Note on the Path-Matching Formula

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Abstract: As a common generalization of matchings and matroid intersections, W.H. Cunningham and J.F. Geelen introduced the notion of path-matchings. They proved a min-max formula for the maximum value. Here, we exhibit a simplified version of their min-max theorem and provide a purely combinatorial proof. © 2002 Wiley Periodicals, Inc. J Graph Theory 41: 110–119, 2002

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1. INTRODUCTION

W.H. Cunningham and J.F. Geelen [2 and 3] introduced the notion of path-matchings as a common generalization of the weighted matching problem and the weighted matroid intersection problem.They proved that this problem is solvable in polynomial time through the ellipsoid method [7]. They also proved the total dual integrality of thecorresponding linear system.

Cunningham and Geelen defined a path-matching as follows. Let $G = (V, E)$ be an undirected graph and T_1, T_2 disjoint sets of G; we call these two sets the terminal sets of G. We denote $V - (T_1 \cup T_2)$ by R. Let M_1 and M_2 be two matroids on T_1 and T_2, respectively. An *independent path-matching* with respect to M_1, M_2 is a set K of edges such that every component ofthe subgraph $G_K = (V, K)$ having at least one edge is a simple path from $T_1 \cup R$ to $T_2 \cup R$, all of whose internal nodes are in R, and such that the set of nodes of T_i in any of these paths is independent in M_i, for $i = 1$and 2. The one-edge components in R are called the *matching edges* of K. The *value* of a path-matching K is defined to be the number $val(K) = |K| + |K'|$, where K' denotes the set of the matching edges of K. (That is, the matching edges count twice.)

A *basic path-matching* is a set K of edges such that the subgraph $G_K = (V, K)$ is a collection of r disjoint paths, all of whose internal nodes are in R, linking a basis of M_1 to a basis of M_2, together with a perfect matching of the nodes of R not in any of the paths. That is, the rank of M_1 and M_2 are the same and equals to r. Note that the value of a basic path-matching is $r + |R|$.

If M_1 and M_2 are free matroids, then we refer to an independent path-matching as a *path-matching* and to a basic path-matching as a *perfect path-matching* (then $|T_1| = |T_2| = r$).

A pair of subsets $I_1 \subseteq T_1 \cup R$, $I_2 \subseteq T_2 \cup R$ is called *stable*, if no edge of G joins a node in $I_1 - I_2$ to a node in I_2 or a node in $I_2 - I_1$ to a node in I_1. Let $c(G)$ denote the number of components of G having an odd number of nodes. For a subset S of nodes of G, $G[S]$ denotes the subgraph of G induced by S. Throughout the study, we do not distinguish between a set ofcardinality one and its only element.

Cunningham and Geelen proved the following min-max formula for the maximum value of a path-matching.

Theorem 1.1. *For the maximum value of a path-matching one has the following formula:*

$$\max_{M,\ a\ path\text{-}matching} val(M) =$$

$$\min_{(I_1,I_2)\ a\ stable\ pair} |T_1 \cup R - I_1| + |T_2 \cup R - I_2| + |I_1 \cap I_2| - c(G[I_1 \cap I_2]).$$

They proved the following theorem as a consequence of the min-max formula.

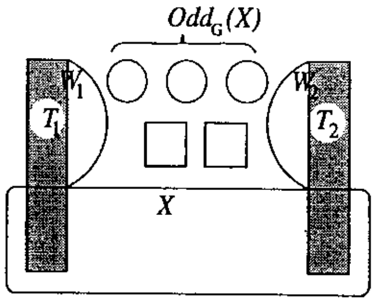

FIGURE 1. A cut X separating T_1 and T_2.

