Editorial: The Harold W. Kuhn Award

NRL has recently established a new "best paper award" sponsored by IBM, General Motors, LogicTools, the MIT Leaders for Manufacturing Program, ONR, SAP, and Wiley. The purpose of the award is to recognize and reward outstanding research published in NRL.

To inaugurate the award, a committee consisting of Professors Egon Balas, Tom Magnanti, George Nemhauser, and Dr. Alan J. Hoffman (chair) was established *to select the best paper representing the journal from among all papers published in the journal since its founding in 1954 and to name the award after the first recipient.*

We are pleased to report that the committee selected the paper **"The Hungarian Method for the Assignment Problem" by Harold W. Kuhn (NRL, pp. 83–97, vol. 2, nos. 1 & 2, March–June 1955).** According to the award citation:

> *"This pioneering paper set a style for both content and exposition of many other algorithms in combinatorial optimization, and also launched and inspired the primal–dual algorithm for more general linear optimization problems. The journal is also pleased to recall that the research was sponsored by the Office of Naval Research Logistics Project at Princeton University. Professor Kuhn's enduring contributions to optimization, discrete and continuous, linear and nonlinear, from its earliest days in the 1950's, are legendary. He is a man who was in the right place, at the right time, with the right stuff."*

The award was presented in the recent INFORMS meeting during the general INFORMS Welcome and Award ceremony, Monday October 25, 2004.

This issue of the journal is celebrating the establishment of the award by publishing:

- A review article by Dr. András Frank. The article relates Professor Kuhn's elegant method to earlier work by two Hungarian mathematicians and identifies the influence of Kuhn's algorithm on the area of Combinatorial Optimization.
- Comments from Professor Kuhn on the origin of the research that led to the paper.

We are also pleased to republish the winning paper.

Starting in 2005, the annual **Harold W. Kuhn** award will be presented for the *best paper published in the journal in the last three years.* The prize includes a monetary award, as well as a certificate acknowledging the award. Papers will be recommended by members of the editorial board, and the winning paper will be selected by a three-person committee.

We trust that the award will help NRL increase its reputation and the quality of papers published in the journal.

David Simchi-Levi

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On Kuhn's Hungarian Method—A Tribute from Hungary

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Abstract: Harold W. Kuhn, in his celebrated paper entitled "The Hungarian Method for the assignment problem" [Naval Res Logist Quart 2 (1955), 83–97] described an algorithm for constructing a maximum weight perfect matching in a bipartite graph. In his delightful reminescences ["On the origin of the Hungarian method," History of mathematical programming—a collection of personal reminiscences, J.K. Lenstra, A.H.G. Rinnooy Kan, and A. Schrijver (Editors), CWI, Amsterdam and North-Holland, Amsterdam, 1991, pp. 77–81], Kuhn explained how the works (from 1931) of two Hungarian mathematicians, D. König and E. Egerváry, had contributed to the invention of his algorithm, the reason why he named it the Hungarian Method. (For citations from Kuhn's account as well as for other invaluable historical notes on the subject, see A. Schrijver's monumental book [Combinatorial optimization: Polyhedra and efficiency, Algorithms and Combinatories 24, Springer, New York, 2003].) In this note I wish to pay tribute to Professor H.W. Kuhn by exhibiting the exact relationship between his Hungarian Method and the achievements of König and Egerváry, and by outlining the fundamental influence of his algorithm on Combinatorial Optimization where it became the prototype of a great number of algorithms in areas such as network flows, matroids, and matching theory. And finally, as a Hungarian, I would also like to illustrate that not only did Kuhn make use of ideas of Hungarian mathematicians, but his extremely elegant method has had a great impact on the work of a next generation of Hungarian researchers. © 2004 Wiley Periodicals, Inc. Naval Research Logistics 52: 2–5, 2005.

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1. RELATIONSHIPS

A little technicality: Though both Egerváry and Kuhn used matrix terminology, here I follow König by working with the equivalent bipartite graph formulation.

