be a non-negative, integer-valued function (with $h(\emptyset) = h(V) = 0$) which is "crossing G-supermodular", that is, $h(X) + h(Y) \leqslant h(X \cap Y) + h(X \cup Y) + d(X, Y)$ whenever $X, Y \subseteq V$ are crossing sets. ($d(X, Y)$ denotes the number of edges between $X - Y$ and $Y - X$.)

Theorem 3.3. *There exists an orientation of the edges of G for which*

$$\rho(X) \geqslant h(X) \quad \textit{for every } X \subseteq V \tag{3.2}$$

if and only if

$$\text{(a)} \quad e_{\mathcal{P}} \geqslant \sum h(V_i), \qquad \text{(b)} \quad e_{\mathcal{P}} \geqslant \sum h(\bar{V}_i) \tag{3.3}$$

for every partition $\mathcal{P} = \{V_1, V_2, \ldots, V_n\}$ *of V where* $e_{\mathcal{P}}$ *denotes the number of edges connecting distinct* V_i*'s.*

(Theorem 3.2 is indeed a special case of Theorem 3.3 since if in this latter theorem function h, in addition, is symmetric, that is $h(X) = h(\bar{X})$ for $X \subseteq V$, then it suffices to require (3.3) only for $n=2$. That is, the required orientation exists if and only if $2d(X) \geq h(X)$ for every $X \subseteq V$).

Proof. The necessity is straightforward. To see the sufficiency let $p''(X) = h(X) + |E(X)|$. Since $|E(X)| + |E(Y)| = |E(X \cap Y)| + |E(X \cup Y)| - d(X, Y)$ and h is crossing G-supermodular, $p''(X)$ is crossing supermodular. By (3.3) for a partition $\{V_1, \ldots, V_n\}$ of V we have

$$\sum p''(V_i) = \sum h(V_i) + |E| - e_{\mathscr{P}} \leq |E| = p''(V)$$

and

$$\sum p''(\bar{V}_i) = \sum h(\bar{V}_i) + (n-1)|E| - e_{\mathscr{P}} \leq (n-1)|E| = (n-1)p''(V).$$

Applying Theorem I.2.11 to $b'' = -p''$ we obtain an integral vector $m: V \to Z$ for which

$$m(X) \geq p''(X) \text{ for every } X \subseteq V \quad \text{and} \quad m(V) = |E|. \tag{3.4}$$

Since $h \geq 0$, m satisfies (3.1) and therefore there is an orientation of G for which $\rho(v) = m(v)$ for every $v \in V$. This orientation satisfies (3.2) since $\rho(X) = \sum(\rho(v): v \in X) - |E(X)| = m(X) - |E(X)| \geq h(X)$. □

The vectors m satisfying (3.4) form a base polyhedron Q. Since the proof of Lemma 3.1 is algorithmic in Frank and Gyárfás (1978) in order to construct the required orientation it suffices to find an integral vector of Q. In Chapter IV we present a version of the greedy algorithm which either finds such a point or finds a partition violating (I.2.4) (providing this way a new constructive proof of Theorem I.2.11). This algorithm will need a certain oracle to minimize $b''(X) - x(X)$ for certain (fixed) vectors x. In special cases such as Theorem 3.2 above and 3.4 below this oracle can be built up from a max flow min cut algorithm.

We note that in Frank (1984a, 1984b) this orientation model was derived from the submodular flow theory. The present approach, including the algorithm given in Chapter IV, is better since it relies on g-polymatroids, a simpler structure than submodular flows. On the other hand, the problem of finding a h-strongly edge-connected orientation of a mixed graph, which was solved also by means of submodular flows by Frank (1984b), does not seem to be reducible to g-polymatroids.

From Theorem 3.3 one can derive (see Frank (1980)) a necessary and sufficient condition for the existence of a h-strong orientation which satisfies $f(v) \leq \rho(v) \leq g(v)$ for every $v \in V$ where f and g are given vectors in $\mathbb{Z}^V$ with $f \leq g$. We do not repeat here this result but only mention the following version of it.