Theorem 1.2. *If $|T_1| = |T_2| = k$, then there exists a perfect path-matching if and only if*

$$|I_1 \cup I_2| + c(G[I_1 \cap I_2]) \leq |R| + k \quad \textit{for all stable pairs } (I_1, I_2).$$

We define a *cut* separating the terminal sets T_1 and T_2 to be asubset $X \subseteq V$ such that there is no path between $T_1 - X$ and $T_2 - X$ in $G - X$. (See Fig. 1)

Let $odd_G(X)$ denote the number of connected components of $G - X$ which are disjoint from $T_1 \cup T_2$ and have anodd number of nodes, and let $Odd_G(X)$ denote the union of these components. In this note, we provide a simplified characterization for the existence of a perfect path-matching, which is a direct extension of Tutte's theorem onperfect matchings. It allows us to provide a combinatorial proof by mimicking Anderson's simple proof of Tutte's theorem [1]. (Cunningham and Geelen gave two proofs for their min-max theorem. The first one uses the Tutte-matrix, while the second approach relies on polyhedral methods.)

Theorem 1.3. *In $G = (V, E)$ there exists a perfect path-matching if and only if $|T_1| = |T_2| = k$ and*

$$|X| \geq odd_G(X) + k \quad \textit{holds for all cuts } X. \tag{1}$$

As a corollary, we are going to deduce the following simplified version of Theorem 1.1.

Theorem 1.4. *For the maximum value of a path-matching one has the following formula:*

$$\max_{M,\ a\,path\text{-}matching} val(M) = |R| + \min_{X,\ a\ cut} (|X| - odd_G(X)). \tag{2}$$

When $T_1 = T_2 = \emptyset$, Theorems 1.3 and 1.4 specialize to Tutte's theorem and the Berge–Tutte formula, respectively.

One can easily derive the following formula [5, 8] for the rank function r of the matching matroid defined by G.

Theorem 1.5. *Let $G = (V, E)$ be a graph and R an arbitrary subset of V. For the maximum size $r(R)$ of a subset of R covered by a matching of G one has*

$$r(R) = |R| + \min_{X \subseteq V}(|X| - odd_G(X)). \tag{3}$$

Proof. Let $T_1 = V - R$ and $T_2 = \emptyset$. For this special choice, it can easily be seen that there is a maximum value path-matching M so that each path of M consists of one edge, i.e., M is a matching on V. The value of such a path-matching is the number of its (one-element) paths plus twice the number of its matching edges. That is, $val(M)$ is exactly the number of nodes in R covered by M. Therefore, Theorem 1.4 implies equation (3). ■

Cunningham and Geelen showed in [3] that Menger's theorem on the number of node-disjoint paths can also be derived from Theorem 1.2 by a simple elementary construction. This derivation is even easier if Theorem 1.3 is used rather than Theorem 1.2.

Note that Theorem 1.4 may be interpreted as a more specific version of Theorem 1.1, which asserts that the minimum in Theorem 1.1 is always attained at a stable pair (I_1, I_2) arising from a cut X by taking $I_1 = V - X - W_2$ and $I_2 = V - X - W_1$, where W_i denotes the set of nodes of $G - X$ reachable from $T_i - X$ $(i = 1, 2)$.

2. PROOFS

A cut X is called *trivial* if $X = T_1$ or $X = T_2$. A cut X is defined to be *tight* if $|X| = odd_G(X) + k$, that is, condition (1) is satisfied by equality.

A graph $G = (V, E)$ is said to be *factor-critical*, if it is connected and each node is missed by a maximum matching.

Lemma 2.1 (Gallai's lemma [6]). *If $G = (V, E)$ is factor-critical, then $|V|$ is an odd number and a maximum matching of G has cardinality* $(|V| - 1)/2$.

