Balanced list edge-colourings of bipartite graphs

Tamás Fleiner* and András Frank**

Abstract

Galvin solved the Dinitz conjecture by proving that bipartite graphs are Δ-edge-choosable. We improve Galvin's method and deduce from any colouring of the edges of bipartite graph G some further list edge-colouring properties of G. In particular, for bipartite graphs, it follows from the existence of balanced bipartite edge-colourings that balanced list edge-colourings exist as well. While the key to Galvin's proof is the stable marriage theorem of Gale and Shapley, our result is based on the well-known "many-to-many" version of the stable matching theorem.

Keywords: chromatic index; list edge-colouring; stable matching

1 Introduction

A *proper edge-colouring* of graph G is the assignment of colours to the edges of G such that no two edges incident with the same vertex have the same colour. Graph G is said to be k-*edge-choosable* if no matter how we assign lists $L(e)$ of k possible colours to each edge e of G, there always exists a proper edge-colouring of G such that each edge e of G is coloured from $L(e)$. The *list chromatic index* $\chi'_l(G)$ of G is the smallest integer k such that G is k-edge-choosable. As we may assign the same list to each edge, the chromatic index is always a lower bound on the list chromatic index: $\chi'(G) \leq \chi'_l(G)$. Dinitz conjectured (in terms of Latin squares) that complete bipartite graph $K_{n,n}$ has list chromatic index $\chi'_l(K_{n,n}) = n$ (see [1]). This is a special case of the famous list colouring conjecture stating that for any finite loopless graph G, we have $\chi'(G) = \chi'_l(G)$. Galvin's celebrated result shows that the list colouring conjecture is true for any finite bipartite graph and this immediately implies the Dinitz conjecture [3].

*Department of Computer Science and Information Theory, Budapest University of Technology and Economics, Magyar tudósok körútja 2., Budapest, Hungary. E-mail: fleiner@cs.bme.hu . Research was supported by the OTKA K 69027 project and the MTA-ELTE Egerváry Research Group.

**Department of Operations Research, Eötvös Loránd University and MTA-ELTE Egerváry Research Group, Pázmány Péter sétány 1/C, H-1117 Budapest, Hungary. E-mail: frank@cs.elte.hu . Research was supported by the OTKA K 60802 project.

The hearth of Galvin's proof is a special case of a Theorem of Maffray [6]. This very special case is equivalent to the stable marriage theorem of Gale and Shapley [2] stating that for any (finite) bipartite graph G and for any linear preference orders on the stars, there exists a so called stable matching M of G with the property that every edge e of $G - M$ has a vertex v such that there is an edge m of M preferred to e by v. (If the two colour classes of G represent boys and girls, respectively, and an edge means that the two corresponding people are not against marriage then a stable matching describes a marriage scheme where no boy and girl have the mutual interest to leave their spouses for one another.)

In this work, we extend Galvin's method to not necessarily proper edge-colourings. For this reason, we define a partial order on edge-colourings such that proper colourings and 1-colourings are at the best and worse elements, respectively. In Section 2, we apply the "many-to-many" generalization of the stable marriage theorem to show that for any given edge-colouring of bipartite graph G there exists a better list edge-colouring provided each edge has a sufficiently large list of possible colours. This result with a Δ-edge-colouring of G (that exists by a theorem of Kőnig) implies Galvin's theorem.

Section 3 is devoted to our main motivation: balanced colourings of bipartite graphs. It is well-known that for any bipartite graph G and for any positive integer k there exists a colouring of the edges of G with k colours such that for any vertex v, at most $\left\lceil \frac{d(v)}{k} \right\rceil$ edges incident with v can have the same colour. We show that if each edge list contains at least k colours then there exists a list edge-colouring with the same property. We conclude in Section 4 by formulating two open questions and indicating that a certain generalization of our result is not possible.