Let us start with a quotation from Kuhn's paper [17, page 83]: "One interesting aspect of the algorithm is the fact that it is latent in the work of D. König and E. Egerváry that predates the birth of linear programming by more than 15 years (hence the name of *Hungarian Method*)." But what is the exact relationship of the algorithm arising from Egerváry's proof technique and Kuhn's method? In a paper [11], written in Hungarian, I exhibited in detail the achievements of König and Egerváry and Kuhn. The following section is an outline of some observations from [11].

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1.1. König's Theorem and Proof Method

The starting point is König's matching theorem [16].

THEOREM 1 (König): In a bipartite graph $G = (S, T; E)$, the maximum cardinality $\nu = \nu(G)$ of a matching is equal to the minimum number $\tau = \tau(G)$ of nodes covering all the edges.

It is useful to restate the theorem in an equivalent form.

THEOREM 2: In a bipartite graph $G = (S, T; E)$, the minimum number $\mu = \mu(G, S)$ of elements of S exposed by a matching is equal to the maximum of the deficit $h(X)$ over the subsets of S, where $h(X) := |X| - |\Gamma(X)|$ and $\Gamma(X)$ denotes the set of elements of T having a neighbor in X. In particular, there is a matching covering S if and only if $|\Gamma(X)| \geq |X|$ holds for every subset $X \subseteq S$.

The outline (in modern terms) of König's constructive proof for the nontrivial $\nu \geq \tau$ direction is as follows. By starting with any matching M, orient the edges in M from T to S and all other edges from S to T. Let R_S and R_T denote the set of nodes of S and of T, respectively, exposed by M.

Let Z denote the set of nodes of the resulting directed graph which can be reached from R_S by a directed path. If $R_T \cap Z \neq \emptyset$, then we have a path P from R_S to R_T that alternates in M and then the symmetric difference of M and P is a matching M' with $|M'| = |M| + 1$. (Technically, one must simply reorient the edges of P in order to obtain the digraph corresponding to M'.) If $R_T \cap Z = \emptyset$, then $L := (T \cap Z) \cup (S - Z)$ is a set of nodes covering all edges and $|M| = |L|$. In the alternative version of König's theorem above, $Z \cap S$ is a subset of S with maximum deficiency.

The following observation will be useful in estimating the efficiency of Egerváry's method. We call a subset $X \subseteq S$ **deficient** if $h(X) > 0$. Let $\mathcal{F} := \{X \subseteq S : h(X) = \mu(G, S)\}$, that is, $\mathcal{F}$ denotes the family of the subsets of S with maximum deficit. The members of $\mathcal{F}$ are called **max-deficient** sets. It can be shown that $\mathcal{F}$ is closed under union and intersection. Therefore, if $\mathcal{F}$ is not empty, there is a unique smallest max-deficient set, and in the constructive proof of König's theorem above, the max-deficient set $Z \cap S$, provided by the algorithm, itself is this unique smallest set.

1.2. Egerváry's Theorem and Proof Method

In 1931 Egerváry [6] extended König's results to weighted bipartite matchings. His fundamental min-max result is as follows.

THEOREM 3: Let $G = (S, T; E)$ be a complete bipartite graph with $|S| = |T|$ and let $c : E \to \mathbf{Z}_+$ be a nonnegative integer-valued weight function. The maximum weight of a perfect matching of G is equal to the minimum weight of a nonnegative, integer-valued, weighted-covering of c where a *weighted-covering* is a function $\pi : S \cup T \to \mathbf{R}$ for which $\pi(u) + \pi(v) \geq c(uv)$ for every edge $uv \in E$ and the weight of π is defined to be $\Sigma[\pi(v) : v \in S \cup T]$.

This theorem seems to be the first appearance of the linear programming duality theorem for the case when the constraint matrix is the incidence matrix of a bipartite graph. The outline of Egerváry's proof is as follows. Let π be a nonnegative integer-valued weighted-covering of c with minimum weight. If there is a perfect matching M in the subgraph G_π of tight edges, where an edge uv is called **tight** if $\pi(u) + \pi(v) = c(uv)$, then M is a maximum weight perfect matching of G whose weight is equal to the weight of π.