It follows directly from Tutte's theorem that

$$\begin{aligned}&\text{a connected } G \text{ is factor-critical if and only if}\\ &odd_G(Y) \leq |Y| - 1 \ \forall\, Y \subseteq V,\ |Y| \geq 1.\end{aligned} \tag{4}$$

The following is an easy corollary of Gallai's lemma for a factor-critical graph.

$$\begin{aligned}u, v \in V \Longrightarrow\ &\text{there exists a } u,v\text{-path such that}\\ &\text{there exists a perfect matching on the nodes not in the path.}\end{aligned} \tag{5}$$

Let $|T_1| = |T_2| = k$. We call a set K of edges a *nearly perfect path-matching*, if the subgraph $G_K = (V, K)$ is a collection of k disjoint paths linking T_1 to T_2, together with even cycles and one-edge-component edges covering all the nodes of R not in any of the k paths. The following claim is straightforward from this definition.

Claim 2.1. *G has a perfect path-matching if and only if G has a nearly perfect path-matching.*

Proof of Theorem 1.3. Necessity of inequality (1). Let us consider a perfect path-matching M consisting of k paths $P_1, P_2, \ldots, P_k$. Let α be the number of the components of $Odd_G(X)$ that are traversed by some P_i, and let $\beta := odd_G(X) - \alpha$. For a path P_i, let t_i denote the number of components of $Odd_G(X)$ which are traversed by P_i.
Now we have

$$k + odd_G(X) = k + \alpha + \beta \leq \sum_{i=1}^{k}(t_i + 1) + \beta \leq |X|, \tag{6}$$

for all cuts X, since orienting each P_i from T_1 to T_2, there is a first node on P_i in X, furthermore after traversing an (odd) component P_i enters X again.

For any tight cut X, we have equality everywhere in Equation (6). This means that M and X have the following properties: any component C of $Odd(X)$ is either traversed by one path P_i and $C \cap P_i$ is connected, or there is exactly one matching edge with one end-node in C and the other in X; there is no edge of M spanned by X and the even components of $G - X$, which are disjoint from T_1 and T_2 are avoided by any path P_i of M. Furthermore, for any path P_i of M, the intersection of W_1 and P_i is connected orempty (same is for W_2).

The proof of sufficiency goes by induction on $|R| + |E|$. If $|R| = 0, |E| \leq 1$, the theorem is obviously true.

Case 1. Every tight cut is trivial.

If $k = 0$, i.e., $T_1 = T_2 = \emptyset$, then $R \neq \emptyset$. For cut $X = \emptyset$ by condition (1), $0 \geq 0 + odd_G(X)$, hence G has an even number of nodes. So every node of R is a nontrivial tight cut. Hence $k > 0$. Let us consider an edge $e = uv$ with $u \in T_1$. Let G' denote $G - e$. If condition (1) is satisfied in G', then we are done by induction. Suppose now that G' does not satisfy (1), i.e., there is a cut X in G' so that $|X| < odd_{G'}(X) + k$. Since $|X| \geq odd_G(X) + k$, $u \in T_1 - X$ and either v is in an odd component of $G' - X$ or else v is in a path from $T_1 - X$ to $T_2 - X$. In the first case, $odd_G(X) + k \leq |X| < odd_{G'}(X) + k = odd_G(X) + 1 + k$, so $|X| = odd_G(X) + k$, and X is tight. Furthermore, $X + u$ is also tight. At least one of these two tight cuts is nontrivial, contradicting our assumption that every tight cut is trivial (Fig. 2a).

In the second case $|X + u| \geq odd_G(X + u) + k = odd_{G'}(X) + k \geq |X| + 1$, and hence $|X + u| = odd_G(X + u) + k$, that is, $X + u$ is a tight cut. An analogous argument shows that $X + v$ is also a tight cut.

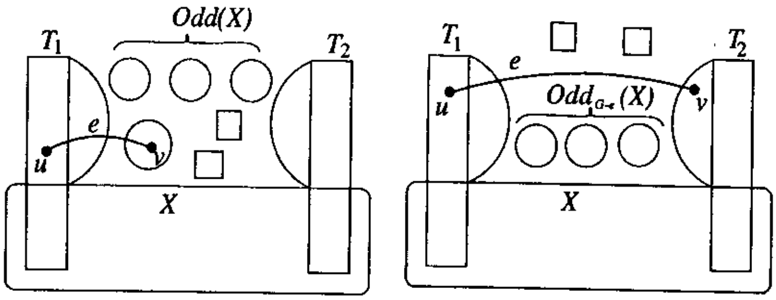

FIGURE 2a and 2b.