2 Edge-colourings and list edge-colourings

To define a partial order on edge-colourings, we start from a little afar. For a nonnegative integer n, a *(number theoretic) partition of* n is a way to decompose n as a sum of positive integers where the order of the summands is indifferent. That is, if two sums only differ in the order of the summands then those determine the same partition. We say that partition π of n is *finer* than partition π' of n (denoted by $\pi \preceq \pi'$) if π' can be obtained from π by grouping certain summands of π. It follows immediately from the definition that for any partition π of n we have $1 + 1 + \ldots + 1 \preceq \pi \preceq n$

Let us turn to edge-colourings now. By a *k-edge-colouring* of graph G we mean a function $c : E(G) \to \{1, 2, \ldots, k\}$, and $c(e)$ is called the *colour* of edge e of G. Each k-edge-colouring c and each vertex v of G induce a partition $\pi(c, v)$ of degree $d(v)$ of v into (at most k) summands that describe how many edges of each colour of c are incident with v. In particular, edge-colouring c is a proper one if and only if $\pi(c, v)$ is the finest partition of $d(v)$ for each vertex v of G. An *edge-colouring* of graph G is a k-edge-colouring of G for some k.

Let c and c' be two edge-colourings of G. We say that edge-colouring c is *finer* than c' if $\pi(c, v) \preceq \pi(c', v)$ holds for each vertex v of G, that is, if c induces a finer partition on each degree than c' does. This definition yields in particular that the finest edge-colourings are the proper ones. Now we can claim our main theorem.

Theorem 2.1. *Let $G = (V, E)$ be a finite bipartite graph and let c be a k-edge-colouring of G. If for each edge $e = uv$ of G, $L(e)$ is a set of k elements (that is, a list of k possible colours) then for each edge e of G, we can pick an element (a colour) $c'(e)$ of $L(e)$ such that edge-colouring c' is finer than c.*

Galvin's theorem is an immediate consequence of Theorem 2.1. Recall that $\Delta(G)$ denotes the maximum degree of G.

Corollary 2.2 (Galvin [3]). *Each bipartite graph G is $\Delta(G)$-edge-choosable.*

Proof. There is a proper $\Delta(G)$-edge-colouring c of G by Kőnig's edge-colouring theorem [5]. So by Theorem 2.1, if each edge list $L(e)$ contains at least $\Delta(G)$ colours then there exists an edge-colouring c' from the edge-lists such that c' is finer than c. As c is a proper edge-colouring, c' must also be a proper one. □

We proceed with justifying our main theorem. A main ingredient of the proof of Theorem 2.1 is the following well-known stable b-matching theorem, the many-to-many version of the stable marriage theorem of Gale and Shapley. (Note that special case $b \equiv 1$ of Theorem 2.3 is equivalent to the stable marriage theorem.)

Theorem 2.3 (See [4] and [7]). *Let G be a finite bipartite graph, let $b : V(G) \to \mathbb{N}$ be an arbitrary quota function and let $\leq_v$ be a linear order on the edges of G incident with v. There always exists a set S of edges of G such that each vertex v of G is incident with at most $b(v)$ edges of S and each edge e of $E \setminus S$ has an endvertex u such that u is incident with $b(u)$ edges of S each of which is smaller in $\leq_u$ than e.*

The edges of subgraph G' in Theorem 2.3 is sometimes called a *stable b-matching*. We shall follow Galvin's method to prove our result but instead of stable matchings, we pick stable b-matchings with appropriate quotas b.

Proof of Theorem 2.1. Let A and B denote the colour classes of bipartite graph G. Choose linear order $\leq_v$ in such a way that if e, e' are incident with v and $c(e) < c(e')$ then $e \leq_v e'$ whenever $v \in A$ and $e' \leq_v e$ for $v \in B$. Such a choice is clearly possible. Let $d_i(v)$ denote the number of edges of colour i incident with v and let E_i be the set of edges of G that can be coloured with colour i:

$$d_i(v) := |\{e = uv \in E : c(e) = i\} \qquad E_i := \{e \in E : i \in L(e)\} \ .$$

To prove the existence of the list edge-colouring c' described in the theorem, we shall c'-colour some edges of E_1 with colour 1, then certain edges of E_2 with colour 2, followed by giving c'-colour 3 to a couple of edges in E_3, and so on. More precisely, we start with defining $b_1 := d_1$ we find a stable b_1-matching S_1 of bipartite graph (V, E_1) (that exists by Theorem 2.3) and we give c'-colour 1 to each edge of S_1. To proceed with colour 2, we define for each vertex v of G

$$b_2(v) = \begin{cases} b_1(v) - |S_1(v)| & \text{if } b_1(v) > |S_1(v)| \\ d_2(v) & \text{if } b_1(v) = |S_1(v)| \end{cases}$$

where for a set S of edges and vertex v of G, $S(v)$ denotes the set of edges of S that are incident with vertex v. Let $E'_2 := E_2 \setminus S_1$ be the set of so far uncoloured edges that can be c'-coloured to 2. There exists a stable b_2-matching S_2 of (V, E'_2), and we c'-colour the edges of S_2 with colour 2.