If there is no perfect matching in G_π, then König's theorem implies that there is a deficient set $X \subseteq S$ in G_π. Increase the π-value of each node in $\Gamma_{G_\pi}(X)$ by 1 and decrease the π-value of each node in X by 1. This way, one obtains another weighted-covering π' of c whose weight is smaller than that of π. In case π' has negative (that is, -1) values, increase the π'-values on the elements of S by 1 and decrease the π'-values on the elements of T by 1. Since G is complete bipartite and $c \geq 0$, the resulting π'' is a nonnegative weighted-covering of c whose weight is smaller than that of π, in a contradiction with the minimum choice of π.

Egerváry noted that his theorem easily extends to rational weights (in the sense that the integrality of the weighted-covering is not required anymore), and, by continuity arguments, the theorem holds for real weight functions as well.

Egerváry, in his paper, did not speak on algorithms at all. But his proof above can easily be turned into an algorithm since it finds, starting with an arbitrary weighted-covering π, either a better weighted-covering or else a maximum weight perfect matching. A natural observation is that the revision of the current potential, as described in the proof above, may be done by $\min\{\pi(u) + \pi(v) - c(uv) : u \in X, v \in T - \Gamma_{G_\pi}(X)\}$, a value possibly larger than 1. Perhaps it is not unfair to call the algorithm described this way **Egerváry's algorithm.** This is clearly finite for integer or rational c. Jüttner [15], however, observed that Egerváry's algorithm in this generic form is not polynomial for integer-valued c and not necessarily finite for real-valued c, even if max-deficient sets are used throughout the run of the algorithm for the revision of the current π. It should be noted, however, that by appropriately specifying the choice of the deficient sets used for revising the current π, the algorithm can be made strongly polynommial. Namely, this is the case if the unique smallest max-deficient set is used throughout. This was proved in [11] directly, but I am almost sure that a proof had appeared earlier in the literature. As mentioned above, the deficient set found by König's algorithm is the unique smallest max-deficient set. Therefore, the specific version of Egerváry's algorithm, when the deficient set found by König's alternating path technique, rather than just taking an arbitrary deficient set, is strongly polynomial.

This situation is analogous to the well-known case of maximum flows: for integer or rational capacities the max-flow min-cut algorithm of Ford and Fulkerson [7] is finite though not polynomial, while for real capacities it is not even finite. On the other hand, if a shortest augmenting path is used at every augmentation step, which is actually automatic when breadth-first-search is applied to find an augmenting path, then the algorithm is strongly polynomial, as was proved by Edmonds and Karp [5] and by Dinits [1]. We stress, however, that in the maximum weight matching problem as well as in the maximum flow problem the proof of strong polynomiality is not at all trivial and certainly needs some work.

1.3. Kuhn's Hungarian Method

In light of Egerváry's proof technique, let us see the novelty of Kuhn's Hungarian Method. Egerváry used

König's theorem as a black box or subroutine, and therefore the algorithm read off from his proof is not polynomial. The striking advantage of Kuhn's algorithm is that it is strongly polynomial; moreover, this immediately follows from the description of the algorithm. The main idea of Kuhn's algorithm is that the two separate parts in Egerváry's proof (computing a deficient set and revising the current π) are combined into one.

In a general step, Kuhn's algorithm also has a weighted-covering π and considers the subgraph G_π of tight edges (on node set $S \cup T$). Let M be a matching in G_π. Orient the elements of M from T to S while all other edges of G_π from S to T. Let $R_S \subseteq S$ and $R_T \subseteq T$ denote the set of nodes exposed by M in S and in T, respectively. Let Z denote the set of nodes reachable in the resulting digraph from R_S by a directed path (that can be computed by a breadth-first search, for example).

