Journal of Combinatorial Theory, Series B 75, 237–244 (1999)
Article ID jctb.1998.1877, available online at http://www.idealibrary.com on IDEAL®

Finding Minimum Generators of Path Systems*

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Received April 29, 1997

A simple algorithmic proof of a min–max theorem of E. Győri on generators of path systems is described. The algorithm relies on Dilworth's theorem. © 1999 Academic Press

1. INTRODUCTION

Let $P=(v_0, j_1, v_1, j_2, v_2, ..., j_n, v_n)$ be a simple directed path where each directed edge j_h has tail v_{h-1} and head v_h. Let $V:=\{v_0, ..., v_n\}$ and $E:=\{j_1, ..., j_n\}$ denote the node-set and edge-set of P, respectively. Path P defines an ordering of the elements of V in which $v_h < v_k$ if $0 \leqslant h < k \leqslant n$. In this case we will say that node v_h *precedes* node v_k and that edge j_h *precedes* edge j_k.

Let $E^* := \{uv : u, v \in V, u < v\}$ be the set of all directed edges whose tail u precedes its head v. For a subpath J of P, let $f(J)$ and $l(J)$ denote the first and last nodes of J, respectively. Also, for a (directed) edge $e = uv \in E^*$ let $f(e) := u$ and $l(e) := v$.

Let $\mathscr{P}$ be a system of distinct subpaths of P. We use the convention that the edge-set of P_i will be denoted by the same letter P_i. The node set of P_i is denoted by $V(P_i)$. We say that a system $\mathscr{G}$ of subpaths of P *generates* a path J if J is the union of some members of $\mathscr{G}$. $\mathscr{G}$ *generates* $\mathscr{P}$ or $\mathscr{G}$ is a *generator of* $\mathscr{P}$ if every member of $\mathscr{P}$ is generated by $\mathscr{G}$. For example, $\mathscr{P}$ is a generator of itself, or the system $\{j_1, ..., j_n\}$ consisting of one-element paths is also a generator of $\mathscr{P}$. Let $\gamma(\mathscr{P})$ denote the minimum cardinality of a generator of $\mathscr{P}$.

* Research supported by the Hungarian National Foundation for Scientific Research Grant, OTKA T17580.

0095-8956/99 $30.00

E. Győri [7] proved a beautiful min-max theorem concerning $\gamma(\mathscr{P})$. To formulate his result, we need the following notions. Let $J \in \mathscr{P}$ be a path and $j \in J$ an edge. We say that the pair (J, j) is a *path-edge pair*. Let $\mathscr{R}$ denote the set of all path-edge pairs (J, j) for which $j \in J \in \mathscr{P}$. Let $(J, j)^-$ denote the set of nodes of J preceding edge j and let $(J, j)^+$ denote the set of nodes of J following j. That is, $(J, j)^-$ and $(J, j)^+$ form a partition of the node-set $V(J)$ of J. A directed edge $uv \in E^*$ *covers* (J, j) if $u \in (J, j)^-$ and $v \in (J, j)^+$. A set C of directed edges *covers* $\mathscr{R}$ (or C is a *covering* of $\mathscr{R}$) if each member of $\mathscr{R}$ is covered by an element of C.

Note that there is a natural correspondence between generators of $\mathscr{P}$ and coverings of $\mathscr{R}$. Namely, by associating with each edge uv of a covering C the subpath of P from u to v, one obtains a generator of $\mathscr{P}$, and conversely, by associating with each member J of a generator $\mathscr{G}$ a directed edge from $f(J)$ to $l(J)$, one obtains a covering of $\mathscr{R}$. Therefore $\gamma(\mathscr{P})$ may be interpreted as the minimum number of directed edges covering $\mathscr{R}$.

Two path-edge pairs (I, i), (J, j) are called *independent* if $(I, i)^- \cap (J, j)^- = \emptyset$ or $(I, i)^+ \cap (J, j)^+ = \emptyset$. A set of path-edge pairs is called *independent* if its members are pairwise independent. Let $\sigma(\mathscr{P})$ denote the maximum cardinality of an independent subsystem of $\mathscr{R}$. Because two independent path-edge pairs cannot be covered by one edge, every covering of $\mathscr{R}$ must have at least $\sigma(\mathscr{P})$ edges, or equivalently, every generator of $\mathscr{P}$ has at least $\sigma(\mathscr{P})$ members, that is, $\sigma(\mathscr{P}) \leqslant \gamma(\mathscr{P})$. The theorem of Győri [7] asserts that here actually equality holds.

