Reprinted from JOURNAL OF COMBINATORIAL THEORY, Series B
All Rights Reserved by Academic Press, New York and London

Vol. 61, No. 2, July 1994
Printed in Belgium

Notes

On Packing T-Cuts*

ANDRÁS FRANK†

Research Institute for Discrete Mathematics,
University of Bonn, Nassestr. 2, Bonn – 1, Germany, D-5300

AND

ZOLTÁN SZIGETI

Department of Computer Science, Eötvös University,
Múzeum krt. 6-8, Budapest, Hungary, H-1088

Received July 2, 1992

A short proof of a difficult theorem of P. D. Seymour on grafts with the max-flow min-cut property is given. © 1994 Academic Press, Inc.

I. INTRODUCTION

The Chinese postman problem, in other words the minimum T-join problem, consists of finding a minimum cardinality subset of edges of a graph satisfying prescribed parity constraints on the degrees of nodes. This minimum is bounded from below by the maximum value of a (fractional) packing of T-cuts. In the literature there are several min-max theorems for cases when equality actually holds. In this paper we list some of these results and exhibit new relationships among them.

To be more specific, P. D. Seymour's theorem [7] on binary matroids with the max-flow min-cut property, when specialized to T-joins, provides a characterization of pairs (G, T) for which the minimum weight of a T-join is equal to the maximum packing of T-cuts for every integer weighting. Motivated by Seymour's theorem, A. Sebő [6] proved a min-max theorem

* Research partly supported by the Hungarian National Foundation for Scientific Research Grants OTKA 2118 and 4217.

† On leave from the Department of Computer Science (until August 1993), Eötvös University, Múzeum krt. 6-8, Budapest, Hungary, H-1088.

0095-8956/94 $6.00

concerning minimum T-joins and maximum packing of T-borders. He also observed that his result, combined with a simple-sounding lemma on bi-critical graphs (Theorem 7 below), immediately implies Seymour's theorem.

The purpose of this note is twofold. We show first that Sebő's theorem is an easy consequence of an earlier min-max theorem [2] and, second, we provide a simple proof of the above-mentioned statement on bi-critical graphs. This way we will have obtained a simple proof of Seymour's theorem. Along the line, we will point out that Tutte's theorem on perfect matchings is a direct consequence of the result from [2].

A *graft* (G, T) is a pair consisting of a connected undirected graph $G=(V, E)$ and a subset T of V of even cardinality. A subset J of edges is called a *T-join* if $d_J(v)$ is odd precisely when $v \in T$. Here $d_J(v)$ denotes the number of elements of J incident to v. J is called a *perfect matching* if $d_J(v)=1$ for every $v \in V$. Note that a perfect matching is a T-join for which $T=V$. Let $G=(V, E)$ be a graph with non-empty edge-set E. G is called *bi-critical* if for every pair of nodes u, v, the graph $G-\{u, v\}$ contains a perfect matching. It follows immediately from Tutte's theorem (see Theorem 0 below) that G is bi-critical if and only if

$$q(X) \leqslant |X|-2 \qquad \text{for every subset} \quad X \subseteq V \text{ with } |X| \geqslant 2, \tag{1}$$

where $q(X)$ denotes the number of odd-cardinality components of $G-X$.

Let us call a set $X \subseteq V$ *T-odd* if $|X \cap T|$ is odd. Given a partition $\mathscr{P}=\{V_1, V_2, ..., V_k\}$ of V, by a *multicut* $B=B(\mathscr{P})$ we mean the set of edges connecting different parts of $\mathscr{P}$. If each V_i is T-odd and induces a connected subgraph, B is called a *T-border*. Then clearly k is even and $\operatorname{val}(B):=k/2$ is called the *value* of the T-border. When $k=2$ a T-border B is called a *T-cut*. Note that the value of a T-cut is one.

The *border graph* G_B of a T-border $B=B(\mathscr{P})$ is one obtained by contracting each V_i into one node. Let us call a T-border *bi-critical* if its border graph is bi-critical.

Note that the cardinality of the intersection of a T-cut and a T-join is always odd, in particular, at least one. Hence the cardinality of the intersection of a T-border B and a T-join J is always at least $\operatorname{val}(B)$ and equality holds precisely when the edges in J connecting distinct V_i's form a perfect matching in the border graph of B.