Corollary 3.4. *A graph G has a h-strongly edge-connected orientation for which $f(v) \leq \rho(v) \leq g(v)$ for $v \in V$ if and only if $f \leq g$ and G has a h-strongly-connected orientation satisfying $f(v) \leq \rho(v)$ for every $v \in V$ and there is a h-strongly edge-connected orientation satisfying $\rho(v) \leq g(v)$ for every $v \in V$.*

This theorem immediately follows from Proposition 1.5 when it is applied to Q.

Edge-disjoint arborescences

The next application concerns the problem of packing arborescences. Let $G = (V, E)$ be a directed graph, k a positive integer and $m: V \to Z_+$ a non-negative integer vector for which $m(V) = k$. We rely on the following theorem of Edmonds.

Theorem 3.5 (Edmonds (1973)). *In $G = (V, E)$ there are k edge-disjoint arborescences, exactly $m(v)$ of which are rooted at v for every $v \in V$, if and only if $m(V) = k$ and*

$$\rho(X) \geq k - m(X) \quad \text{for } \emptyset \neq X \subseteq V. \tag{3.5}$$

We call a vector m in (3.5) a *root vector* (of k arborescences). Since ρ is fully submodular the function p' defined by $p'(X) = k - \rho(X)$ for $\emptyset \neq X \subseteq V$ and $p'(\emptyset) = 0$ is intersecting supermodular. Thus the root vectors are precisely the integer points of the g-polymatroid $Q = \{x \in \mathbb{R}^V: x(A) \geq p'(A) \text{ for } A \subseteq V, x \geq 0, x(V) = k\}$.

Let f and g be non-negative integral vectors in $\mathbb{R}^V$ with $f \leq g$ and let $B_1 = \{x \in \mathbb{R}^V: f \leq x\}$, $B_2 = \{x \in \mathbb{R}^V: x \leq g\}$. Both $B_1 \cap Q$ and $B_2 \cap Q$ are g-polymatroids (see III.1.4). By Proposition II.2.6 $B_1 \cap Q$ is non-empty iff $\sum_{i=1}^t p'(V_i) + f(V_0) \geq k$ for every partition $\{V_0, V_1, \ldots, V_t\}$ of V.

Similarly, $B_2 \cap Q$ is non-empty if $\sum_{i=1}^t p'(V_i) \geq k$ for every collection $\{V_1, \ldots, V_t\}$ of pairwise disjoint subsets of V and $p'(X) \geq g(X)$ for every $X \subseteq V$. Finally, by Proposition 1.5 $B_1 \cap B_2 \cap Q$ is non-empty if and only if neither $B_2 \cap Q$ nor $B_2 \cap Q$ is empty.

From these observations one can obtain the following theorem:

Theorem 3.6. *In $G = (V, E)$ there are k edge-disjoint arborescences such that:*

a. (Cai Mao-Cheng (1983)). *At least $f(v)$ of them are rooted at v for every $v \in V$ if and only if*

$$\sum_{i=1}^{t} \rho(V_i) - k(t-1) \geq f(V_0) \tag{3.6}$$

for every partition $\{V_0, V_1, \ldots, V_t\}$ of V (where only V_0 may be empty),

b. (Frank (1981b)). *At most $g(v)$ of them are rooted at v for every $v \in V$ if and only if*

$$\rho(X) + g(X) \geq k \quad \text{and} \quad \sum \rho(V_i) \geq k \cdot (t-1) \tag{3.7}$$

for every family $\{V_1, V_2, \ldots, V_t\}$ of pairwise disjoint nonempty subsets,

c. *At least $f(v)$ and at most $g(v)$ of them are rooted at v for every $v \in V$ if and only if (3.6) and (3.7) hold.* □

We remark that Cai Mao-Cheng (1983) found another characterization for c.

Since the 0–1 vectors of a base polyhedron form the characteristic vectors of bases of a matroid and the root vectors are the integral points of a base polyhedron we have:

Corollary 3.7. *Given a digraph $G=(V,E)$, the set $B=\{X\subset V: |X|=k$, there are k edge-disjoint arborescences with distinct roots from $X\}$, if non-empty, is the family of bases of a matroid.*

From an algorithmical point of view the above argument reduces the problem to deciding whether a g-polymatroid defined by a weak pair is empty or not. In Chapter IV we are going to show how the greedy algorithm can be used for this purpose. (On the other hand each of the methods in Cai Mao-Cheng (1983) and Frank (1978, 1976), is algorithmic).