In general, if $S_1, S_2, \ldots, S_i$ are the set of i-coloured edges of c', respectively then the set S_{i+1} of edges that we c'-colour with colour $i+1$ is a stable b_{i+1}-matching of (V, E'_{i+1}), where $E'_{i+1} := E_{i+1} \setminus \bigcup_{j=1}^{n} S_j$ is the set of $(i+1)$-colourable, yet c'-uncoloured edges and

$$b_{i+1}(v) = \begin{cases} b_i(v) - |S_i(v)| & \text{if } b_i(v) > |S_i(v)| \\ d_s(v) & \text{if } \sum_{j=1}^{i} |S_i(v)| = \sum_{j=1}^{s-1} d_j(v) \ . \end{cases} \tag{1}$$

That is, if stable b_i-matching S_i did not saturate vertex v then the new quota $b_{i+1}(v)$ will be the unused part of the previous quota, otherwise, when we have coloured previously exactly $d_1(v)+d_2(v)+\ldots+d_{s-1}(v)$ edges of $E(v)$ with colours $1, 2, \ldots, s-1$, then we start to fill up the next degree-quota $d_s(v)$.

We show that c' is an edge-colouring. Consider edge e of G and assume that $c(e) = t$. What does it mean that edge e is not c'-coloured with colour i of $L(e)$? It can happen only because one of the following two reasons. Either $e \notin E'_i$, that is, e is c'-coloured by some colour $j < i$, or e is dominated by stable b_i-matching S_i. That is, e has a *dominating* vertex u such that u is incident with $b_i(u)$ edges of S_i and each of these edges precede e in $\leq_u$. Note that this latter case can occur for at most $t-1$ indices i where the dominating vertex is in colour class A and for at most $k-t$ indices with the dominating vertex in B. So if the above procedure does not determine $c'(e)$ then there can be altogether at most $t-1+k-t = k-1$ colours of $L(e)$ that is not assigned to edge e. This contradicts the assumption that $L(e)$ contains k colours, so c' is indeed a genuine edge-colouring of G.

To finish the proof, we show that c' is finer than c, that is $\pi(c', v) \preceq \pi(c, v)$ holds for each vertex v of G. By the construction of c', $d_1(v) = |S_1(v)| + |S_2(v)| + \ldots + |S_{i(1)}(v)|$ where i_1 is the first colour such that stable b-matching S_{i_1} is saturated at vertex v. Again, by the construction $d_2(v) = |S_{i(1)+1}(v)| + |S_{i(1)+2}(v)| + \ldots + |S_{i(2)}(v)|$ for some $i_2 > i_1$, and in general $d_j(v) = |S_{i(j-1)+1}(v)| + |S_{i(j-1)+2}(v)| + \ldots + |S_{i(j)}(v)|$ holds. But this follows that $c' \preceq c$, just as we claimed. □

3 Balanced list edge-colourings

Another application of Theorem 2.1 has to do with balanced colourings.

Corollary 3.1. *Assume that G is a bipartite graph and for each edge e of G, list $L(e)$ contains at least k colours. Then it is possible to pick a colour $c(e) \in L(e)$ for each edge e of G such that no vertex v is incident with more than $\left\lceil \frac{d(v)}{k} \right\rceil$ edges of the same colour.*

To prove Corollary 3.1, we use a special edge-colouring provided by the following well-known observation.

Lemma 3.2. *For any bipartite graph G and positive integer k there exists a k-edge-colouring c of G in such a way that each summand in $\pi(c, v)$ is either $\left\lceil \frac{d(v)}{k} \right\rceil$ or $\left\lfloor \frac{d(v)}{k} \right\rfloor$ for each vertex v of G.*

We prove Lemma 3.2 for the sake of self-containedness. For this reason we need the following well-known fact.