By the assumption, both $X + u$ and $X + v$ are trivial cuts, it follows that $X = \emptyset$ and $T_1 = \{u\}$, $T_2 = \{v\}$. Since $Y = \{u, v\}$ is not tight, for any component C of $G - Y$ has even cardinality. We claim that C has a perfect matching which implies that G has a perfect path-matching. For any subset Z of the node-set of C, $Z \cup Y$ is a nontrivial cut of G, hence the number of odd components of $C - Z$ is at most $|Z|$, and we are done by Tutte's theorem. (Our proof uses Tutte's theorem but by induction, we could avoid doing so.)

Case 2. There exists a nontrivial tight cut.
Let us consider a maximal nontrivial tight cut X.

Claim 2.2. *Each component of $G - X$ lying entirely in R is factor-critical.*

Proof. If a component C has an even number of nodes, then $X + v$ is also a tight cut for any node $v \in C$ contradicting the maximality of X. If C has an odd number of nodes, then let $Y \subseteq C$ be a subset with maximal value of $odd_C(Y) - |Y|$. Since $X \cup Y$ is a nontrivial tight cut, $Y = \emptyset$, hence C is factor-critical according to Equation (4). ■

Let us contract each component of $Odd_G(X)$ to a node. Let G_Q denote the graph obtained this way. Q denotes the set of new nodes. Let $R_Q := R - Odd(X) \cup Q$.

Claim 2.3. *If G_Q has a perfect path-matching, then so has G.*

Proof. Let K_Q denote a perfect path-matching of G_Q. Let K denote the set of edges of G corresponding to K_Q. We claim that K can be completed in G to be a perfect path-matching. To this end, let C denote a component of $Odd_G(X)$, and let c denote its corresponding node in G_Q. By Claim 2.2, C is factor-critical.

If K_Q covers c by a matching egde, then K covers one node, say v, of C, and by Gallai's lemma, there is a perfect matching on $C - v$. If K_Q covers c by a path, then K covers either one node v of C or two distinct nodes, say u and v, of C. In the first case, Gallai's lemma applies again, while in the second one, by Equation (5), there is a path P in C connecting u and v and a perfect matching on $C - V(P)$, where $V(P)$ denotes the nodes of P. ■

We are going to show that G_Q has a perfect path-matching. Recall the definition of sets W_1, W_2 after Theorem 1.5. Let $G^l = (V^l, E^l)$, where $V^l := V - ((X \cap T_1) \cup W_2 \cup Odd_G(X)) \cup Q$ and E_l is the set of the induced edges after the deletions and contractions. Let $T_1^l := (T_1 - X) \cup Q, T_2^l := X - T_1$. Similarly, let $G^r = (V^r, E^r)$, where $V^r := V - ((X \cap T_2) \cup W_1 \cup Odd_G(X)) \cup Q$ and E_r is the set of the induced edges after the deletions and contractions. Let $T_1^r := X - T_2, T_2^r := (T_2 - X) \cup Q$ (See Fig. 3). Note that these two graphs may have nodes in common. Let $R^l := V^l - (T_1^l \cup T_2^l)$. R^r is defined similarly.

Claim 2.4. *G^l has a perfect path-matching M_1 with respect to the terminal sets T_1^l and T_2^l.*

Proof. By definition $|T_1^l| = |T_1| - |T_1 \cap X| + |Q|$, $|T_2^l| = |X| - |T_1 \cap X|$. We have $|T_1^l| = |T_2^l|$, since X is tight, i.e., $|X| = odd_G(X) + k$ (recall that $|T_1| = k$ and $|Q| = odd_G(X)$). Let $k_l := |T_1^l|$.