A list $\mathscr{B}=\{B_1, ..., B_l\}$ of T-borders is called a *packing* (*2-packing*) if each edge of G belongs to at most one (two) member(s) of $\mathscr{B}$. The *value* of a packing is $\sum(\operatorname{val}(B): B \in \mathscr{B})$ and the *value* of a 2-packing is $\sum(\operatorname{val}(B): B \in \mathscr{B})/2$. Note that a T-border of value t determines a 2-packing of T-cuts of value t.

For an edge $e=uv$ we define the *elementary T-contraction* as a graft (G', T'), where G' arises from G by contracting e and $T':=T-\{u, v\}$ if $|\{u, v\} \cap T|$ is even and $T':=T-\{u, v\}+x_{uv}$ if $|\{u, v\} \cap T|$ is odd, where x_{uv} denotes the contracted node. The *T-contraction* of a graph means a sequence of elementary T-contractions. If $X \subseteq V$ induces a connected subgraph of G, then by T-contracting X we mean the operation of T-contracting a spanning tree of X.

Let $\mathbf{K}_4$ denote the graft $(K_4, V(K_4))$, where K_4 is a complete graph on four nodes. Note that a graft (G, T) can be T-contracted to $\mathbf{K}_4$ precisely when there is a partition $\{V_1, V_2, V_3, V_4\}$ of V into T-odd sets so that each V_i induces a connected subgraph and there is an edge connecting V_i and V_j whenever $1 \leqslant i<j \leqslant 4$.

For a general account on matchings and T-joins, see [4].

II. Results on T-Cuts and T-Joins

Our starting point is Tutte's theorem [9] on perfect matchings.

Theorem 0. *A graph $G=(V, E)$ contains no perfect matching if and only if there is a set X of nodes so that $G-X$ includes more than $|X|$ components of odd cardinality.*

The perfect matching problem can be reformulated in terms of T-joins. Namely, by chosing $T:=V$, one observes that G has a perfect matching precisely if the minimum cardinality of a T-join is $|V|/2$. Therefore it was natural to ask for theorems concerning the minimum cardinality of a T-join. Let us list some known results concerning this minimum. The first one was proved by L. Lovász [3] (and was stated earlier in a more general form by J. Edmonds and E. Johnson [1]).

Theorem 1. *The minimum cardinality of a T-join is equal to the maximum value of a 2-packing of T-cuts.*

For example, in $\mathbf{K}_4$ a perfect matching is a T-join of two elements and a 2-packing of T-cuts with value 2 is provided by taking each T-cut once. Note that the value of the best T-cut packing is 1.

Although this theorem, when applied to $T:=V$, provides a good characterization for the existence of a perfect matching (namely, *a graph $G=(V, E)$ with $|V|$ even has no perfect matching if and only if there is a list of more than $|V|$ V-cuts so that every edge belongs to at most two of them*), Tutte's theorem does not seem to follow directly.

For bipartite graphs P. D. Seymour [8] proved a stronger statement.

Theorem 2. *In a bipartite graph the minimum cardinality of a T-join is equal to the maximum number of disjoint T-cuts.*

This theorem immediately implies Theorem 1 by subdividing each edge by a new node. In [2] the following sharpening of Theorem 2 was proved.

THEOREM 3. *In a bipartite graph* $D=(U, V; F)$ *the minimum cardinality of a T-join is equal to* $\max \sum q_T(V_i)$, *where the maximum is taken over all partitions* $\{V_1, ..., V_l\}$ *of* V *and* $q_T(X)$ *denotes the number of T-odd components of* $D-X$.

Let $G=(V, E)$ be an arbitrary graph. Subdivide each edge by a new node and let $D=(V, U; F)$ denote the resulting bipartite graph (where U denotes the set of new nodes). By applying Theorem 3 to this graph one can easily obtain the following.

THEOREM 4. *In a graph* $G=(V, E)$ *the minimum cardinality of a T-join is equal to* $\max \sum q_T(V_i)/2$, *where the maximum is taken over all partitions* $\{V_1, ..., V_l\}$ *of* V.

Observe that Theorem 3 implies Seymour's Theorem 2. In [2] we pointed out via an elementary construction that Theorem 3 also implies the Berge–Tutte formula, a slight generalization of Tutte's theorem. Let us show now an even simpler derivation of the (non-trivial part of) Tutte's theorem.

THEOREM 4 → THEOREM 0.

