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PACKING PATHS IN PLANAR GRAPHS

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A generalization of P. Seymour's theorem on planar integral 2-commodity flows is given when the underlying graph G together with the demand graph H (a graph having edges that connect the corresponding terminal pairs) form a planar graph and the demand edges are on two faces of G.

1. Introduction

Let us be given an undirected graph $G = (V, E)$ and k pairs of nodes (s_1, t_1), $(s_2, t_2), \ldots, (s_k, t_k)$. *The edge-disjoint paths problem* is to find k pairwise edge-disjoint paths connecting the corresponding pairs (s_i, t_i). The pairs (s_i, t_i) are called *terminal pairs*.

It is convenient to mark each terminal pair to be connected by an edge, called a *demand edge*. The graph $H = (V, F)$ formed by the demand edges is called a *demand graph* while the original graph $G = (V, E)$ is the *supply graph*. (Of course, H may not be connected.)

The edge-disjoint paths problem is **NP**-complete even if H consists of two sets of parallel edges (Even et al. [1]) but there are important special cases when it is tractable. For a survey see Frank [2]. P. Seymour [5] settled the case when H consists of two sets of parallel edges and G and H together are planar. The purpose of the present paper is to present a generalization of Seymour's theorem for the case when $G + H$ is planar and the demand edges are placed on at most two faces of G.

Throughout the paper we work with an undirected connected graph $G = (V, E)$ that contains no loops but parallel edges are allowed. Here V denotes the node set of G. We do not distinguish between an element v and the one-element set $\{v\}$. The set of edges between A and $V - A$ is called a *cut* and is denoted by $\nabla(A)$. If both A and $V - A$ induce a connected subgraph, then $\nabla(A)$ is called a *bond*. It is well-known that any cut can be partitioned into bonds.

An element e of E with endpoint u and v is denoted by $e = uv(= vu)$. (Such a notation is not precise because G may have parallel edges, but no ambiguity will arise from this sloppiness.)

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For sets $X, Y \subseteq V$ let $d_G(X,Y)$ denote the number of edges with one end in $X - Y$ and one end in $Y - X$. We use $d_G(X)$ for $d_G(X, V - X)$. (When it is not ambiguous we leave out the subscript G.) For two graphs $G = (V, E)$ and $H = (V, F)$ (where E, F are disjoint but may contain elements that are parallel) $G + H$ denotes the graph $(V, E \cup F)$.

2. The Theorem

A natural necessary condition for the solvability of the edge-disjoint paths problem is the cut criterion:

CUT CRITERION $\quad d_G(X) \geq d_H(X) \quad$ for every $\quad X \subseteq V$.

Since any cut of G can be partitioned into bonds, the cut criterion holds if we require the inequality above only for subsets X for which $\nabla(X)$ is a bond.

We call the difference $s(X) := d_G(X) - d_H(X)$ the *surplus* of cut $\nabla(X)$.

The cut criterion is equivalent to saying that the surplus of every cut is non-negative. A cut $\nabla(X)$ is called *tight* if $s(X) = 0$.

The cut criterion is sufficient if the demand graph consists of one set of parallel edges (this is the edge-disjoint undirected version of Menger's theorem). The cut criterion is not sufficient, in general, as the following simple example shows.

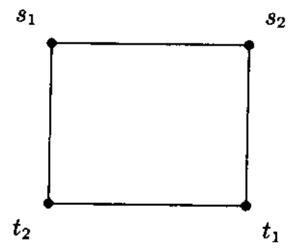

Fig. 1

Let us call a set $X \subseteq V$ *odd* (or the cut $\nabla_G(X)$ odd) if $d_{G+H}(X)$ is odd. Clearly, the number of odd nodes is always even and a set X is odd if and only if X contains an odd number of odd nodes.

The crucial observation concerning odd cuts is that, given an odd set X and any solution to the edge-disjoint paths problem, an odd number of edges of $\nabla_G(X)$, in particular at least one edge, cannot be used by the paths in the solution. Since in a tight cut every edge must be used by a solution, the following criterion is obviously necessary.

INTERSECTION CRITERION $\quad \left\{ \begin{array}{l} d_{H+G}(S \cap T) \text{ is even for} \\ \text{any two tight sets } S, T \subseteq V. \end{array} \right.$

Observe that in Figure 1 the intersection criterion is violated.

P. Seymour [5] proved the following. Suppose that $G+H$ is planar and H consists of two sets of parallel edges. Suppose furthermore that the cut criterion holds but the edge-disjoint paths problem does not have a solution. Then some edges of G can be contracted in such a way that the resulting graph G' has at most four nodes and the corresponding edge-disjoint paths problem still has no solution.