The phenomenon which appeared in Corollary 3.4 and Theorem 3.6c has been known for a long time. An old result of this type is due to Ford and Fulkerson (1962): Let $G=(V,E)$ be a digraph, f_1, g_1, f_2, g_2 functions on V. There is a subgraph of G for which (a) $f_1(v)\leq \rho(v)$, $\delta(v)\leq g_1(v)$ and (b) $\rho(v)\leq g_2(v)$, $\delta(v)\geq f_2(v)$ if and only if there is one satisfying (a) and one satisfying (b). The reader will easily show this theorem to be a consequence of Proposition 1.5. So is the following result: Where M is a matroid on S and $\{S_1, S_2, \ldots, S_t\}$ is a fixed partition of S there is a basis B of M for which (a) $f_i\leq |B\cap S_i|$ and (b) $|B\cap S_i|\leq g_i$ $(i=1,2,\ldots,t)$ if and only if there is one satisfying (a) and one satisfying (b). (Here $f_i\leq g_i$ $(i=1,2,\ldots,t)$ are integers.) To see this apply Proposition 1.5 to the g-polymatroid which arises from the matroid basis polyhedron by applying homomorphic image defined by partition $\{S_1,\ldots,S_t\}$.

Matroid reinforcement

In Remark 1.20 we derived a known formula for the rank function of the sum of matroids. From this it follows that a matroid M has k disjoint bases if and only if $k\cdot t(X)\leq |X|$ where $t(X)$ is the co-rank function (i.e., $t(X)=r(S)-r(S-X)$) due to Edmonds (1965b).

Let us consider the following optimization problem. Suppose there are no k disjoint bases in M and we want to adjoin parallel elements in order for M to have k disjoint bases. What is the minimum cardinality (or more generally, the minimum cost) of the required new elements? This *matroid reinforcement* problem was introduced and solved for graphic matroids by W. Cunningham (1985). To describe a solution z let $z(s)$ denote the number of new elements parallel to s to be adjoined to S $(s\in S)$. We call z *feasible* if the enlarged matroid possesses k disjoint bases. z is a feasible solution if and only if $z(A)\geq k\cdot t(A)-|A|$. Thus feasible vectors are precisely the integral points of the g-polymatroid $Q=\{x\in\mathbb{R}^S,\ x\geq 0,\ x(A)\geq k\cdot t(A)-|A|\}$.

Consequently, the greedy algorithm for g-polymatroids (Chapter IV) provides a solution to the problem.

G-matroids

We briefly summarize some applications taken from a recent paper of Tardos (1985).

Parallel to the relation between matroids and polymatroids the notion of g-matroids was introduced by Tardos (1985). We say the set $\mathcal{I}$ of integer vectors of a g-polymatroid Q in the 0–1 unit cube is a *g-matroid*. A simple example for g-matroids is the following. Let $S_1, S_2, \ldots, S_k$ be a partition of S and $f_i\leq g_i$ non-negative integers $(i=1,2,\ldots,k)$. Then $\mathcal{F}=\{X: f_i\leq |X\cap S_i|\leq g_i\}$ is a g-matroid.

If (p,b) is the strong pair defining Q, then there are two matroids M, M' on S with rank functions r, r', respectively, such that $b=r$, $p(X)=r'(S)-r'(S-X)$ and M' is the strong map of M (for a definition see Welsh (1976)). Moreover, the integer points of Q correspond to a set system $\mathcal{I}=\{X\subseteq S$, X is independent in M and a spanning set of $M'\}$. Here is a relation between g-matroids and Higgs' (1968) theorem on factorization of strong maps, see also in Welsh (1976).

(A *strong map* (induced by the identity function) is an ordered pair (M', M) of matroids on the same ground set S such that $r(X)-r(Y)\geq r'(X)-r'(Y)$ whenever $Y\subseteq X\subseteq S$. It is *elementary* if $r(M)-r'(M)=1$.)