Lemma 3.3. *If f is an st-flow on digraph D then there exists an integral st-flow $\overline{f}$ on D (a* rounding *of f) such that $\overline{f}(a) \in \{\lfloor f(a) \rfloor, \lceil f(a) \rceil\}$ holds for each arc a of D.*

Proof. Let $A' := \{a \in A(D) : f(a) \notin \mathbb{N}\}$ denote the set of arcs of D with nonintegral flow. From flow conservation, it follows that no vertex of D different from s and t can be incident with exactly one arc of A'. Consequently, if A' is nonvoid then some subset A'' of A' forms a cycle or an st path in the weak (unoriented) sense. It is trivial to modify flow f along A'' in such a way that for each arc a of A'' the new flow value on a belongs to closed interval $[\lfloor f(a) \rfloor, \lceil f(a) \rceil]$ and the modified flow becomes integral on at least one arc of A''. This means that step by step we can decrease the number of noningegral arcs in such a way that each integral flow value is a rounding of the original one. The Lemma directly follows from this latter observation. □

Proof of Lemma 3.2. We prove the lemma by induction on k. Clearly, $c \equiv 1$ is feasible 1-edge-colouring of G for $k = 1$. Assume now that we have already justified Lemma 3.2 for $1, 2, \dots, k-1$ and we shall prove that it holds for k.

It is easy to see that the Lemma is equivalent to the statement that any bipartite graph G has a so called balanced k-edge-colouring c such that for any vertex v of G and any colours i and j the difference between the number of i-coloured edges incident with v and the number of j-coloured edges incident with v is at most one. We shall prove that that there exists a subset F of the edges of G such that

$$|F(v)| \in \left\{ \left\lceil \frac{d(v)}{k} \right\rceil, \left\lfloor \frac{d(v)}{k} \right\rfloor \right\} \tag{2}$$

for each vertex v of G. This implies Lemma 3.2 as $G - F$ has a balanced $(k-1)$-edge colouring c' by the inductive assumption and we can extend it to a balanced k-edge-colouring c of G by colouring each edge of F to colour k.

Orient each edge e of bipartite graph G from colour class X to colour class Y, let $\vec{e}$ denote the arc corresponding to edge e and let $D = (V, A)$ denote the resulted digraph. To create digraph D' from D, introduce new vertices s and t, add arcs from s to each vertex of X and from each vertex of Y to t. Define st-flow f on D' by letting $f(a) = \frac{1}{k}$ for each arc of A and choose the f-values on the remaining arcs (incident with s or t) so as flow conservation holds.

By Lemma 3.3, there is a rounding $\overline{f}$ of f such that $\overline{f}(sx) \in \left\{ \left\lceil \frac{d(x)}{k} \right\rceil, \left\lfloor \frac{d(x)}{k} \right\rfloor \right\}$ and $\overline{f}(yt) \in \left\{ \left\lceil \frac{d(y)}{k} \right\rceil, \left\lfloor \frac{d(y)}{k} \right\rfloor \right\}$ for any $x \in X$ and $y \in Y$. So if $F := \{e \in E(G) : \overline{f}(\vec{e}) = 1\}$ denotes those edges that correspond to arcs with rounded flow value 1 then F has property (2), and this is exactly what we wanted to prove. □

Proof of Corollary 3.1. Applying Theorem 2.1 to k-edge-colouring c of G as in Lemma 3.2 yields an edge-colouring c' that is finer than c and picks the colours of each edge e from its list $L(e)$. As no vertex v is incident with more than $\left\lceil \frac{d(v)}{k} \right\rceil$ edges of the same colour in c, this has to be true also for finer edge-colouring c'. □

4 Conclusion

The list colouring conjecture can be interpreted in such a way that if k colours are enough to properly colour the edges of a graph then from arbitrarily given edge lists of size k it is possible to pick a colour for each edge to form a proper edge-colouring. The list colouring conjecture is known to be true for bipartite graphs due to [3] by Galvin. Our Theorem 2.1 shows that for bipartite graphs, an even stronger statement is true: if we fix some k-edge-colouring of G, it never hurts if we assign *different* lists of k colours to the edges in the sense that we can always find a finer edge-colouring from the lists than our fixed colouring. It is a natural question whether the following more general form of the list colouring conjecture is true.