Proof. Apply Theorem 4 with the choice $T := V$. Note that in this case a subset of V is T-odd if its cardinality is odd. If there is no perfect matching, then the minimum cardinality of a T-join is larger than $|V|/2$. By Theorem 4 there is a partition $\{V_1, ..., V_l\}$ of V so that $\sum q_T(V_i)/2 > |V|/2$, that is, $\sum q_T(V_i) > \sum |V_i|$. Therefore there must be a subscript i so that $q_T(V_i) > |V_i|$; that is, the number of components in $G - V_i$ with odd cardinality is larger than $|V_i|$, as required. ∎

A. Sebő [6] determined the minimal totally dual integral linear system defining the conical hull of T-joins. As a by-product, he derived the following integer min-max theorem concerning T-joins.

THEOREM 5. *In a graph* $G=(V, E)$ *the minimum cardinality of a T-join is equal to the maximum value of a T-border packing* $\{B_1, ..., B_r\}$. *Furthermore, if an optimal packing is chosen in such a way that* r *is as large as possible, then each* B_i *is bi-critical.*

Note that both Theorems 4 and 5 imply Theorem 1. The last theorem of our list is also due to P. D. Seymour [7].

THEOREM 6. *If a graft* (B, T) *cannot be T-contracted to* $\mathbf{K}_4$, *then the minimum cardinality of a T-join is equal to the maximum number of disjoint T-cuts.*

This theorem is a special case of a very difficult result of Seymour concerning binary matroids with the max-flow min-cut property. It can be formulated in an apparently stronger form:

> *A graft* (G, T) *cannot be T contracted to* $\mathbf{K}_4$ *if and only if for every integer weight-function w the minimum weight of a T-join is equal to the maximum number of T-cuts so that every edge belongs to at most* $w(e)$ *T-cuts.*

Note, however, that the "if" part is trivial and the "only if" part easily follows from Theorem 6 if we contract each edge e with $w(e)=0$ and subdivide each edge e by $w(e)-1$ new nodes when $w(e)>0$.

III. PROOFS

We show first that Sebő's Theorem 5 is also an easy consequence of Theorem 3 and, second, using Sebő's theorem we provide a simple proof of Seymour's Theorem 6.

Let $G=(V, E)$ be an arbitrary graph and let $D=(V, U; F)$ be a bipartite graph arising from G by subdividing each edge by a new node. Here sets E and U are in a one-to-one correspondence and we will not distinguish between their corresponding elements. In particular, a subset of U will be considered as a subset of E and vice versa.

Observe that in Theorem 3 the two parts U and V of the bipartite graph play an asymmetric role. When one applies Theorem 3 to D and the maximum is taken over the partitions of V, Theorem 4 can be obtained. Sebő's theorem will follow from Theorem 3 by taking the maximum over the partitions of U.

Proof of Theorem 5. We have already seen that the value of a T-border packing is a lower bound for the minimum cardinality of a T-join. We are going to prove that there is a T-join J of G and a packing $\mathscr{F}$ of T-borders of G so that

$$|J| = \mathrm{val}(\mathscr{F}). \tag{2}$$

By Theorem 3 there is a partition $\mathscr{U}$ of U and a T-join J' of D for which

$$|J'| = \sum (q_T(X) : X \in \mathscr{U}). \tag{3}$$

Assume that $l := |\mathscr{U}|$ is a large as possible and let Z be an arbitrary member of $\mathscr{U}$ with $q_T(Z) > 0$. Let $K_1, K_2, ..., K_h$ be the components of $D - Z$, $V_i := V \cap K_i$ and $\mathscr{P} := \{V_1, ..., V_h\}$.

Clearly, $Z \supseteq B(\mathscr{P})$ and, in fact, we have equality here since if an edge e induced by V_i belonged to Z, then $|Z| \geqslant 2$ and in $\mathscr{U}$ we could replace Z by two sets $Z - e$ and $\{e\}$ without destroying (3), contradicting the maximality of l. We also claim that each V_i is T-odd for otherwise $|Z| \geqslant 2$ and for an edge $e \in Z$ leaving V_i we could replace Z by $Z - e$ and $\{e\}$ without destroying (3), contradicting again the maximality of l.

Let $\mathscr{F} := \{Z \in \mathscr{U} : q_T(Z) > 0\}$. We have seen that each member Z of $\mathscr{F}$ is a T-border of G with $\text{val}(Z) = q_T(Z)/2$. Hence (2) and the first half of Theorem 5 follow by noting that J' corresponds to a T-join J of G with $|J| = |J'|/2$.

To prove the second half of the theorem let $\mathscr{B}$ be a T-border packing of maximum value such that $r := |\mathscr{B}|$ is as large as possible. Suppose indirectly, that a member $B \in \mathscr{B}$ is not bi-critical. That is, the border graph G_B of B includes a subset X of nodes with $|X| \geqslant 2$ for which $q(X) \geqslant |X|$. (Here $q(X)$ denotes the number of odd-cardinality components of $G_B - X$.)

For any odd component K of $G_B - X$ let us define a partition of $V(G_B)$ consisting of the elements of K as singletons and a set $V(G_B) - K$. This partition defines a T-border of G with value $(|K| + 1)/2$. For any even component L of $G_B - X$ let us define a partition of $V(G_B)$ consisting of the elements of $L - v$ as singletons and the set $V(G_B) - (L - v)$, where v is an arbitrary element of L having a neighbour in X. This partition defines a T-border of G with value $|L|/2$. The T-borders defined this way are pairwise disjoint subsets of B and their total value is $|V(G_B)|/2$, the value of B. This contradicts the maximal choice of r. ∎

The following Theorem 7, interesting for its own right, was stated by A. Sebő. He noted that it follows from Seymour's Theorem 6 and observed that, conversely, Theorem 6 is an easy consequence of Theorems 5 and 7. We provide here a simple proof.

THEOREM 7. *The node set of an arbitrary bi-critical graph G_B on $k \geqslant 4$ nodes can be partitioned into four subsets V_1, V_2, V_3, V_4 of odd cardinality so that each V_i induces a connected subgraph and there is an edge connecting V_i and V_j whenever $1 \leqslant i < j \leqslant 4$.*

Proof. Let M be a perfect matching of G_B, $uv \in M$, and $M_{uv} := M - uv$. Let z $(\neq v)$ be a neighbour of u. Since G_B is bi-critical $G_B - \{v, z\}$ contains a perfect matching M_{vz}. The symmetric difference $M_{uv} \oplus M_{vz}$ consists of node-disjoint circuits and a path P connecting z and u. Now $C := P + uz$ is an odd circuit of G_B so that, starting at u and going along C, every second edge of C belongs to M.

Let $u, u_1, ..., u_h$ be the nodes of C (in this order). Because of the existence of M, the component K of $G_B - V(C)$ containing v is of odd cardinality while all the other components are of even cardinality.

Let $V_1 := K$. It follows from (1) that G_B is two-connected and, moreover, contains no separating set X of two elements for which $q(X) > 0$. Hence K must have at least three distinct neighbours u, u_i, u_j in C.

If there is a matching edge $xy \in M$ on C so that u, u_i, x, y, u_j reflects the order of these nodes around C (where both $u_i = x$ and $u_j = y$ are possible), then define $V_2' := \{u_1, u_2, ..., x\}$, $V_3' := \{y, ..., u_{h-1}, u_h\}$, $V_4' := \{u\}$.

If there is no such matching edge, that is, $j = i + 1$ and i is even, then define $V_2' := \{u_i\}$, $V_3' := \{u_{i+1}\}$, $V_4' := V(C) - \{u_i, u_{i+1}\}$.

In both cases $\{V_2', V_3', V_4'\}$ is a partition of $V(C)$. Let $\mathscr{L}$ denote the set of even components of $G_B - V(C)$. For each $L \in \mathscr{L}$ choose a subscript $s = s(L)$ $(= 2, 3, 4)$ so that L is connected to a node in V_s'. For $t = 2, 3, 4$ define $V_t := V_t' \cup \bigcup (L \in \mathscr{L} : s(L) = t)$. The partition $\{V_1, V_2, V_3, V_4\}$ constructed this way satisfies the requirements. ∎

Proof of Theorem 6. Let $\mathscr{B}$ be an optimal packing of bi-critical T-borders provided by Theorem 5. We claim that each member of $\mathscr{B}$ is a T-cut. Indeed, if $B \in \mathscr{B}$ is a T-border determined by a partition $\mathscr{P}$ of V $(|\mathscr{P}| \geqslant 4)$ into T-odd sets, then the graft $(G_B, V(G_B))$ arises from (G, T) by T-contracting each member of $\mathscr{P}$ and then, by Theorem 7, (G, T) can be T-contracted to $\mathbf{K}_4$, a contradiction. ∎

In order for the paper to be self-contained, we include here a proof of Theorem 3, due to A. Sebő [5].

Proof of Theorem 3. We prove only the non-trivial direction max $\geqslant$ min. Let J be a T-join of minimum cardinality. Let w denote a weighting on F for which $w(e) = -1$ if $e \in J$ and $w(e) = 1$ if $e \in F - J$. Then w is clearly *conservative*; that is, there is no circuit of negative total weight. Actually, we prove the following.