Proposition 3.8 (Tardos (1985)). *For a g-matroid $\mathcal{I}$ and an integer k, $r'(S)\leq k\leq r(S)$, the family $\mathcal{I}_k=\{X\in\mathcal{I}: |X|=k\}$ is the collection of bases of a matroid M_k. Moreover M_k is a strong map of M_{k+1} $(r'(S)\leq k<r(S))$ and these matroids yield a factorization of the strong map (M', M) through elementary strong maps.*

The second application from Tardos (1985) concerns supermodular colourings introduced by Schrijver (1985). Let $p': 2^S\to Z\cup\{-\infty\}$ be an intersecting supermodular function for which $p'(X)\leq |X|$ $(X\subseteq S)$ and $p'(X)\leq k$.

A partition $\{S_1, S_2, \ldots, S_k\}$ of S is called a good colouring of S if every subset $X\subseteq S$ meets at least $p'(X)$ colour classes.

Proposition 3.9 (Tardos (1985)). *For every j, $1\leq j\leq k$, the family $\mathcal{I}_j=\{X, X=S_1\cup S_2\cup\cdots\cup S_j$ where $\{S_1, S_2, \ldots, S_j, \ldots, S_k\}$ is a good colouring$\}$ is a non-empty g-matroid.*

This proposition can be proved by showing that there exists one good colouring and then using the characterization given for the set of integral points of a g-polymatroid (Proposition 2.7). (Another, more direct proof was provided by Tardos (1985).) Proposition 3.9 will be used to prove Schrijver's supermodular colouring theorem. See Section V.

Matchable subsets

Balas and Pulleyblank (1983) described the convex hull Q of perfectly matchable subsets of nodes of a bipartite graph $G=(A,B;E)$. Define $Q_1 := \{x\in\mathbb{R}^{A\cup B}: 0\leq x\leq$

$1, x(A)=x(B), x(X)\leq x(\Gamma(X))$ for $X\subseteq A\}$ where $\Gamma(X)=:\{v\in B: uv\in E$ for some $u\in X\}$.

Theorem 3.10 (Balas and Pulleyblank). $Q=Q_1$.

Proof. Since obviously $Q\subseteq Q_1$, it suffices to show that the vertices of Q_1 are integer-valued. (An integer-valued vertex x of Q is 0–1 valued and the set $X:=\{v\in A\cup B: x(v)=1\}$ is perfectly matchable by Hall theorem). Let Q_2 be the reflection of Q_1 through $\mathbb{R}^B$, that is $Q_2:=\{(x_A, x_B): x_A\in\mathbb{R}^A, x_B\in\mathbb{R}^B, (x_A, -x_B)\in Q_1\}$. Now we show that Q_2 is an (integral) g-polymatroid from which the integrality of Q_1 follows. Let us consider G as a directed graph with every edge directed from A to B. Obviously, $\mathcal{F}=\{X\subseteq A\cup B$, no directed edge leaves $X\}$ is a ring family and $Q_3:=\{x\in\mathbb{R}^{A\cup B}: x(A\cup B)=0,\ x(X)\leq 0$ for $X\in\mathcal{F}\}$ is a g-polymatroid. Now Q_2 is the intersection of Q_3 and the box

$$\{x: -1\leq x(v)\leq 0 \text{ for } v\in B \text{ and } 0\leq x(v)\leq 1 \text{ for } v\in A\}. \qquad \square$$

Arrangement polyhedron

Let $a_1>a_2>\cdots>a_n>0$ be n numbers. By an *m-arrangement* $(m\leq n)$ we mean a vector of m dimension whose components are distinct numbers among $a_1, a_2, \ldots, a_n$.

Theorem 3.11 (Yemelichev-Kovalev-Kratsov (1984)). *The convex hull Q of m-arrangements is described by $\{x\in\mathbb{R}^m: p(A)\leq x(A)\leq b(A)$ for $A\subseteq\{1,2,\ldots,n\}, |A|\leq m\}$ where $p(A)=a_n+a_{n-1}+\cdots+a_{n-|A|+1}$ and $b(A)=a_1+a_2+\cdots+a_{|A|}$.* $\square$

Observe that Q is a generalized polymatroid. The above result was proved for $m=n$ by Edmonds and Giles (1977) and Balas (1975).