We claim that $|Y| \geq odd_{G^l}(Y) + k_l$ for every cut Y of G^l. Let Y be a cut of G^l. Notice that $Z := (Y - Q) \cup (T_1 \cap X)$ is a cut of G, hence $|Z| \geq odd_G(Z) + k$ (see Fig. 4). Since the nodes of $Q - Y$ are isolated in $G - Z$, $odd_G(Z) \geq odd_{G^l}(Y) + |Q - Y|$. Hence

$$|Y| - |Q \cap Y| + |T_1 \cap X| = |Z| \geq odd_G(Z) + k \geq odd_{G^l}(Y) + |Q - Y| + k.$$

That is,

$$\begin{aligned}|Y| \geq |Q \cap Y| - |T_1 \cap X| + odd_{G^l}(Y) + |Q - Y| + k = \\ odd_{G^l}(Y) + (k - |T_1 \cap X| + |Q|) = odd_{G^l}(Y) + k_l.\end{aligned}$$

Since X is nontrivial, either $X \cap R \neq \emptyset$ or $T_1 \cap X \neq \emptyset$, hence $|R^l| + |E^l| < |R| + |E|$. Consequently, by induction, G^l has a perfect path-matching with respect to T_1^l and T_2^l. ∎

Analogously, G^r has a perfect path-matching M_2 with respect to T_1^r and T_2^r. We claim that $M_1 \cup M_2$ is a nearly perfect path-matching in G_Q. Indeed, every node of $R^l \cup R^r \cup Q \cup (X \cap R) = R_Q$ either has degree 2 in $M_1 \cup M_2$ or it is covered by a matching edge. Furthermore, there is no odd cycle and every path has one

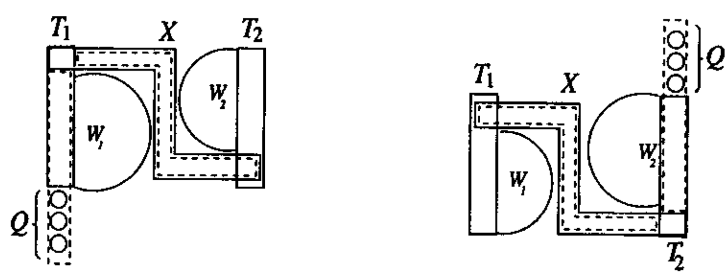

FIGURE 3. G^l and G^r.

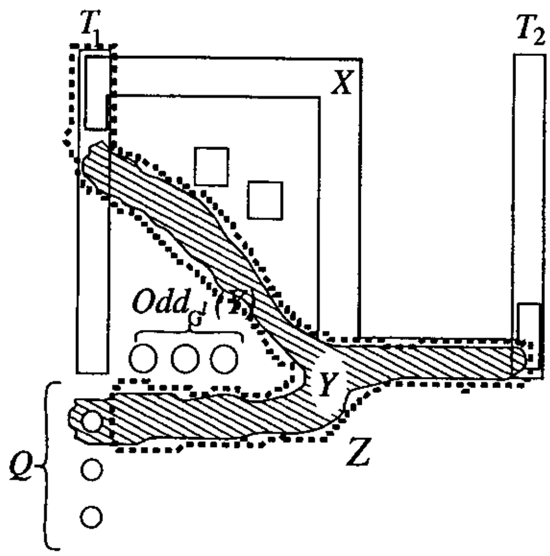

FIGURE 4.

end-node in T_1 and the other in T_2. By Claims 2.1 and 2.3, G has a perfect path-matching. (See Fig. 5) ∎