> **Generalized list colouring conjecture.** Is it true that any graph G and for any k-edge-colouring c of G, no matter how sets $L(e)$ of size k are assigned to each edge e of G, there always exist elements $c'(e)$ of $L(e)$ such that c' is a finer edge-colouring of G than c is?

The key notion for our results is the partial order on edge-colourings of graphs. This partial order is based on number theoretic partitions. However, it is possible to define it for set theoretic partitions as well, moreover, it is more usual to define the "finer" relation for those. This would give a coarser partial order on edge-colourings: edge colouring c' of G is *strictly finer* than c if for each vertex v of G and each colour i there exist some set I of colours such that $\{e \in E(v) : c(e) = i\} = \{e \in E(v) : c'(e) \in I\}$. A very natural question is whether our main result, Theorem 2.1 also holds if we require that edge-colouring c' must be strictly finer than c.

Unfortunately, this stronger version of Theorem 2.1 does not hold in general. Consider a balanced k-edge-colouring c of complete bipartite graph $K_{1,k(k+1)}$ and assign k-lists to the edges such that for any k-subset L of $\{1, 2, \ldots, k+1\}$ and for each colour i there is an edge e with $c(e) = i$ and $L(e) = L$. As any list edge-colouring must use at least two colours for each c-colour-class, no list edge-colouring is strictly finer than c.

Note that there is a well-known stronger form of Lemma 3.2, namely, for any positive integer k and any bipartite graph G along with a nested set-system on each colour class there is a k-edge-colouring of $G = (V, E)$ in such a way that beyond the consequence of Lemma 3.2, we have that for any element X of one of the nested systems and for any colour i in the colouring the number of i-coloured edges incident with X is either $\left\lceil \frac{|E(X)|}{k} \right\rceil$ or $\left\lfloor \frac{|E(X)|}{k} \right\rfloor$, where $E(X)$ stands for the set of edges of G incident with X. To prove this generalization, it is enough to modify the construction in the proof of Lemma 3.2 in such a way that instead of direct sx and yt edges, we introduce a two trees with roots s and t and leaves in the corresponding colour classes such that the edge cuts of these trees determine the nested set systems, and orient the edges of these trees in the natural way. It is a natural question whether a stronger form of Corollary 3.1 is true, that is, if positive integer k and bipartite graph G with edge-lists of size k is given there always exists a list edge-colouring such that for any vertex v of G no more than $\lceil \frac{d(v)}{k} \rceil$ edges of the same colour is incident with v and for any set X of the nested systems at most $\lceil \frac{|E(X)|}{k} \rceil$ edges of $E(X)$ can have the same colour. We leave this open question to the reader.

References

[1] Paul Erdős, Arthur L. Rubin, and Herbert Taylor. Choosability in graphs. In *Proceedings of the West Coast Conference on Combinatorics, Graph Theory and Computing (Humboldt State Univ., Arcata, Calif., 1979)*, Congress. Numer., XXVI, pages 125–157, Winnipeg, Man., 1980. Utilitas Math.

[2] D. Gale and L.S. Shapley. College admissions and stability of marriage. *Amer. Math. Monthly*, 69(1):9–15, 1962.

[3] Fred Galvin. The list chromatic index of a bipartite multigraph. *J. Combin. Theory Ser. B*, 63(1):153–158, 1995.

[4] Dan Gusfield and Robert W. Irving. *The stable marriage problem: structure and algorithms*. MIT Press, Cambridge, MA, 1989.

[5] D. Kőnig. Graphok és alkalmazásuk a determinánsok elméletére [Hungarian]. *Mathematikai és Természettudományi Értesítő*, 34:104–119, 1916. [German translation: Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre, *Mathematische Annalen* 77 (1916) 453–465].

[6] Frédéric Maffray. Kernels in perfect line-graphs. *J. Combin. Theory Ser. B*, 55(1):1–8, 1992.

[7] Alvin E. Roth and Marilda A. Oliveira Sotomayor. *Two-sided matching*. Cambridge University Press, Cambridge, 1990. A study in game-theoretic modeling and analysis, With a foreword by Robert Aumann.