Alternating vectors

Let $G=(V,E)$ be a directed graph. A vector $x: E\to\{0,\pm 1\}$ is called an *alternating vector* if every node $v\in V$ has an incident edge e with $x(e)=-1$ and $E:=\{e\in E: x(e)\geq 0\}$ is a forest each component of which contains exactly one positive edge.

Theorem 3.12 (Gröflin and Liebling (1979)). *The convex hull Q of alternating vectors is $\{x\in\mathbb{R}^E: x(A)\leq|V(A)|-|A|$ for $A\subseteq E\}$ where $V(A)$ denotes the set of nodes incident to some edges of A.* $\square$

Observe that Q is a g-polymatroid. Gröflin and Liebling also proved an intersection theorem concerning the convex hull of alternating vectors. This turns out to be a special case of the g-polymatroid intersection theorem (see Theorem V.1.4).

Chapter IV. THE GREEDY ALGORITHM AND ITS APPLICATIONS

1. Introduction

The greedy algorithm is one of the most studied procedures in combinatorial optimization. In this chapter we briefly summarize the greedy algorithm for g-polymatroids, but our main purpose is to show some apparently new applications of the greedy algorithm. (The emphasis will be on the existence of (simple) combinatorial algorithms with polynomial complexity and we do not go into details to obtain the best complexity results.)

The greedy algorithm stems from a procedure of Boruvka (1926) to find a maximum weight spanning tree of an edge-weighted connected graph. Extending this R. Rado (1957) showed that a maximum weight independent set of a matroid can be found in a greedy way. (See also Edmonds (1971), Gale (1968) and Welsh (1968).) Namely, order the elements of the ground set so that $w(1)\geq w(2)\geq\cdots\geq w(k)\geq 0> w(k+1)\geq\cdots\geq w(n)$ (throughout this section we adopt the notation $w(i)$ for $w(v_i)$) and, one by one in this order, consider the elements of $\{v_1, v_2, \ldots, v_k\}$. Choose or discard an element according to the rule that the already chosen elements form an independent set.

Edmonds (1970) observed that the greedy algorithm extends to polymatroids. See Theorem I.2.14. That theorem tells us that a linear objective function can be maximized over a polymatroid P in a greedy fashion if P is defined by its unique polymatroid function. An important consequence of this result is the following geometrically more transparent version of the greedy algorithm. Let us be given a polymatroid P and a weight function $w=(w(1),\ldots,w(n))$ for which $w(1)\geq w(2)\geq\cdots\geq w(k)\geq 0> w(k+1)\geq\cdots\geq w(n)$, $z\in\mathbb{R}^S$, is a solution to $\max(wx: x\in P)$ if z is defined as follows. Suppose that $z(1), z(2), \ldots, z(j)$ have already been defined $(j<k)$ and set $z(j+1)=\max(x(j+1): x(i)=z(i)$ for $i=1,2,\ldots,j,\ x\in P)$. For $i>k$ let $z(i)=0$.

To distinguish between the two versions let us call this second one the greedy principle. Observe that the greedy principle is a statement concerning P as a polyhedron and has nothing to do with the linear system defining P.

It is not surprising that the greedy principle and algorithm can be further generalized to g-polymatroids. This was done by R. Hassin (1982) for bounded and by S. Fujishige and N. Tomizawa (1983) for arbitrary g-polymatroids.

This extension goes along the same line except that a minor difficulty has to be overcome. This difficulty arises from non-boundedness and consists of finding an appropriate ordering of the elements.

After reviewing the algorithm we shall show special cases where the values $z(j)$ can be computed somehow but not so trivially as above. In other words these special cases are non-trivial instances where the greedy principle can be turned into a polynomial-time algorithm.

2. Greedy algorithm and principle

Let Q be a g-polymatroid and w a weight function. The problem is to maximize wx over $x \in Q$.

We can suppose that Q is a 0-base polyhedron since each g-polymatroid is a one-coordinate projection of a 0-base polyhedron.