Proof of Theorem 1.4. First we show for each cut X that any path-matching M has value at most $|R| + |X| - odd_G(X)$. This can be done by a refinement of the argument used to prove the necessity of condition (1), but we exhibit a shorter inductive way. If X is empty, then there is no path between T_1 and T_2 and the statement is obviously true in this case. Let v be an element of X. If $v \in T_1 \cup T_2$, then in $G - v$, we have $val(M') \leq |R| + |X - v| - odd_{G-v}(X) = |R| + |X| - odd_G(X) - 1$ for any path-matching M' by induction. Let M^* denote the path-matching obtained from M by deleting node v. Since $val(M) \leq val(M^*) + 1$, $val(M) \leq |R| + |X| - odd_G(X)$. If $v \in R$, then in $G - v$, we have for any path-matching M' by induction $val(M') \leq |R - v| + |X - v| - odd_{G-v}(X) = |R| + |X| - odd_G(X) - 2$ for any path-matching M' by induction. Let M^* denote the

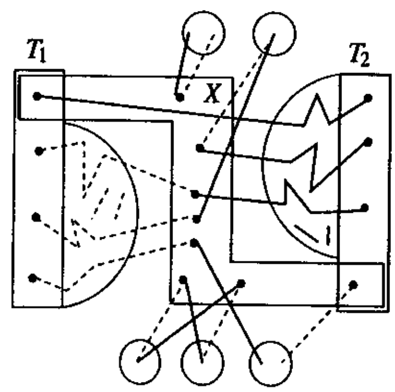

FIGURE 5. Combining M_1 and M_2.

path-matching obtained from M by deleting node w. Since $val(M) \leq val(M^*)+2$, $val(M) \leq |R| + |X| - odd_G(X)$.

Now we prove that there is a cut and a path-matching for which equality holds in Equation (2). Suppose that $|T_1| = l \geq k = |T_2|$, and let $m := \min_{X,\, a\ cut} (|X| - odd_G(X))$. Let T_2' be a set obtained from T_2 by adding $l-k$ new nodes each of which is connected by an edge with every node in $R \cup T_1$. Let R' be a set obtained from R by adding $k-m$ new nodes each of which is connected by an edge with every node in $R' \cup T_2' \cup T_1$. We added $(l-k)+(k-m) = l-m$ new nodes to G. Let G' denote the graph obtained this way.

Since T_2 is a cut in G, $m \leq k$. If $m = k$, then $|Y| - odd_{G'}(Y) \geq l$ holds, since any cut Y in G' includes either T_1 or $T_2' - T_2$. If $m < k$, then any cut Y in G' includes either T_1 or T_2' or $(T_2' - T_2) \cup (R' - R)$, hence $|Y| - odd_{G'}(Y) \geq l$ holds.

A cut X of G together with the new nodes $((T_2' - T_2) \cup (R' - R))$ form a cut of G'. So $\min_{Y,\, a\ cut} |Y| - odd_G(Y) = m + (l-m) = l$, and by Theorem 1.3, there exists a perfect path-matching M' in G', i.e., the value of M' is $val(M') = |R'| + l = |R \cup (R' - R)| + l = |R| + (k-m) + l$. $E \cap M'$ is a path-matching in G with value $val(E \cap M') = (|R| + k - m + l) - |T_2' - T_2| - 2|R' - R| = (|R| + k - m + l) - (l-k) - 2(k-m) = |R| + m = |R| + \min_{X,\, a\ cut}(|X| - odd_G(X))$. ■

Finally, we mention that a corresponding min-max formula for the maximum value of an independent path-matching is as follows. Let r_1 and r_2 denote the rank-function of M_1 and M_2.

Theorem 2.1. *The maximum value of an independent path-matching is equal to*

$$\min_{X,\, a\ cut} r_1(T_1 \cap X) + r_2(T_2 \cap X) + |R \cap X| - odd_G(X).$$

This formula can be proved by using standard matroidal techniques along with the above proofs.Theorem 2.1 contains as a special case Edmonds' theorem on the maximum cardinality of a common independent set of two matroids [4].

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