If $R_T \cap Z$ is nonempty, then we have obtained a path P consisting of tight edges that alternates in M. The symmetric difference of P and M is a matching M' of G_π consisting of one more edge than M does. The procedure is then iterated with this M'. If $R_T \cap Z$ is empty, then revise π as follows. Let $\Delta := \min\{\pi(u) + \pi(v) - c(uv) : u \in Z \cap S,\ v \in T - Z\}$. Decrease (increase, respectively) the π-value of the elements of $S \cap Z$ (of $T \cap Z$, resp.) by Δ. The resulting π' is also a weighted-covering. Construct the subgraph of $G_{\pi'}$ and iterate the procedure with π' and with the unchanged M.

The wonderful thing is that Kuhn's algorithm can be seen with no effort to be strongly polynomial. Indeed, observe first that there may be at most $|S|$ cases of matching augmentation. Second, in a phase when the current matching M is unchanged, the set of nodes reachable from R_S in G_π is properly included in the set of nodes reachable from R_S in $G_{\pi'}$. Hence, this situation may occur at most $|S|$ times, that is, after at most $|S|$ consecutive changes of the weighted-covering, a matching augmentation must follow. Since a breadth-first-search needs $O(|E|)$ steps, the overall complexity of Kuhn's Hungarian Method may be bounded by $O(|E||S|^2)$.

At that time, complexity consideration was not an issue beyond finiteness and therefore it is not surprising that Kuhn was content with proving the finiteness of his algorithm. We stress that the foregoing proof of strong polynomiality is basically automatic.

2. INFLUENCE

The main merit of Kuhn's Hungarian Method is that in the past half a century it has became the starting point of a fast developing area of efficient combinatorial algorithms, now called Combinatorial Optimization. Its seminal ideas, developed originally for the weighted bipartite matching problem (that is, the assignment problem) have been applied by Ford and Fulkerson to the transportation problem and, more generally, to minimum cost flows, as well (see [7]). In all of these cases, as Hoffman and Kruskal [14] discovered, the integrality of the optimal solutions is due to the total unimodularity of the underlying constraint matrix: the incidence matrix of a bipartite graph or a digraph. In 1965, Edmonds [2] was able to generalize the approach of the Hungarian Method to nonbipartite matchings, as well, a much more complex situation where the constraint matrix is not totally unimodular. Edmonds' weighted matroid intersection algorithm [3] was another fundamental breakthrough of a similar vein where the spirit of the Hungarian Method was used and extended.

Harold Kuhn could use ideas of Hungarian mathematicians. A next generation of Hungarian researchers, in turn, highly profited from his method and achieved important results in Combinatorial Optimization. For example, Tardos [22] was the first to construct a strongly polynomial algorithm for the minimum cost circulation problem. Sebö [21] found fundamental structural results on edge-weighted undirected graphs with no negative cycles. Lovász's deep theory on matroid parity [19] was also affected by the Hungarian Method.

Finally, I would like to make some personal remarks. The Hungarian Method caught my heart and imagination very early. I have been teaching it in regular courses for decades, and I am still fascinated at every occasion by its clean elegance and beauty. The method has had a great impact on my research, too. For example, [9] describes a weighted matroid intersection algorithm that may be considered as a straight extension of the original algorithm of Kuhn because, instead of working with dual variables assigned to subsets of the ground-set, as earlier matroid intersection algorithms did, it uses only node-numbers, just as Kuhn's algorithm does. The same idea could be carried over in [10] to submodular flows, a wonderfully general and flexible framework, due to Edmonds and Giles [4]. The theoretical and practical efficiency and the wide range of applicability of the Hungarian Method are only one side of its far-reaching effect. Another one is that the method is an effective proof technique. For example, the version of Kuhn's Hungarian Method developed by Ford and Fulkerson (see [7]) for solving the min-cost flow problem could be used in [8] to prove a common generalization of a theorem of Greene [12] and a theorem of Greene and Kleitman [13] on maximum chain and antichain families of a partially ordered set. These theorems are deep generalizations of Dilworth's classical chain-covering theorem. Based on ideas of the Hungarian Method, one can compute a maximum cardinality subset of a poset that is the union of k chains (or k antichains).