A slight refinement of Seymour's theorem asserts that when $G+H$ is planar and H consists of two sets of parallel edges, then the edge-disjoint paths problem has a solution if and only if the cut criterion and the intersection criterion hold. Our main result is the following generalization:

Theorem. *Suppose that $G + H$ is planar and the demand edges are on at most two faces of G. The edge-disjoint paths problem has a solution if and only if the cut criterion and the intersection criterion hold.*

Remark 1. Note that in the theorem no parity restriction is imposed on the degrees of nodes of $G + H$. If $G + H$ is planar and Eulerian, then the cut criterion itself is sufficient for the solvability of the edge-disjoint paths problem, irrespective of the number of faces of G necessary to include the demand edges. This is another theorem of P. Seymour from the same paper [1981] we have cited above.

Remark 2. M. V. Lomonosov [3] proved a maximization form of Seymour's above-mentioned feasibility-type theorem on planar integer two-commodity flows. With some effort Lomonosov's theorem can be derived from Seymour's (see Frank [2]). It would be interesting to find a maximization form of our result.

3. The Proof

The following equality will prove useful. For $A, B \subseteq V$

(3.1) $\qquad d_G(A) + d_G(B) = d_G(A \cap B) + d_G(A \cup B) + 2d_G(A, B).$

We will use a more complicated relation due to G. Tardos [6]. Suppose that the node set V is partitioned into 5 sets; A, M, N, X, Y. Then

(3.2) $\quad d(X \cup M) + d(Y \cup M) + 2d(A, N) = d(X \cup N) + d(Y \cup N) + 2d(A, M).$

The proof of both (3.1) and (3.2) consists of showing that the contribution of any of the edges to the two sides of the identity is the same.

Lemma 1. *Suppose that the cut criterion holds.*
(a) If A and B are tight and $d_H(A, B) = 0$, then both $A \cap B$ and $A \cup B$ are tight and $d_G(A, B) = 0$.
(b) If A and B are tight and $d_H(A, V - B) = 0$, then both $A - B$ and $B - A$ are tight and $d_G(A, V - B) = 0$.

Proof. By applying (3.1) to G and H we have

$$d_H(A) + d_H(B) = d_G(A) + d_G(B) = d_G(A \cap B) + d_G(A \cup B) + 2d_G(A, B) \geq$$
$$d_H(A \cap B) + d_H(A \cup B) + 2d_G(A, B) = d_H(A) + d_H(B) + 2(d_G(A, B) - d_H(A, B))$$

from which the first part follows. We obtain part (b) if (a) is applied to A and $\overline{B} := V - B$. ∎

Let $\nabla(K)$ be a bond and C a facial circuit of G. Because of planarity $\nabla(K)$ and $E(C)$ have zero or two edges in common. This property will be extensively used in the proof.

Let us recall that if the cut criterion does not hold, then there is a bond violating it. An analogous statement holds for the intersection criterion.

Lemma 2. *Suppose that the cut criterion holds with respect to G and H but the intersection criterion does not. Then there are sets S and T violating the intersection criterion for which $\nabla(S)$ and $\nabla(T)$ are both bonds.*

Proof. Let S and T be two tight sets violating the intersection criterion for which $k(S)+k(T)$ is minimal where $k(X)$ denotes the number of components of $G-\nabla(X)$. We show that $\nabla(S)$ is a bond. If not, then at least one of S and $V-S$, say S, partitions into two non-empty parts S' and S'' with $d_G(S',S'')=0$. Since $\nabla(S)$ is tight, both $\nabla(S')$ and $\nabla(S'')$ are tight, Moreover, exactly one of $d_{G+H}(S'\cap T)$ and $d_{G+H}(S''\cap T)$, say the first one, is odd. Therefore S' and T also violate the intersection criterion. However, as is easily seen, $k(S')<k(S)$ contradicting the minimal choice of S and T. ∎

Let us turn to the proof of the main theorem. We have seen the necessity of the cut and intersection criteria. To prove the sufficiency let us assume that $G+H$ is a minimal counterexample. Then G is 2-connected since otherwise the problem can easily be decomposed into smaller problems.

Assume that the terminal pairs are positioned on faces C_1 and C_2.

Let us call a demand edge and its two end nodes *of type i* $(i=1,2)$ if it lies in face C_i. By symmetry we can assume that there are at least as many edges of type 1 as of type 2. We assume that C_1 is the outer face of G. Since G is 2-connected every face of G is bounded by a circuit. It will cause no confusion that we use the term C_i to denote a face of G and the circuit of G bounding this face.

Since $G+H$ is planar there are two internally disjoint subpaths P' and P'' of C_1 such that the two endpoints s', t' of P' and the two endpoints s'', t'' of P'' are terminal of type 1 and none of P' and P'' contains a terminal node of type 1 as an inner node. (It may happen that $\{s',t'\}=\{s'',t''\}$.)