THEOREM 1.1 [E. Győri]. *For a family $\mathscr{P}$ of subpaths of a directed path P, the minimum cardinality of a generator of $\mathscr{P}$ is equal to the maximum cardinality of an independent set of path-edge pairs, that is,* $\gamma(\mathscr{P}) = \sigma(\mathscr{P})$.

The original proof of Győri is complicated. Although his proof is not algorithmic, its ideas could be used to construct a polynomial-time algorithm to compute the two extrema in question. This was done by D. S. Franzblau and D. J. Kleitman [6]. Their method was later extended by A. Lubiw [9] to a weighted version of Győri's theorem. The paper of D. E. Knuth [8] provides a clarified and simplified version of the algorithm of Franzblau and Kleitman along with a computer code for computing the extrema in question. In Frank and Jordán [5] a general min-max theorem was derived for the minimum number of directed edges covering a crossing bi-supermodular function. That result gave rise to further extensions of Győri's theorem. The proof-method of Frank and Jordán [5], when specialized, gives rise to a simple proof of Győri's theorem which is, however, not algorithmic. The main purpose of the present paper is to provide a short algorithmic proof of the theorem of Győri.

2. ALGORITHMIC PROOF

A member (J, j) of $\mathscr{R}$ will be called *essential* if there is no member J' $(\neq J)$ of $\mathscr{P}$ for which $j \in J' \subset J$. Let $\mathscr{E}$ denote the set of essential path-edge pairs of $\mathscr{R}$. We will say that an essential pair (J, j) is *sitting* at j.

As we have seen already that $\sigma \leqslant \gamma$, we prove only the non-trivial inequality $\sigma \geqslant \gamma$. The proof is an algorithm constructing a covering C of $\mathscr{R}$ (that is, a generator of $\mathscr{P}$) and an independent subset $\mathscr{I}$ of $\mathscr{R}$ for which $|C| = |\mathscr{I}|$.

We introduce a partial order on $\mathscr{E}$. For (I, i), $(J, j) \in \mathscr{E}$ we write

$$(I, i) \leqslant (J, j) \quad \text{if both} \quad (I, i)^- \subseteq (J, j)^- \quad \text{and} \quad (I, i)^+ \supseteq (J, j)^+ \quad \text{hold.} \tag{2.1}$$

This is indeed a partial order. We say that two members (I, i), (J, j) of $\mathscr{E}$ are *crossing* or that (I, i) *crosses* (J, j) if they are uncomparable in this partial order and they are not independent. Since (I, i) and (J, j) are essential, provided that j precedes i, this is equivalent to requiring that $\{i, j\} \subseteq I \cap J$, $f(J) < f(I)$, and $l(J) < l(I)$, (that is, $f(J) < f(I) \leqslant f(j) < f(i) < l(J) < l(I)$). A subset $\mathscr{K}$ of $\mathscr{E}$ is called *cross-free* if $\mathscr{K}$ contains no two crossing members.

The algorithm consists of three phases. In the first one we construct a cross-free subset $\mathscr{K}$ of $\mathscr{E}$. In the second phase we apply Dilworth's theorem to $\mathscr{K}$ and compute a minimum chain-partition of $\mathscr{K}$ along with a maximum subset $\mathscr{I}$ of pairwise uncomparable members of $\mathscr{K}$. The chain-decomposition will correspond to a set C of directed edges covering every member of $\mathscr{K}$. In the third phase we modify C, without changing its cardinality, so as to obtain a covering of $\mathscr{R}$.

Phase 1. Consider, one by one, the edges $j_1, ..., j_n$ of path P in this order and assign each member of $\mathscr{E}$ to one of two groups $\mathscr{K}$ and $\mathscr{T}$. Once a member is assigned to $\mathscr{K}$ or to $\mathscr{T}$, its status will never change. The final $\mathscr{K}$ and $\mathscr{T}$ will form a partition of $\mathscr{E}$.