In 2001, encouraged by these precedents, young and senior researchers in Budapest, the city of König and Egerváry, felt obliged to establish the Egerváry Research Group, supported by the Hungarian Academy of Sciences. (For its homepage, see http://www.cs.elte.hu/egres). Our main goal has been to work on combinatorial algorithms and structures in the spirit of Kuhn's Hungarian Method and of the min-max theorems of König and Egerváry.

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Statement for *Naval Research Logistics*

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I am extremely pleased that "The Hungarian Method for the Assignment Problem" has been selected as a paper representing the best of *Naval Research Logistics* in its first 50 years. This paper has always been one of my favorite "children," combining as it does elements of the duality of linear programming and combinatorial tools from graph theory. It may be of some value to tell the story of its origin.[1]

I spent the summer of 1953 at the Institute for Numerical Analysis housed on the U.C.L.A. campus. This Institute was the home of the SWAC (Standards Western Automatic Computer), which had been designed by John von Neumann and had a memory composed of 256 Williamson (cathode ray) tubes. The formulation of the assignment problem as a linear program was well known, but a 10 by 10 assignment problem has 100 variables in its primal statement and 100 constraints in the dual and so was too large for the SWAC. The SEAC (Standard Eastern Automatic Computer), housed in the National Bureau of Standards, could solve linear programs with about 25 variables and 25 constraints.

During that summer, while working on a number of problems including the Traveling Salesman Problem, I was reading König's book on graph theory. I recognized the following theorem of König to be a pre-linear programming example of duality:

> If the numbers of a matrix are 0's and 1's, then the minimum number of rows and columns that will contain all of the 1's is equal to the maximum number of 1's that can be chosen, with no two in the same row or column.

Indeed, the primal problem is the special case of an assignment problem in which the ratings of the individuals in the jobs are only 0's and 1's. In a footnote, König refers to a paper of E. Egerváry (in Hungarian), which seemed to contain the treatment of a more general case.

When I returned to Bryn Mawr, where I was on the faculty in 1953, I took out a Hungarian grammar and a large Hungarian–English dictionary and taught myself enough Hungarian, to translate Egerváry's paper. I then realized that Egerváry's paper gave a computationally trivial method for reducing the general assignment problem to a 0–1 problem. Thus, by putting the two ideas together, the Hungarian Method was born.

I tested the algorithm by solving 12 by 12 problems with random 3-digit ratings by hand. I could do any such problem, with pencil and paper, in no more than 2 hours. This seemed to be much better than any other method known at the time.

The paper was published in *Naval Research Logistics Quarterly*. This was a natural choice since the project in Game Theory, Linear and Nonlinear Programming, and Combinatorics at Princeton, with which Al Tucker and I were associated from 1948 to 1972, was supported by the Office of Naval Research Logistics Branch. Many mathematicians were beneficiaries of the wise stewardship of Mina Rees as head of the ONR and Fred Rigby as chief of the Logistics branch. We were also fortunate to have Jack Laderman, the first editor of the journal, as our project supervisor. They share the credit for this award for creating a congenial framework in which such research could be done.

[1]After writing this acceptance statement, I looked up an account written 14 years ago:

> "On the origin of the Hungarian Method," History of mathematical programming; a collection of personal reminiscences, Jan Karel Lenstra, Aexander H. G. Rinnooy Kan, and Alexander Schrijver (Editors), North Holland, Amsterdam, 1991, pp. 77–81.

This account is substantially the same as the statement that I have just written but contains more detail. Large sections are reproduced in the book by Alexander Schrijver:

> Combinatorial optimization: polyhedra and efficiency, Vol. A. Paths, Flows, Matchings, Springer, Berlin, 2003.

Schrijver's account places the Hungarian Method in the mathematical context of combinatorial optimization and rephrases the concepts in graph-theoretical language.

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