Delete the edges of P' from G and remove one demand edge connecting s' and t' from H (that is, we remove a circuit from $G+H$). Let the resulting supply and demand graph be G' and H'. Let G'' and H'' be defined analogously.

Lemma 3. *The cut criterion holds for at least one of (G',H') and (G'',H'')*

Proof. If the cut criterion does not hold for G' and H', then there is a set K violating the cut criterion such that K and $V-K$ both induce connected subgraphs of G'. Then $\nabla_G(K)$ is a bond of G.

Therefore P' and $\nabla_G(K)$ have at most two edges in common. Since the cut criterion holds for G and H, $\nabla_G(K)$ does not separate s' and t' and $s(K)\le 1$. By interchanging K and $V-K$, if necessary, we can assume that $s',t'\notin K$ and $K\cap V(P')\neq\emptyset$. By the choice of P' no terminal pairs of type 1 are separated by $\nabla(K)$. Exploiting that $d_G(K)\ge 2$ (G being 2-connected) and $s(K)\le 1$ we see that $\nabla(K)$ separates a terminal pair of type 2.

Analogously, if the cut criterion does not hold for G'' and H'', there is a set L for which $s(L)\le 1$, $s'',t''\notin L$, $L\cap V(P'')\neq\emptyset$ and L separates a terminal pair of type 2.

Since both K and L contain a node of C_2, by planarity, s' and t' cannot be in the same component of $G-(K\cup L)$. Obviously, there is a subpath Q of C_1 connecting s' and one of s'' and t'', say s'', such that Q and $K\cup L$ are disjoint. That is, s' and s'' are in the same component of $G-(K\cup L)$, Therefore $V-(K\cup L)$ can be partitioned

into two sets A and N such that $s',s''\in N$, $t',t''\in A$ and $d_G(A,N)=0$. Let us introduce the following notation: $M:=K\cap L$, $X:=K-L$, $Y:=L-K$.

Claim. *At least one of $d_H(A,M)$ and $d_H(N,M)$ is zero.*

Proof. Suppose that $d_H(A,M)>0$. Then there is a terminal pair (s_1,t_1) such that $t_1\in M$ and $s_1\in A$. Since $M\cap V(C_1)=\emptyset$, terminal pair (s_1,t_1) is of type 2. Analogously, if $d_H(N,M)>0$, then there is a terminal pair (s_2,t_2) of type 2 such that $t_2\in M$ and $s_2\in N$. $\nabla(K)$ and $E(C_2)$ have two edges in common. Since t_1 and t_2 are in K while s_1 and s_2 are not, the planarity of $G+H$ implies that the cyclic order of these four nodes around C_2 is t_1,t_2,s_2,s_1.

Since $\nabla(L)$ and $E(C_2)$ have two edges in common, $t_1,t_2\in L$ and $s_1,s_2\notin L$, we see that the subpath of C_2 between s_1 and s_2 which does not contain t_1 is entirely in $G-(K\cup L)$. But this is impossible since $s_1\in A$, $s_2\in N$ and $d_G(A,N)=0$. ∎

By the claim we can suppose that $d_H(A,M)$, say, is zero. We also know that $d_G(A,N)=0$ and $d_H(A,N)>0$. By applying (3.2) to G and H we get

$$\begin{aligned}&2=1+1\ge s(X\cup M)+s(Y\cup M)=\\&s(X\cup N)+s(Y\cup N)+2[d_G(A,M)-d_H(A,M)]-2[d_G(A,N)-d_H(A,N)]=\\&s(X\cup N)+s(Y\cup N)+2[d_G(A,M)+d_H(A,N)]\ge 0+0+2[0+1]=2.\end{aligned}$$

Therefore we have equality throughout and, in particular, $s(K)=s(L)=1$, $s(X\cup N)=s(Y\cup N)=0$, $d_G(A,M)=0$, $d_H(A,N)=1$. It follows that there is exactly one demand edge of type 1. Consequently, there is at most one demand edges of type 2. Actually there is exactly one demand edge of type 2 since $G+H$ is a counterexample. Therefore $d_G(K)=2=d_G(L)$ and $d_H(K)=1=d_H(L)$. (This means that the two edges of G leaving K are common edges of C_1 and C_2 and the same holds for L).

Now $M=K\cap L$ must be empty for if a node v is in $K\cap L$, then there is a path in K from v to P'. But such a path leaves L along an edge that is not in C_1. So we would have $d_G(L)\ge 3$. We see that $K=X$ and $L=Y$.