At the beginning both groups $\mathscr{K}$ and $\mathscr{T}$ are empty. For every edge $j \in P$, let $\mathscr{E}_j$ denote the set of elements of $\mathscr{E}$ sitting at j. At a general step, when edge $j \in P$ is processed, consider each member (J, j) of $\mathscr{E}_j$ which has not been assigned to $\mathscr{T}$ and put it into $\mathscr{K}$. At the same time put every essential pair $(J', j') \in \mathscr{E}_{j'}$ into $\mathscr{T}$ for which j precedes j' and (J, j) crosses (J', j').

CLAIM. *The final $\mathscr{K}$ is cross-free.*

Proof. Suppose, indirectly, that $\mathscr{K}$ contains two crossing members (I, i) and (J, j) and assume that j precedes i. But this is impossible since then

(J, j) would be put into $\mathscr{K}$ first, and when (J, j) was put into $\mathscr{K}$ the rule of Phase 1 required putting (I, i) into $\mathscr{T}$. ∎

Phase 2. Consider the partial order defined on $\mathscr{K}$ by (2.1) and apply Dilworth's theorem to $\mathscr{K}$. We obtain that there exist an antichain $\mathscr{I} \subseteq \mathscr{K}$ and a chain-decomposition $\{\mathscr{C}_1, ..., \mathscr{C}_t\}$ of $\mathscr{K}$ so that $|\mathscr{I}| = t$. It is well known (see, for example, Ford and Fulkerson [2]) that these configurations can be efficiently computed via a bipartite matching algorithm. Since $\mathscr{K}$ is cross-free, so is $\mathscr{I}$ and therefore $\mathscr{I}$ is an independent family of path-edge pairs.

CLAIM. *The members of a chain $\mathscr{C}$ of $\mathscr{K}$ can be covered by one edge.*

Proof. By re-indexing, if necessary, we may assume that the members $(I_1, i_1), ..., (I_k, i_k)$ of $\mathscr{C}$ are indexed in such a way that $(I_1, i_1)^- \subseteq (I_2, i_2)^- \subseteq \cdots \subseteq (I_k, i_k)^-$ and $(I_1, i_1)^+ \supseteq (I_2, i_2)^+ \supseteq \cdots \supseteq (I_k, i_k)^+$. Let $e \in E^*$ be a directed edge from $f(I_1)$ to $l(I_k)$. Then e covers every member of $\mathscr{C}$. ∎

By the claim each chain $\mathscr{C}_i$ $(i = 1, ..., t)$ can be covered by an edge e_i. Then $C := \{e_1, ..., e_t\}$ covers every member of $\mathscr{K}$ and $|C| = |\mathscr{I}|$.

Phase 3. We say that two elements $f_1 = x_1y_1$, $f_2 = x_2y_2$ of C are *exchangeable* if $x_2 < x_1 < y_1 < y_2$ and

$$C' = C - \{f_1, f_2\} \cup \{f'_1, f'_2\} \tag{2.2}$$

is also a covering of $\mathscr{K}$ where $f'_1 := x_2y_1, f'_2 := x_1y_2$. Replacing C by C' is called an *exchange step*. Phase 3 consists of applying the exchange step as long as there are exchangeable members of the current covering of $\mathscr{K}$.

For an edge $f = xy$ let $h(f)$ denote the number of edges of the underlying path P between x and y. It is easy to see that $\sum (h^2(f) : f \in C') < \sum (h^2(f) : f \in C)$. Therefore Phase 3 terminates after at most $|C| |P|^2 \leqslant |P|^3$ applications of the exchange step.

At termination we are left with a covering C of $\mathscr{K}$ possessing no exchangeable edges. The crucial point of the present proof of Győri's theorem is the fortunate fact that such a covering of $\mathscr{K}$ will automatically be a covering of the whole set $\mathscr{R}$ of path-edge pairs.

LEMMA. *If C is a covering of $\mathscr{K}$ with no exchangeable edges, then C covers $\mathscr{R}$.*

Proof. Suppose indirectly that C does not cover a member (I, i) of $\mathscr{R}$. Let us choose I and i so that $|(I, i)^-|$ is minimum, and, subject to that, $|(I, i)^+|$ is minimum.