Let $Q = \{x\colon x(S) = 0, x(A) \leq b(A)$ for every $A \subseteq S\}$ where b is a fully submodular function with $b(S) = 0$. Later we discuss what can be said if Q is defined by a crossing submodular function b''.

Since $x(S) = 0$ for every $x \in Q$ we can suppose that $w \geq 0$. The following claim is simple:

$$\max(wx\colon x \in Q) \quad \text{is finite if and only if } \hat{b}(w) < \infty \tag{2.1}$$

(i.e., $b(X) < \infty$ for every level X of w). Recall the notation of digraph $G = G(\mathscr{F}(b))$. To describe b we suppose an evaluation oracle (that tells us the value $b(A)$ for any required set $A \subseteq S$) along with G (that tells us the places where b is finite).

Call two elements $u, v \in S$ *equivalent* if both uv and vu are in G. Since G is transitive this is an equivalence relation. Denote the equivalence classes by $S_1, S_2, \ldots, S_k$. These are precisely the strongly connected components of G. We can suppose that

$$\text{each } S_i \text{ has cardinality one.} \tag{2.2}$$

For otherwise let $S' = \{s_1, s_2, \ldots, s_k\}$ be a set and let $\gamma\colon S \to S'$ be a mapping defined by $\gamma(x) = s_i$ if $x \in S_i$. The homomorphic image $\gamma(Q)$ of Q is a 0-base polyhedron and from an optimal solution to $\max(\gamma(w) \cdot x'\colon x' \in \gamma(Q))$ an optimal solution to $\max(w \cdot x\colon x \in Q)$ can easily be constructed. (Notice that both the evaluation oracle for the fully submodular function b_1 defining $\gamma(Q)$ and the graph $G(\mathscr{F}(b_1))$ can be obtained from those belonging to b).

If both (2.1) and (2.2) hold, then one can easily find an ordering $v_1, v_2, \ldots, v_n$ of the elements of S for which $w(v_i) \geq w(v_{i+1})$ and $b(v_1, \ldots, v_i) < \infty$ for $i = 1, 2, \ldots, n-1$. Call such an ordering *compatible*.

Remark 2.1. In a later section we discuss how the greedy algorithm can be extended if Q is given by a crossing (in particular, an intersecting) submodular function b''. To show how a compatible ordering can be found in this case let us denote by b the bi-truncation of b''. By Proposition 2.5 $G(\mathscr{F}(b)) = G(\mathscr{F}(b''))$. Consequently if $G(\mathscr{F}(b''))$ can be constructed (for example, if a minimizing oracle for b'' is available), then both (2.1) and (2.2) can be assumed and a compatible ordering can be constructed.

Let $X_0 = \emptyset$ and $X_i = \{v_1, \ldots, v_i\}$ and $z(i) = b(X_i) - b(X_{i-1})$ for $i = 1, 2, \ldots, n$. Then every $z(i)$ is a well-defined finite number.

Proposition 2.2 (Fujishige and Tomizawa (1983)). *$z = (z(1), z(2), \ldots, z(n))$ is an optimum solution to* $\max(wx\colon x \in Q)$.

Proof. First we show that $z \in Q$. Obviously $z(S) = 0$. To prove $z(A) \leq b(A)$ we use induction on $|A|$. Let i be the maximum subscript for which $v_i \in A$. By definition $z(X_i) = b(X_i)$, $z(X_{i-1}) = b(X_{i-1})$. By the induction hypothesis $b(A - v_i) \geq z(A - v_i)$ and we have $b(A) + b(X_{i-1}) \geq b(A \cap X_{i-1}) + b(A \cup X_{i-1}) = b(A - v_i) + b(X_i)$ from which $b(A) \geq z(A)$ follows.

To see that z maximizes wx over Q let us consider the following dual pair of linear programs:

$$\begin{array}{ll|ll} x(A) \leq b(A) & \text{for } A \in \mathscr{F}(b) & \sum_A y_A = w & \\ x(S) = 0 & & y_A \geq 0 & \text{for } A \in \mathscr{F}(b) - \{S\} \\ \hline \max wx & & \min \sum y_A b(A) & \end{array}$$

Define

$$y_A = \begin{cases} w(n) & \text{if } A = S, \\ w(i) - w(i+1) & \text{if } A = X_i,\ i = 1, 2, \ldots, n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, this is a dual feasible solution and $wz = \sum y_A b(A)$. This shows that z is primal, y is dual optimum. (Observe that y is nothing but the chain vector of w and $\sum y_A b(A) = \hat{b}(w)$ if $w \geq 0$ is integral.) □

Note that the proof above is an alternative (algorithmic) proof for Proposition II.2.2.

Let us suggest the reader to deduce Edmonds' original greedy algorithm, in particular, to show how the rule, that $z(i)$ has to be chosen 0 if $w(i) < 0$, follows from the general procedure.

The greedy algorithm discussed above is applicable only if an evaluation oracle is available for the defining fully submodular function. There are special cases where in order to construct an evaluation oracle one needs other, more sophisticated algorithms. Let us exhibit such an application. Recall the minimum-weight matroid reinforcement problem (see Section III.3). We have seen that feasible vectors form the integral points of a g-polymatroid

$$Q = \{x \in \mathbb{R}^S, x \geq 0, x(A) \geq k(r(S) - r(S - A)) - |A|\}$$

where r is the rank function of the matroid M in question. The strong pair (p, b) defining Q is $p(A) = \max(k \cdot (r(S) - r(S - X)) - |X|\colon X \subseteq A)$ and $b(A) = \infty$ (for $A \subseteq S$). The greedy algorithm can be applied provided that $p(A)$ can be computed. To this end one has to minimize $kr(S - X) + |X|$ over all subsets X of A.

Let r_1 be the rank function of the matroid $M_1 = M/(S-A)$, that is, $r_1(Z) = r(Z \cup (S-A)) - r(S-A)$ for $Z \subseteq A$.

For $X = A - Z$ we have $k \cdot r_1(Z) + |A-Z| = k \cdot r(S-X) + |X| - k \cdot r(S-A)$ so it suffices to minimize $(k \cdot r_1(Z) + |A-Z|)$ over all subsets Z of A. This minimum is exactly the maximum cardinality of the union of k independent sets of M_1 which can be computed by a matroid partitioning algorithm (Edmonds (1965a)).

Parallel to polymatroids (see Edmonds (1970)) Proposition 2.2 implies:

Corollary 2.3 (Greedy principle). *Let Q be an arbitrary g-polymatroid, $w \geq 0$ a weight function such that $w(i) \geq w(i+1)$ for $i = 1, 2, \ldots, n-1$. Define the components $z(1), z(2), \ldots, z(n)$ of a vector z as follows.*

$$z(i) = \max(y(i)\colon y \in Q, y(j) = z(j) \quad \text{for } j \leq i-1) \tag{2.3}$$

If every $z(i)$ is finite, the vector z is an optimal solution to $\max(wx\colon x \in Q)$.

Remark. The greedy principle can be used in a concrete situation if there is a way to compute the values $z(i)$ and to decide for every $u, v \in S$ whether there are $u\bar{v}$-sets X, Y with $b(X) < \infty$, $p(Y) > -\infty$. (This latter requirement is needed to compute a compatible ordering of the elements of S.)

In the next sections we shall provide some consequences of the greedy principle.

3. Truncation algorithm and applications

Let b' be an intersecting submodular function and let b denote its truncation. That is, for $A \subseteq S$

$$b(A) = \min\left(\sum_i b'(A_i)\colon \{A_i\} \text{ a partition of } A\right). \tag{3.1}$$

We present a method, called *truncation algorithm*, that computes $b(A)$ for a specified subset $A \subseteq S$ provided that a minimizing oracle for b' is available. The algorithm also constructs a partition of A for which $b(A) = \sum b'(A_i)$.

Let $p \equiv -\infty$. Then (p, b') is a weak pair, (p, b) is a strong pair and by the Truncation theorem $Q(p, b') = Q(p, b)$. By Proposition II.2.3 $b(A) = \max(x(A)\colon x \in Q(p, b'))$.