Since M is empty $d_H(N,M)=0$. Therefore (3.2) can be applied with A and N interchanged. We obtain that $Y\cup A$ is tight. Let $S:=V-(Y\cup A)$ $(=X\cup N)$ and $T:=V-(Y\cup N)$ $(=X\cup A)$. Now S and T violate the intersection criterion since S and T are tight, $K=S\cap T$ and $d_{G+H}(K)$ is 3, an odd number. ∎

By Lemma 3 we can suppose that the cut criterion holds for G' and H'.

Lemma 4. *The intersection criterion holds for G' and H'.*

Proof. Let S and T violate the intersection criterion with respect to G' and H'. Then by Lemma 2 we can assume that each of S, $V-S$, T, $V-T$ induces a connected subgraph of G'. Therefore $\nabla_G(S)$ and $\nabla_G(T)$ are bonds and they have at most two edges in common with $E(C_1)$. ∎

Claim. *Both $\nabla(S)$ and $\nabla(T)$ separate terminal pairs of type 1 and type 2.*

Proof. If S and T are tight and $d_{G+H}(S\cap T)$ is odd, then $d_{G+H}(S-T)$ is odd. Therefore neither $S\cap T$ nor $S-T$ can be tight. By Lemma 1 $d_H(S,T)>0$ and $d_H(S,V-T)>0$. Thus there are terminal pairs (s_1,t_1) and (s_2,t_2) for which $s_1\in S-T$, $t_1\in T-S$, $s_2\in S\cap T$, $t_2\in V-(S\cap T)$. These two terminal pairs cannot be of the same type. Indeed, let both terminal pairs be of type 1, say. If s_1 and s_2 are not separated by t_1 and t_2 on C_1, then $\nabla(T)$ and $E(C_1)$ have more than 2 edges in common. If s_1 and s_2 are separated by t_1 and t_2, then $\nabla(S)$ and $E(C_1)$ have more than 2 edges in common. ∎

If S and P' have no node in common, then $\nabla(S)$ is tight for G and H. If S and P' are not disjoint then, by the claim and the choice of P', $\nabla(S)$ separates s' and t' and therefore S is tight for G and H in this case as well. Similarly T is tight for G and H. Since $d_{G+H}(X)$ has the same parity as $d_{G'+H'}(X)$ for any set X, we conclude that S and T violate the intersection criterion for G and H, a contradiction. ∎

So far we have proved that both the cut criterion and the intersection criterion hold for $G' + H'$. Thus there is a solution to the edge-disjoint paths problem for $G' + H'$. But this solution along with path P' is a solution to the edge-disjoint paths problem for $G + H$, a contradiction. ∎

A direct consequence of the main theorem is that the problem has a solution if the cut criterion holds with strict inequality on every cut. The following example of E. Korach shows that this statement, and therefore the main theorem is not true if the demand edges are on three faces of G. Here the cut criterion holds and so does the intersection criterion since there is no tight cut at all. On the other hand the edge-disjoint paths problem has no solution.

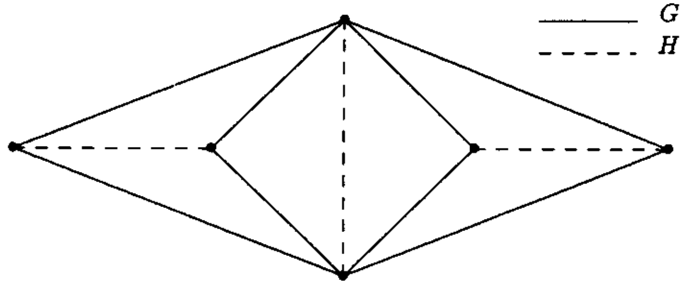

Fig. 2

Remarks. Observe that the proof of the theorem gives rise to a polynomial time algorithm provided that an oracle is available to test the cut criterion. But such an oracle can easily be constructed. Indeed, we have mentioned that it suffices to check the cut criterion for subsets X intersecting C_i in a subpath $(i = 1, 2)$. First specify subpaths P_i of C_i and check the cut criterion for sets X for which $X \cap C_i = P_i$. This can be done by a max flow-min cut computation. There are $(|C_i| - 1)|C_i|/2$ subpaths of C_i, and therefore at most $O(|C_1|^2|C_2|^2)$ max flow-min cut computations are necessary. (Actually, this bound can be replaced by $O(n_1^2 n_2^2)$ where n_i denotes the number of distinct terminals on C_i.)

We note that using matching theory one can construct more efficient methods to test the cut criterion and this approach works in the more general case when we only require that $G + H$ is planar (Lovász–Plummer [4]).

Finally we mention that, with some care, the algorithm suggested by the proof can be made strongly polynomial in the (integer) capacitated version of the problem.

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