First, observe that (I, i) is essential. For if there is a path $J \in \mathscr{P}$ with $i \in J \subset I$, then the minimal choice of $(I, i)^-$ and $(I, i)^+$ implies that C covers (J, i). Since $(J, i)^- \subseteq (I, i)^-$ and $(J, i)^+ \subseteq (I, i)^+$, we obtain that C covers (I, i), as well, contradicting the assumption that (I, i) is not covered by C.

Since C covers $\mathscr{K}$, (I, i) does not belong to $\mathscr{K}$. By the rule of Phase 1, there is a member (J, j) of $\mathscr{K}$ for which j precedes i and (J, j) crosses (I, i). Let us choose (J, j) in such a way that $|(J, j)^+|$ is minimum. Since both (I, i) and (J, j) are essential, so are (I, j) and (J, i). It follows from the minimality of $|(I, i)^-|$ that C covers (I, j), that is, there is an edge $e_1 = x_1y_1 \in C$ covering (I, j).

CLAIM. $(J, i) \in \mathscr{K}$.

Proof. If, indirectly, (J, i) belongs to $\mathscr{T}$, then the rule of Phase 1 implies the existence of a member (J', j') of $\mathscr{K}$ so that j' precedes i and (J', j') crosses (J, i). j' cannot precede j for otherwise (J', j') would cross (J, j) and hence (J, j) would belong to $\mathscr{T}$. Therefore either $j' = j$ or else j precedes j'. In both cases (J', j') crosses (I, i) and $(J', j')^+ \subset (J, j)^+$, contradicting the minimal choice of $|(J, j)^+|$. ∎

Since C covers $\mathscr{K}$ and $(J, i) \in \mathscr{K}$ by the Claim, there is an element $e_2 = x_2y_2$ of C covering (J, i). Since neither e_1 nor e_2 covers (I, i), we have

$$f(J) \leqslant x_2 < f(I) \leqslant x_1 \leqslant f(j) < y_1 \leqslant f(i) < y_2 \leqslant l(J) < l(I).$$

By the hypothesis of the lemma, e_1 and e_2 are not exchangeable, that is, there is a member (K, k) of $\mathscr{K}$ which is not covered by C' where C' is defined in (2.2). Then (K, k) must be covered by e_1 but not covered by e'_1 and e'_2. Hence $f(J) \leqslant x_2 < f(K) \leqslant x_1 < y_1 \leqslant l(K) < y_2 \leqslant l(J)$, that is, $j \in K \subset J$, contradicting that (J, j) is essential. This contradiction proves the lemma. ∎

Associate with each edge uv in C the subpath of P from u to v and let $\mathscr{G}_C$ denote the family of these associated subpaths. As we have noted in the Introduction, $\mathscr{G}_C$ is a generator of $\mathscr{P}$ and $|\mathscr{G}_C| = |\mathscr{I}|$, as required for Győri's theorem. ∎

3. CONCLUSIONS: COMPLEXITY AND EXTENSIONS

The constructive proof above for Győri's theorem gives rise to a polynomial time algorithm for computing a minimum generator of a path system and a maximum independent family of path-edge pairs. To obtain a bound on the complexity, let us consider the three phases separately. Observe first that for indices i, h $(0 \leqslant i < h \leqslant n)$ there is at most one path J of $\mathscr{P}$ with

$f(J) = v_i$ for which (J, j_h) is essential. Therefore the set $\mathscr{E}_{j_h}$ of essential pairs sitting at an edge j_h has at most $h \leqslant n$ members and hence $|\mathscr{E}| \leqslant n^2$.

It is not difficult to compute $\mathscr{E}_{j_h}$ in $O(n^2)$ steps and hence $\mathscr{E}$ can be determined in $O(n^3)$ steps. Since $|\mathscr{E}_{j_h}| \leqslant n$, a stage of Phase 1, when a specified edge of P is processed, can be carried out in $O(n^2)$ steps and hence the whole $\mathscr{K}$ can be computed in $O(n^3)$ steps.