Apply the greedy principle to the weight vector $w = \chi_A$.

By Remark 2.1 one can check ahead of time whether $b(A)$ is finite. If it is, we can construct a compatible ordering, that is an ordering $v_1, v_2, \ldots, v_{|A|}$ of the elements of A such that $b(\{v_1, v_2, \ldots, v_i\}) < \infty$ for $i = 1, 2, \ldots, |A|$.

For the value $z(i)$ in (2.3) we have

$$z(i) = \min(b'(B) - z(B - v_i)\colon B \subseteq \{v_1, \ldots, v_i\}, v_i \in B) \tag{3.2}$$

and therefore, using the minimizing oracle for b', $z(i)$ can be computed. Since by Theorem I.2.14 $z(i) = b(\{v_1, \ldots, v_i\}) - b(\{v_1, \ldots, v_{i-1}\})$ and $v_1, v_2, \ldots, v_{|A|}$ is a compatible ordering we see that each $z(i)$ $(v_i \in A)$ is finite.

Having vector z at hand we can determine a partition $\{A_1, A_2, \ldots, A_k\}$ of A for which $b(A) = \sum (b'(A_i)\colon i = 1, 2, \ldots, k)$ in the following way.

Consider the set B_i where the min is attained on the right-hand side of (3.2). Let $A_1, A_2, \ldots, A_k$ be the components of the hypergraph formed by the hyperedges B_i, $i = 1, 2, \ldots, |A|$. $A_1, A_2, \ldots, A_k$ is a partition of A. By definition each B_i is tight, that is $z(B_i) = b'(B_i)$. Since the union of intersecting tight sets is tight every A_i is tight. Thus

$$b(A) = z(A) = \sum (z(A_i)\colon i = 1, 2, \ldots, k) = \sum (b'(A_i)\colon i = 1, 2, \ldots, k),$$

as required.

Note that the ellipsoid method provides a polynomial algorithm (as shown by Grötschel, Lovász and Schrijver (1981)) both for the problem of minimizing $b'(X) - m(X)$ over $X \subseteq S$ and for the problem of minimizing $\sum b'(X_i)$ over partitions $\{X_1, \ldots, X_k\}$ of S. The main content of the truncation algorithm above is that the latter minimization problem can be solved combinatorially whenever the first one can be. We show two applications where this first minimizing oracle is available.

A. *Generic freedom*

L. Lovász and Y. Yemini (1982) proved that the generic freedom (see Lovász and Yemini (1982), for definition) of a graph $G = (V, E)$ with n nodes is

$$2n - 3 - \min(\sum (2|V(E_i)| - 3)\colon i = 1, 2, \ldots, k)$$

where the minimum ranges over all partitions $\{E_1, E_2, \ldots, E_k\}$ of $E (E_i \neq \emptyset)$. Here $V(E_i)$ denotes the set of nodes met by the elements of E_i.

Since $b_1(X) = 2|V(X)|$ $(X \subseteq E)$ is fully submodular, $b'(X) = b_1(X) - 3$ for $X \neq \emptyset$ is an intersecting submodular function. Consequently, the truncation algorithm can be applied. A minimizing oracle for b' in this special case can be constructed as follows. Let $m \in \mathbb{R}^E$ be a fixed vector. We have to minimize $b'(X) - m(X)$ $(\emptyset \neq X \subseteq E)$ or, equivalently, to minimize $b_1(X) - m(X)$.

Build up a network N with a source s, a sink t and an intermediate node-set $V_E \cup V$. Here the elements of V_E correspond to the elements of E. Define an edge with ∞ capacity from $v_e \in V_E$ to $v \in V$ if the corresponding edge $e \in E$ is incident to v in G.

Define an edge from s to $v_e \in V_E$ with capacity $m(e)$ if $m(e) > 0$, define an edge from v_e to t with capacity $-m(e)$ if $m(e) < 0$, and finally define an edge from $v \in V$ to t with capacity 2.

It is easy to see that there is a one-to-one correspondence between the minimizing sets X for $b_1(X) - m(X)$ and the minimum $s-t$ cuts of N. This latter can be found by a max-flow-min-cut computation.