THEOREM 3′. *Let $D = (U, V; F)$ be a bipartite graph and $w: F \to \{+1, -1\}$ a conservative weighting. There is a partition $\mathscr{L}$ of V so that for each part $P \in \mathscr{L}$ and for each component C of $D - P$ there is at most one negative edge connecting P and C.*

Proof. We use induction on $|J|$, where J denotes the set of negative edges. If J is empty, $\mathscr{L} := \{V\}$ will do. Assume that J is non-empty and let s be an arbitrary node incident to an element of J. Let P be a path of D starting at s so that its weight $m := w(P)$ is minimum and, in addition, P has as few edges as possible. Let t denote the other end-node of P, xt the last edge of P and B the set of edges of D incident to t. Since B is a cut of D, the graph $D' := D/B := (U', V; F')$ arising from D by contracting the elements of B is bipartite. Let t' denote the contracted node of D' corresponding to t and let w' denote the weighting of D' determined by w. We call a subpath $P[y, t]$ of P an *end-segment*. Clearly $m < 0$ by the choice of s and

$$\text{each end-segment of } P \text{ has negative weight,} \qquad (*)$$

in particular, $w(xt) < 0$.

CLAIM. (i) *xt is the only negative edge incident to t.* (ii) *In $D - t$ there is no negative path R connecting two neighbours u, v of t.*

Proof. (i) Let tz be another negative edge. If $z \in P$, then $P[z, t] + tz$ would form a negative circuit contradicting that w is conservative. If $z \notin P$, then $P' := P + tz$ would be a path with $w(p') < w(P)$, contradicting the minimal choice of P. Thus (i) follows.

(ii) Let R be a path in $D - t$ connecting u, v for which $w(R)$ is minimum and suppose for a contradiction that $w(R) < 0$. Clearly u and v are distinct from x since otherwise $R + ut + tv$ would form a negative circuit in G.

An arbitrary node y of R subdivides R into two segments $R[y, u]$ and $R[y, v]$. Since $w(R) < 0$, at least one of the two segments has negative weight.

Suppose first that P and R have a node y in common. Choose y so that $P[y, t]$ has as few edges as possible. Assume that $w(R[u, y]) < 0$. Property $(*)$ implies that $P[t, y] + R[y, u] + ut$ is a negative circuit in D, a contradiction.

Now let P and R be disjoint. Since D is bipartite, R has even length from which $w(R) \leqslant -2$. Hence $P' := P + tu + R$ is a simple path starting at s such that $w(P') < m$, contradicting the minimal choice of P. ∎

The claim is equivalent to saying that w' is a conservative weighting of D'. By the inductional hypothesis, there is a partition $\mathscr{L}'$ of V' satisfying the requirement of the theorem with respect to w'. If $t \in U$ (that is, $t' \in V'$), then $\mathscr{L}'$ determines a partition $\mathscr{L}$ of V. If $t \in V$, then define $\mathscr{L} := \mathscr{L}' \cup \{t\}$. In both cases it is easily seen that $\mathscr{L}$ satisfies the requirements of the theorem. ∎

REFERENCES

1. J. EDMONDS AND E. JOHNSON, Matching: A well-solved class of linear programs, *in* "Combinatorial Structures and Their Application," pp. 89–92, Gordon & Breach, New York, 1970.
2. A. FRANK, A. SEBŐ, AND E. TARDOS, Covering directed and odd cuts, *Math. Programming Study* **22** (1984), 99–112.

3. L. LOVASZ, 2-matchings and 2-covers of hypergraphs, *Acta Sci. Math. Hungar.* **26** (1975), 433–444.
4. L. LOVÁSZ AND M. D. PLUMMER, "Matching Theory," North-Holland, Amsterdam, 1986.
5. A. SEBŐ, A quick proof of Seymour's theorem on t-joins, *Discrete Math.* **64** (1987), 101–103.
6. A. SEBŐ, The Schrijver-system of odd-join polyhedra, *Combinatorica* **8**, No. 1 (1988), 103–116.
7. P. D. SEYMOUR, The matroids with the max-flow min-cut property, *J. Combin. Theory Ser. B* **23** (1977), 189–222.
8. P. D. SEYMOUR, On odd cuts and planar multicommodity flows, *Proc. London Math. Soc. Ser. III* **42** (1981), 178–192.
9. W. T. TUTTE, The factorization of linear graphs, *J. London Math. Soc.* **22** (1947), 107–111.

Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium