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A Weighted Matroid Intersection Algorithm

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Two matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$, and a weight function s on S (possibly negative or nonintegral) are given. For every nonnegative integer k, find a k-element common independent set of maximum weight (if it exists).

This problem was solved by J. Edmonds [3, 4] both theoretically and algorithmically. Since then the question has been investigated by a number of different authors; see, for example, [1, 6–10]. The purpose of this note is to make a simpler primal–dual algorithm and thereby give a clearer constructive proof for Edmonds' matroid polyhedral intersection theorem.

The idea behind the procedure is that the meaning of the dual part in Lawler's primal–dual algorithm can be made much simpler. We shall not need the dual variables assigned to the closed sets of the two matroids. Instead, we are working by splitting the weights of the elements. At the end of the algorithm the optimal dual variables can simply be computed from the final splitting.

The reader is assumed to be familiar with such basic concepts of matroid theory as "independent set," "circuit," "greedy algorithm," etc. [9, 11].

The *weight* of a subset X of S is $s(X) = \Sigma(s(x): x \in X)$. If $\mathcal{F}$ is a family of subsets of S we say that $F \in \mathcal{F}$ is s-maximal in $\mathcal{F}$ if $s(F) \geq s(X)$ for $X \in \mathcal{F}$.

Before describing the algorithm we need some simple lemmas. The main content of the Greedy Algorithm theorem [2] is:

LEMMA 1. *For a given matroid $M = (S, \mathcal{I})$, let $\mathcal{I}^k = \{X: X \in \mathcal{I}, |X| = k\}$. $I \in \mathcal{I}^k$ is s-maximal in $\mathcal{I}^k$ if and only if*

(1) $x \notin I, I + x \notin \mathcal{I}$ *imply* $s(x) \leq s(y)$, *for every* $y \in C(I, x)$ *and*

(2) $x \notin I, I + x \in \mathcal{I}$ *imply* $s(x) \leq s(y)$, *for every* $y \in I$,

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where $C(I, x)$ denotes the unique circuit in $I + x$. □

LEMMA 2. *Let B be s-maximal in $\mathcal{I}^k$. Let $x_1, x_2, \ldots, x_l$ and $y_1, y_2, \ldots, y_l$ be distinct elements, $y_i \in B$, $x_i \notin B$ $(i = 1, 2, \ldots, l)$ such that*

(3) $B + x_i \notin \mathcal{I}$ *and* $y_i \in C(B, x_i)$,

(4) $s(x_i) = s(y_i)$,

(5) $s(y_i) = s(y_j)$ *and* $i < j$ *imply* $y_i \notin C(B, x_j)$.

Then $B' = B - \{y_1, y_2, \ldots, y_l\} \cup \{x_1, x_2, \ldots, x_l\}$ is also s-maximal in $\mathcal{I}^k$.

Proof. Since $s(B') = s(B)$ and $|B'| = k$ we have to prove the independence of B'. By induction on l. The case $l = 1$ is trivial so let $l > 1$. Let y_i be that element which minimizes $(s(y_j), j)$ lexicographically. Then $i \neq j$ implies $y_i \notin C(B, x_j)$; the contrary case would imply $s(y_i) \geq s(y_j)$ by (1) and (4), and so $s(y_i) = s(y_j)$ because of the choice of y_i, whence $i > j$ by (5), contradicting the inequality $(s(y_i), i) < (s(y_j), j)$.

Now the induction hypothesis holds for $B_1 = B - y_i + x_i$ and $x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_l, y_1, y_2, \ldots, y_{i-1}, y_{i+1}, \ldots, y_l$ from which the lemma follows. □

Denote $\mathcal{I}_{12}^k = \mathcal{I}_1^k \cap \mathcal{I}_2^k$.

LEMMA 3. *Let $I \in \mathcal{I}_{12}^k$ and s_1, s_2 be functions on S with the property that $s_1 + s_2 = s$ and I is s_i-maximal in $\mathcal{I}_i^k$ $(i = 1, 2)$. Then I is s-maximal in $\mathcal{I}_{12}^k$.*

Proof. Trivial. □

For each possible k, the algorithm constructs I, s_1, s_2 satisfying the hypotheses of Lemma 3.

The procedure starts with $k = 0$. Then k is increased one by one. An essential property of the algorithm is that, in every stage, the current $I \in \mathcal{I}_{12}^k$ is s-maximal in $\mathcal{I}_{12}^k$.

Suppose we have $I \in \mathcal{I}_{12}^k, s_1, s_2$ which satisfy the hypotheses of Lemma 3 and which have been constructed by the algorithm previously. From these we shall make $I' \in \mathcal{I}_{12}^{k+1}, s_1', s_2'$ satisfying again the hypotheses of Lemma 3. At the beginning $k = 0$, $I = \emptyset$, $s_1 \equiv 0$, $s_2 = s$.

Let $m_i = \max(s_i(y): y \notin I, I + y \in \mathcal{I}_i)$ $(i = 1, 2)$.

Let $X_i = \{x: x \notin I, I + x \in \mathcal{I}_i, s_i(x) = m_i\}$ $(i = 1, 2)$.

Define an auxiliary digraph G on S as follows:

I. If $x \notin I$, $I + x \notin \mathcal{I}_1$, $y \in C_1(I, x)$, $s_1(x) = s_1(y)$ then let (xy) be an edge.

II. If $x \notin I$, $I + x \notin \mathcal{I}_2$, $y \in C_2(I, x)$, $s_2(x) = s_2(y)$ then let (yx) be an edge.

By the well-known labeling technique [9], decide whether there exists a path from the set X_2 to X_1.

Case 1. If the path in question exists let U be a path of minimum number of vertices. (U is considered as a vertex set, and we shall need only that U is minimal.)

Let $I' = I \oplus U$, where $\oplus$ denotes the symmetric difference, and let $\mathbf{s}_i' = \mathbf{s}_i$ $(i = 1, 2)$.

CLAIM 1. *$I', \mathbf{s}_1', \mathbf{s}_2'$ satisfy the conditions of Lemma* 3 *for* $k+1$.

Proof. Let us denote the vertices of U by $x_0, y_1, x_1, y_2, x_2, \ldots, y_l, x_l$ $(x_0 \in X_2, x_l \in X_1)$. By Lemma 1, $B = I + x_0$ is $\mathbf{s}_2$-optimal in $\mathscr{I}_2^{k+1}$.

Observe that the hypotheses of Lemma 2 hold for $k+1$ and for $B \in \mathscr{I}_2^{k+1}$, $x_1, x_2, \ldots, x_l, y_1, y_2, \ldots, y_l$. (Properties (3) and (4) are true because of the definition of G, (5) follows from the minimality of U.) Thus I' is $\mathbf{s}_2$-optimal in $\mathscr{I}_2^{k+1}$. That I' is $\mathbf{s}_1$-optimal in $\mathscr{I}_1^{k+1}$ can similarly be proved with the difference that one should rename the vertices of U just in reverse order (i.e., its last vertex will be x_0 while the first one x_l). □

CLAIM 2. $\mathbf{s}(I') - \mathbf{s}(I) = m_1 + m_2$.

Proof. Obvious. □

Case 2. If there is no path let T consist of vertices having reached from X_2. Let

$$\mathbf{s}_1'(x) = \mathbf{s}_1(x) + \delta \qquad \text{if } x \in T$$
$$= \mathbf{s}_1(x) \qquad \text{if } x \notin T$$

and $\mathbf{s}_2'(x) = \mathbf{s}(x) - \mathbf{s}_1'(x)$. $\delta = \min(\delta_1, \delta_2, \delta_3\, \delta_4)$, where

$$\delta_1 = \min(\mathbf{s}_1(y) - \mathbf{s}_1(x): \quad I + x \notin \mathscr{I}_1, x \in T - I, y \in C_1(I, x) - T),$$
$$\delta_2 = \min(m_1 - \mathbf{s}_1(x): \quad I + x \in \mathscr{I}_1, x \in T - I),$$
$$\delta_3 = \min(\mathbf{s}_2(y) - \mathbf{s}_2(x): \quad I + x \notin \mathscr{I}_2, x \in S - (T \cup I),$$
$$y \in C_2(I, x) \cap T),$$
$$\delta_4 = \min(m_2 - \mathbf{s}_2(x): \quad I + x \in \mathscr{I}_2, x \in S - (T \cup I)).$$

(The minimum is defined to be ∞ when it is taken over the empty set.)

CLAIM 3. $\delta > 0$.

We prove that δ_1 and $\delta_4 > 0$. That $\delta_2, \delta_3 > 0$ can be proved similarly. If $y \in C_1(I, x) - T$ then $\mathbf{s}_1(y) \geq \mathbf{s}_1(x)$ by Lemma 1. But $\mathbf{s}_1(y) = \mathbf{s}_1(x)$ would mean that (xy) is an edge in G leaving T, which is impossible. So $\delta_1 > 0$. If $x \in S - (T \cup I)$ then $x \notin X_2$; thus the definition of m_2 implies $\delta_4 > 0$. □

CLAIM 4. *$I' = I$, $\mathbf{s}_1'$ and $\mathbf{s}_2'$ satisfy the conditions of Lemma* 3.

Proof. We prove only that $I' = I$ is $\mathbf{s}_1'$-optimal in $\mathscr{I}_1^k$. The $\mathbf{s}_2'$-optimality can be proved similarly. By Lemma 1, we have to prove that (1) and (2) hold for $\mathbf{s}_1'$.

Choose elements x, y so that $y \in C_1(I, x)$. If, indirectly, $\mathbf{s}_1'(y) < \mathbf{s}_1'(x)$ then, because of $\mathbf{s}_1(y) \geq \mathbf{s}_1(x)$, $\mathbf{s}_1'(x) = \mathbf{s}_1(x) + \delta$ and $\mathbf{s}_1'(y) = \mathbf{s}_1(y)$ are implied. But $\delta \leq \delta_1 \leq \mathbf{s}_1(y) - \mathbf{s}_1(x)$, that is, $\mathbf{s}_1'(x) \leq \mathbf{s}_1'(y)$, a contradiction. Thus (1) holds.

Choose elements x, y so that $y \in I$, $x \notin I$ and $I + x \in \mathscr{I}_1$. If, indirectly, $\mathbf{s}_1'(y) < \mathbf{s}_1'(x)$ then, because of $\mathbf{s}_1(y) \geq \mathbf{s}_1(x)$, we have $\mathbf{s}_1'(y) = \mathbf{s}_1(y)$ and $\mathbf{s}_1'(x) = \mathbf{s}_1(x) + \delta$. But $m_1 \leq \mathbf{s}_1(y)$ and $\delta \leq \delta_2 \leq m_1 - \mathbf{s}_1(x)$, from which $\mathbf{s}_1'(x) \leq \mathbf{s}_1'(y)$, a contradiction. Thus (2) holds. □

Now again apply the algorithm starting with $I', \mathbf{s}_1', \mathbf{s}_2'$. Observe that the new T' (if Case 2 occurs again) properly includes T; furthermore $X_i' \supseteq X_i$ $(i = 1, 2)$. Consequently, after no more than $|S|$ applications of this loop of the algorithm, either Case 1 is attained or δ becomes ∞. The latter case means that the current I is of maximum cardinality since $k = |I| = \mathbf{r}_1(T) + \mathbf{r}_2(S - T)$. (Obviously, $|I'| \leq \mathbf{r}_1(T') + \mathbf{r}_2(S - T')$ for any common independent set I' and $T' \subseteq S$.)

COMPLEXITY OF THE ALGORITHM

The matroids are defined by the help of an oracle, which decides, in at most g steps, for an independent set I and an element $x \notin I$, whether $I + x$ is independent or not and in the latter case, determines the fundamental circuit $C(I, x)$.

The addition, subtraction, and comparison of two real numbers are considered as one step each.

Let $|S| = n$ and K denote the maximum cardinality of a common independent set (yet to be determined).

The labeling technique requires at most n^2 steps to find a path or the subset T. However, if Case 2 occurs the current labels can be used again, because $T' \supset T$, $X_1' \supseteq X_1$, $X_2' \supseteq X_2$. Consequently, if Case 1 has occurred at any time, after no more than $g \cdot n^2$ steps, Case 1 will have occurred again. Therefore the complexity of the algorithm can be bounded by $O(g \cdot K \cdot n^2) \leq O(gn^3)$.

Remark. If the algorithm starts with $\mathbf{s}_1 \equiv 0$ then $m_1 = 0$, and $\delta_2 = \infty$ throughout the process. We have not exploited this simplification, in order to keep the symmetry between M_1 and M_2 and to provide the possibility of starting with any $I, \mathbf{s}_1, \mathbf{s}_2$ satisfying the conditions of Lemma 3.

Weighted Intersection Algorithm

Input: Matroids M_1, M_2 on S. Weight function $\mathbf{s}$ on S.
Output: Maximum cardinality K of a common independent set and $T \subseteq S$, for which $\mathbf{r}_1(T) + \mathbf{r}_2(S - T) = K$ and
I_k; a k-element common independent set of maximum weight $(0 \le k \le K)$ and
$\mathbf{s}_1^k$ and $\mathbf{s}_2^k$: for which $\mathbf{s}_1^k + \mathbf{s}_2^k = s$ and I_k is $\mathbf{s}_i^k$-maximal in $\mathcal{J}_i^k$ $(i = 1, 2)$.

Step 1

1.0. $\mathbf{s}_1 \equiv 0, \mathbf{s}_2 = \mathbf{s}, k = 0, I_k = \varnothing$.

1.1. Make the auxiliary graph. Determine X_1 and X_2.

1.2. Find a path U from X_2 to X_1 by the labeling technique, using the labels having defined but not deleted previously. If it exists, go to step 3.

Step 2

2.0. Let T denote the set of vertices having labels.

2.1. Count δ. If $\delta = \infty$, Let $K = k$, the current I_k has maximum cardinality and $K = \mathbf{r}_1(T) + \mathbf{r}_2(S - T)$. HALT.

2.2. Let $\mathbf{s}_1(x) = \mathbf{s}_1(x) + \delta, \mathbf{s}_2(x) = \mathbf{s}_2(x) - \delta$ whenever $x \in T$.

2.3. Go to 1.1.

Step 3

3.0. Let $I_{k+1} := I_k \oplus U$. Let $k := k + 1$. I_k is optimal in $\mathcal{J}_{12}^k$, record it.

3.1. Delete all the labels.

3.2. $\mathbf{s}_1^k := \mathbf{s}_1, s_2^k := s_2$.

3.3. Go to 1.1.

Now we show some consequences of the algorithm. Since neither m_1 nor m_2 increases during the algorithm Claim 2 implies a result of Krogdahl [7, 9].

Corollary. $s_{k+1} - s_k \le s_k - s_{k-1}$, *where s_j denotes the weight of an* **s**-*maximal member of* $\mathcal{J}_{12}^j$.

We have proved the following.

Theorem. $I \in \mathcal{J}_{12}^k$ *is* **s**-*maximal if and only if there exist two weightings* $\mathbf{s}_1$ *and* $\mathbf{s}_2$ *such that* $\mathbf{s}_1 + \mathbf{s}_2 = \mathbf{s}$ *and I is* $\mathbf{s}_i$-*maximal in* $\mathcal{J}_i^k$ $(i = 1, 2)$. *If, in addition,* **s** *is integer valued then* $\mathbf{s}_i$ *can be chosen to be integer valued.*

The main consequence of the algorithm is a min–max theorem on the maximum weight s_k of a k-element common independent set. The most convenient way to formulate it is to describe it as a linear program.

Let $\mathbf{A}_i$ denote a 0–1 matrix, the columns of which correspond to the elements of S while the rows correspond to the closed subsets of M_i. Let **e** denote a row vector consisting of $|S|$ 1's. The following result is a version of Edmonds' matroid polyhedral intersection theorem [3].

Theorem. *Consider the dual pair of linear programs:*

$$(*)\quad \begin{bmatrix}\mathbf{A}_1\\ \mathbf{A}_2\end{bmatrix}\mathbf{x} \le \begin{bmatrix}\mathbf{r}_1\\ \mathbf{r}_2\end{bmatrix},\quad \mathbf{e}\mathbf{x} = k,\quad \mathbf{x} \ge 0,\quad \max \mathbf{s}\mathbf{x}$$

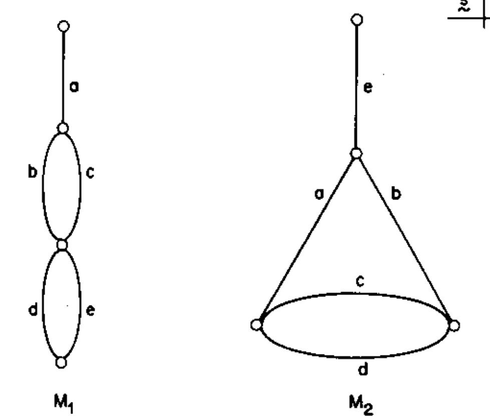
$$(\mathbf{y}_1, \mathbf{y}_2, t)\begin{bmatrix}A_1\\ A_2\\ \mathbf{e}\end{bmatrix} \ge \mathbf{s},\quad \mathbf{y}_1, \mathbf{y}_2 \ge 0,\quad \min \Sigma \mathbf{y}_1(z)\mathbf{r}_1(z) + \Sigma \mathbf{y}_2(z)\mathbf{r}_2(z) + t\cdot k \quad (**)$$

If the primal program $(*)$ *has a feasible solution it has an integral optimal solution. Moreover, if* **s** *is integral the dual program has an integral optimal solution.*

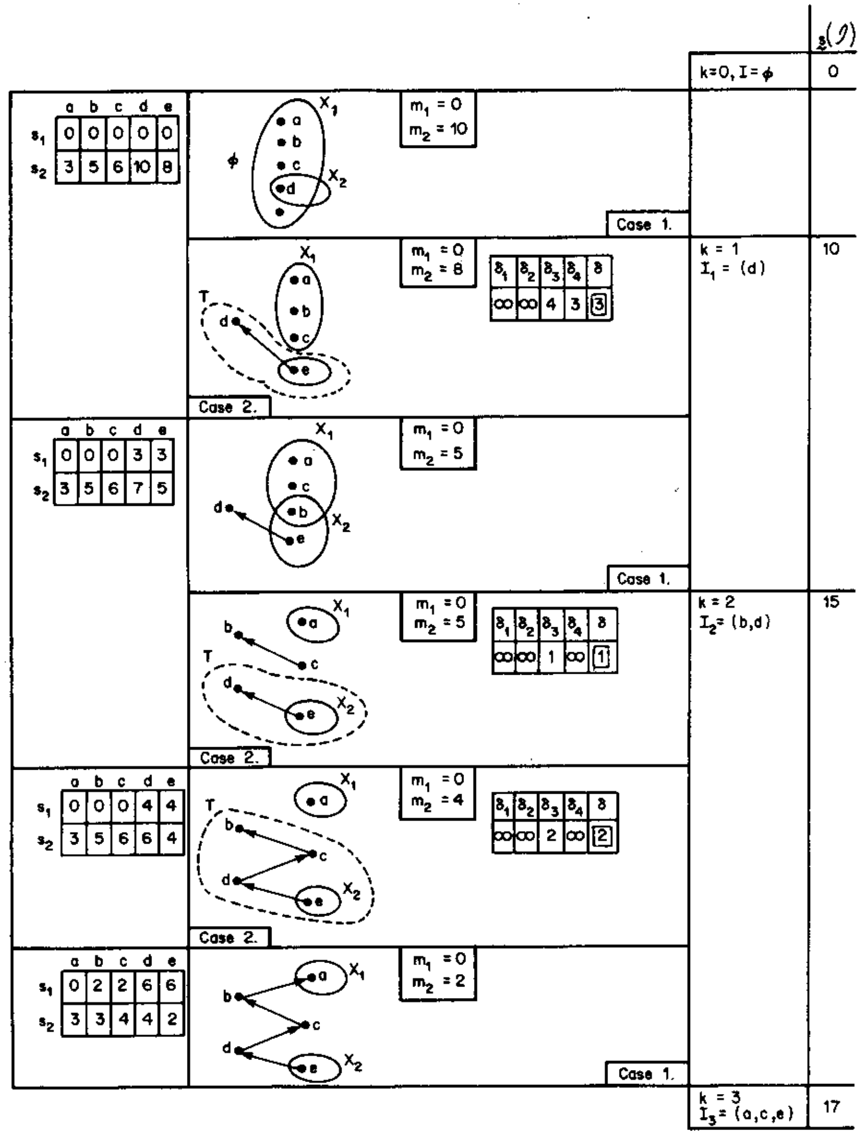
Proof. The primal optimal solution is defined as the incidence vector of I_k, where I_k is the optimum element of $\mathcal{J}_{12}^k$, constructed by the algorithm.
The dual optimal solution can be obtained from $\mathbf{s}_1$ and $\mathbf{s}_2$.
Let $I_k = \{e_1, e_2, \ldots, e_k\} = \{f_1, f_2, \ldots, f_k\}$ so that

$$\mathbf{s}_1(e_1) \ge \mathbf{s}_1(e_2) \ge \cdots \ge \mathbf{s}_1(e_k) \quad \text{and} \quad \mathbf{s}_2(f_1) \ge \mathbf{s}_2(f_2) \ge \cdots \ge \mathbf{s}_2(f_k).$$

	a	b	c	d	e
s	3	5	6	10	8



Example for weighted matroid intersection algorithm.



Let $E_i = \mathrm{sp}_1(e_1, e_2, \ldots, e_i)$ and $F_i = \mathrm{sp}_2(f_1, f_2, \ldots, f_i)$. Define

$$\mathbf{y}_1(E_i) = \mathbf{s}_1(e_i) - \mathbf{s}_1(e_{i+1}),$$
$$\mathbf{y}_2(F_i) = \mathbf{s}_2(f_i) - \mathbf{s}_2(f_{i+1}) \qquad \text{for } i = 1, 2, \ldots, k-1.$$

Let $t = \mathbf{s}_1(e_k) + \mathbf{s}_2(f_k)$. Using (1) and (2), a simple counting shows that

$(\mathbf{y}_1, \mathbf{y}_2, t)$ is a feasible solution to $(**)$. The value of the objective function on $(\mathbf{y}_1, \mathbf{y}_2, t)$ is

$$\begin{aligned} &1(\mathbf{s}_1(e_1) - \mathbf{s}_1(e_2)) + 2(\mathbf{s}_1(e_2) - \mathbf{s}_1(e_3)) + \cdots \\ &\quad + (k-1)(\mathbf{s}_1(e_{k-1}) - \mathbf{s}_1(e_k)) + 1(\mathbf{s}_2(f_1) - \mathbf{s}_2(f_2)) \\ &\quad + 2(\mathbf{s}_2(f_2) - \mathbf{s}_2(f_3)) + \cdots + (k-1)(\mathbf{s}_2(f_{k-1}) - \mathbf{s}_2(F_k)) + k \cdot t \\ &\qquad = \sum_{i=1}^{k} \mathbf{s}_1(e_i) - k\mathbf{s}_1(e_k) + \sum_{i=1}^{k} \mathbf{s}_2(f_i) - k\mathbf{s}_2(f_k) + k \cdot t \\ &\qquad = \sum_{i=1}^{k} \mathbf{s}(e_i) = \mathbf{s}(I_k). \end{aligned}$$

This proves the optimality of the primal and dual solutions. Moreover, if $\mathbf{s}$ is integer then so are $\mathbf{s}_1$ and $\mathbf{s}_2$ and thus $\mathbf{y}_1, \mathbf{y}_2$, as well. □

Another consequence of the algorithm deserving mention is:

COROLLARY. *For every $k \geq 0$, if $\mathcal{G}_{12}^{k+1} \neq \varnothing$, there exists an optimal solution to $(**)$ which, at the same time, is an optimal solution to $(**)$ for $k+1$ instead of k.*

(This is the so called t-phenomenon; see [5].)

Proof. Consider that stage of the algorithm when I_{k+1} is arising. The vector $(\mathbf{y}_1, \mathbf{y}_2, t)$ belonging to the current $\mathbf{s}_1$, $\mathbf{s}_2$ satisfies the requirements.

The original version of Edmonds' theorem can similarly be obtained:

THEOREM. [3]. *Consider the dual pair of linear programs*

$$\begin{array}{cc} \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix} \mathbf{x} \leq \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{pmatrix}, & (\mathbf{y}_1, \mathbf{y}_2) \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix} \geq \mathbf{s} \\ \mathbf{x} \geq 0, & \mathbf{y}_1, \mathbf{y}_2 \geq 0 \\ \hline \max \mathbf{s}\mathbf{x} & \min \Sigma \mathbf{y}_1(z)\mathbf{r}_1(z) + \Sigma \mathbf{y}_2(z)\mathbf{r}_2(z) \end{array}$$

The primal program has an integral optimal solution. If $\mathbf{s}$ is integer valued the dual program has an integral optimal solution.

Proof. For simplicity suppose that none of the matroids contains loops. Let us slightly modify the algorithm. Suppose that the algorithm starts with $\mathbf{s}_1 \equiv 0, \mathbf{s}_2 = \mathbf{s}$, in which case $m_1 = 0$ and $\delta_2 = \infty$ throughout the algorithm. Furthermore when Case 2 occurs set $\delta = \min(\delta_1, \delta_3, \delta_4, m_2)$. If $\mathbf{s} \leq 0$ the zero vectors appropriately dimensioned satisfy the requirements. If $\mathbf{s} \nleq 0$, m_2 is strictly positive when the algorithm is starting. Obviously the algorithm works in the same way as before until the value of δ takes the current m_2. (This case will certainly occur.) Now performing the changes in $\mathbf{s}_1$ and $\mathbf{s}_2$ m_2 becomes 0 (first time during the algorithm). Let $I_k, \mathbf{s}_1, \mathbf{s}_2$ denote the corresponding common independent set and weightings belonging to this stage. Now we have $m_1 = m_2 = 0$ and $\mathrm{s}_1(e_k) \geq 0, \mathrm{s}_2(f_k) \geq 0$. (Of course $e_1, \ldots, e_k$ and $f_1, \ldots, f_k$ may change when $\mathrm{s}_1, \mathrm{s}_2$ are changed.) Let $\mathrm{y}_1(E_i)$ and $\mathrm{y}_2(F_i)$ be defined as before but now for $i = 1, 2, \ldots, k$. Then $\mathrm{y}_1 \geq 0, \mathrm{y}_2 \geq 0$ and, as can be simply checked, the incidence vector $\mathbf{x}$ of I_k and $\mathrm{y}_1, \mathrm{y}_2$ are optimal solutions to the primal and dual programs. Moreover, if s is integer valued then so are y_1 and y_2.

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