Ford and Fulkerson [2] showed that a bipartite matching algorithm, applied to a bipartite graph on $2p$ nodes, can be used to compute a minimum chain-decomposition and a maximum cardinality antichain of a partially ordered set on p elements. Since there are matching algorithms of complexity $O(p^{2.5})$ and in our case $p := |\mathscr{K}| \leqslant n^2$, Phase 2 terminates in $O(n^5)$ steps. There is, however, a tighter bound for the complexity of Phase 2. A. Benczúr, J. Förster, and Z. Király [1] proved by a clever counting argument that any cross-free set of essential pairs, and hence the cross-free set $\mathscr{K}$ computed by Phase 1, can have at most $n \log n$ members. Therefore Phase 2 terminates in $O((n \log n)^{2.5}) \leqslant O(n^3)$ steps.

As we have noted already, Phase 3 needs at most $O(n^3)$ exchange steps, and hence we can conclude that the overall complexity of the algorithm can be estimated by $O(n^3)$.

Benczúr *et al.* [1] actually provided a detailed description of the algorithm, made further improvements, gave a careful analysis of the running time, and obtained a bound of $O(nm + n^2 \sqrt{n \log n})$ for the complexity where $m = |\mathscr{P}|$. Their paper also reports on computational results along with comparisons to Knuth's algorithm.

Before turning to extensions, let us mention that Győri's theorem has a surprising application in combinatorial geometry. (As a matter of fact, this problem was the starting point of investigations of Győri.)

COROLLARY. *Suppose we are given a (bounded) region R in the plane bounded by horizontal and vertical segments which is vertically convex in the sense that each vertical straight line intersects R in an interval. Then the minimum number of rectangles (of horizontal-vertical sides) whose union is R is equal to the maximum number of points of R such that no two of them can be covered by a rectangle belonging to R.*

Since the derivation is simple and algorithmic, the algorithm described above may be used to compute the optima in the corollary. As far as possible extensions are concerned, it was shown in Frank and Jordán [5] that Győri's theorem is a special case of a general framework concerning a crossing family $\mathscr{F}$ of pairs (!) of sets. Namely, we proved the following.

THEOREM 3.1. *Given a crossing family $\mathscr{F}$ of pairs of subsets and a crossing bi-supermodular function p on $\mathscr{F}$, the minimum number of directed edges covering each member $X = (X^-, X^+)$ of $\mathscr{F}$ at least $p(X)$ times is equal to $\max_{\mathscr{I} \subseteq \mathscr{F}} (\sum (p(X) : X \in \mathscr{I}))$ where $\mathscr{I}$ is a subset of pairwise independent pairs. In particular (when $p \equiv 1$), the minimum number of edges covering each member of $\mathscr{F}$ is equal to the maximum number of pairwise independent pairs of $\mathscr{F}$.*

This includes not only Győri's theorem but its extensions, as well. For example, suppose that, in addition to the underlying path P and the path system $\mathscr{P}$, we are given another system $\mathscr{G}_0$ of subpaths of P. We want to find a minimum generator of $\mathscr{P}$ provided that the members of $\mathscr{G}_0$ can be used freely. Then (already the second part of) Theorem 3.1 implies:

THEOREM 3.2. *Given two families $\mathscr{P}$ and $\mathscr{G}_0$ of subpaths of a directed path P, the minimum number of subpaths of P whose addition to $\mathscr{G}_0$ yields a generator of $\mathscr{P}$ is equal to the maximum cardinality of an independent set of path-edge pairs which are not covered by $\mathscr{G}_0$.*

When $\mathscr{G}_0$ is empty, we are back at Győri's theorem. With a slight modification of the algorithmic proof we described above for Győri's theorem, one can easily prove this extension, as well. Namely, the set $\mathscr{E}$ of essential path-edge pairs should be replaced by those essential path-edge pairs which are not covered by $\mathscr{G}_0$.

Theorem 3.1 actually implies more general forms of Theorem 3.2. One possible extension is when the underlying directed path P is replaced by a directed circuit. A further one is when there is a certain weight function on path-edge pairs (including as a special case results of A. Lubiw [9]). By extending the ideas above we provided an algorithmic solution to these extensions in Frank [4] but this solution is significantly more complicated than the one above.

Another important application of Theorem 3.1 provides a min-max formula for the minimum number of new edges whose addition to a given directed graph G results in a K-connected directed graph. One of the main motivations behind our investigations was to find a combinatorial algorithm for computing the minimum. Although this is still open in general, in the special case, however, when the starting digraph G is $K-1$ connected, it was possible to construct such an algorithm (Frank [3]).

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Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium