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Independent arborescences in directed graphs

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ABSTRACT

As a vertex-disjoint analogue of Edmonds' arc-disjoint arborescences theorem, it was conjectured that given a directed graph D with a specified vertex r, there are k spanning arborescences rooted at r such that for every vertex v of D the k directed walks from r to v in these arborescences are internally vertex-disjoint if and only if for every vertex v of D there are k internally vertex-disjoint directed walks from r to v. Whitty (1987) [10] affirmatively settled this conjecture for $k \leq 2$, and Huck (1995) [6] constructed counterexamples for $k \geq 3$, and Huck (1999) [7] proved that the conjecture is true for every k when D is acyclic. In this paper, we generalize these results by using the concept of "convexity" which is introduced by Fujishige (2010) [4].

1. Introduction

Let D be a directed graph with n vertices and m arcs, where we assume that D has no loop but may have parallel arcs. We denote by $V(D)$ and $A(D)$ the vertex set and the arc set of D, respectively. For $a \in A(D)$, let $\partial^+ a$ and $\partial^- a$ be the tail and the head of a, respectively. For $v \in V(D)$, define

$$\Gamma^+(v) := \{a \in A(D) \mid \partial^+ a = v\}, \qquad \Gamma^-(v) := \{a \in A(D) \mid \partial^- a = v\}.$$

For a subgraph H of D and $v \in V(H)$, define

$$\Gamma^+_H(v) := \Gamma^+(v) \cap A(H), \qquad \Gamma^-_H(v) := \Gamma^-(v) \cap A(H).$$

For $X \subseteq V(D)$, we denote by $D[X]$ the subgraph of D induced by X. For $B \subseteq A(D)$, let $D \setminus B$ be the graph obtained by removing all the arcs of B from D. For $v \in V(D)$, define $D - v := D[V(D) \setminus \{v\}]$.

A *directed walk* P is an alternating sequence $v_0, a_1, v_1, \ldots, a_l, v_l$ of vertices $v_0, v_1, \ldots, v_l$ and arcs $a_1, a_2, \ldots, a_l$ such that $a_i = v_{i-1}v_i$. In this paper, we allow $v_i = v_j$ and $a_i = a_j$ for distinct i, j. We call v_0 and v_l the *initial vertex* and the *terminal vertex* of P, respectively. A directed walk with an initial vertex u and a terminal vertex v is called a (u, v)-*walk*. Notice that a vertex v is a (v, v)-walk. A vertex v is said to be *reachable* from a vertex u in D, if there is a (u, v)-walk in D. A (u, v)-walk P containing at least one arc is called a *directed cycle*, if $u = v$. A directed graph with no directed cycle is said to be *acyclic*. A subset X of $V(D)$ is said to be *convex*, if for every (u, v)-walk P such that $u, v \in X$, all the intermediate vertices of P are also in X. Notice that the intersection of convex sets is convex. Suppose that we are given distinct (u_i, v_i)-walks P_i $(i = 1, 2, \ldots, k)$. Walks $P_1, P_2, \ldots, P_k$ are said to be *internally disjoint*, if

$$W_i \cap W_j = (\{u_i\} \cap \{u_j\}) \cup (\{v_i\} \cap \{v_j\})$$

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for every $i, j = 1, 2, \ldots, k$ such that $i \neq j$, where W_i and B_i represent the sets of vertices and arcs that P_i traversed, respectively. If D is acyclic, then there is a *topological ordering* $\pi : V(D) \to \{1, 2, \ldots, n\}$ such that $\pi(\partial^- a) < \pi(\partial^+ a)$ for every $a \in A(D)$. For $v \in V(D)$, define $\hat{\pi}(v) := n + 1 - \pi(v)$

An acyclic graph T is called an *arborescence*, if there is $r \in V(T)$ such that $|\Gamma_T^-(r)| = 0$, and $|\Gamma_T^-(v)| = 1$ for every $v \in V(T) \setminus \{r\}$. We call such an arborescence T an *r-arborescence*. For an r-arborescence T and $v \in V(T)$, we denote by rTv the unique (r, v)-walk in T. We say that r_i-arborescences $T_i (i = 1, 2, \ldots, k)$ are *independent*, if for every vertex v belonging to any two of them, the walks from the roots to v in those two arborescences are internally disjoint, i.e., for every $i, j = 1, 2, \ldots, k$ such that $i \neq j$ and every $v \in V(T_i) \cap V(T_j)$, r_iT_iv and r_jT_jv are internally disjoint.

1.1. Edmonds' theorem and its extensions

Edmonds [2] proved the following fundamental theorem about existence of arc-disjoint arborescences.

Theorem 1 (*Edmonds [2]*). *Let D be a directed graph with a specified vertex r. There are k arc-disjoint spanning r-arborescences if and only if for every $v \in V(D)$ there are k arc-disjoint (r, v)-walks.*

Kamiyama, Katoh and Takizawa [8] generalized Theorem 1 to the multiple roots case by using the concept of reachability. Furthermore, Fujishige [4] extended the results of [8] by employing the concept of convexity instead of reachability (for related topics see [1]).

In Theorem 1, an obvious necessary condition is also sufficient. So, as a vertex-disjoint analogue, the following question naturally arises.

Question 1 (*Frank [9, p. 235]*). *Let D be a directed graph with a specified vertex r. There are k independent spanning r-arborescences if and only if for every $v \in V(D)$ there are k internally disjoint (r, v)-walks.*

Whitty [10] affirmatively settled Question 1 for $k \leq 2$. Huck [6] constructed counterexamples for $k \geq 3$. Furthermore, Huck [7] proved that if D is acyclic, then Question 1 is true for every k.

1.2. Our problem and results

The goal of this paper is to generalize the results about Question 1 in the same manner as Fujishige's extension of Edmonds' theorem. More precisely, we consider the following Question 2.

Question 2. *Let D be a directed graph with (possibly not distinct) specified vertices $r_1, r_2, \ldots, r_k$ and convex subsets $C_1, C_2, \ldots, C_k \subseteq V(D)$ such that $r_i \in C_i$. There are independent r_i-arborescences $T_i (i = 1, 2, \ldots, k)$ such that $V(T_i) = C_i$ if and only if for every $v \in V(D)$ there are internally disjoint (r_i, v)-walks $P_i (i \in I(v))$, where $I(v)$ is the set of i such that $v \in C_i$.*

Question 2 is a generalization of Question 1 in the sense that there may be multiple roots and the arborescence need not span $V(D)$. By the result of Huck [6], Question 2 is in general not true for the case where there is a vertex contained in more than two of $C_1, C_2, \ldots, C_k$ even if $r_1, r_2, \ldots, r_k$ are identical. In this paper, we prove that Question 2 is true for the following three cases.

Case 1. $r_1 = r_2 = \cdots = r_k (=: r)$ and every vertex of $V(D) \setminus \{r\}$ is contained in at most two of $C_1, C_2, \ldots, C_k$.
Case 2. $r_1 = r_2 = \cdots = r_k$ and D is acyclic.
Case 3. D is acyclic and every vertex of $V(D)$ is contained in at most two of $C_1, C_2, \ldots, C_k$.

If $k = 2$, then every vertex is automatically contained in at most two convex sets. Thus, the result for Case 1 generalizes the result of Whitty [10] in the sense that each arborescence does not necessarily span all vertices. The result for Case 2 also generalizes the result of Huck [7] in the same sense. The result for Case 3 is a proper generalization in the sense that a given directed graph has multiple roots.

2. Case 1

In this section, we prove the following theorem.

Theorem 2. *Let D be a directed graph with a specified vertex r and convex subsets $C_1, C_2, \ldots, C_k \subseteq V(D)$ such that $r \in C_i$ and every vertex of $V(D) \setminus \{r\}$ is contained in at most two of $C_1, C_2, \ldots, C_k$. There are independent r-arborescences $T_1, T_2, \ldots, T_k$ such that $V(T_i) = C_i$ if and only if for every $v \in V(D)$ there are $|I(v)|$ internally disjoint (r, v)-walks, where $I(v)$ is the set of i such that $v \in C_i$.*

Proof. Since the *only if* part is immediate, we prove the other direction. If $k = 1$, then the theorem immediately follows from the definition of a convex set. So, we assume that $k \geq 2$. Let V_1 be the set of vertices of $V(D)$ that are contained in exactly one of $C_1, C_2, \ldots, C_k$. Define

$$\mathcal{X} := \{C_i \cap C_j \mid i, j = 1, 2, \ldots, k,\ i \neq j,\ C_i \cap C_j \neq \{r\}\}.$$

Since every vertex of $V(D) \setminus \{r\}$ is contained in at most two of $C_1, C_2, \ldots, C_k$, $X \cap Y = \{r\}$ for distinct $X, Y \in \mathcal{X}$. For $X \in \mathcal{X}$, let I_X be the unique pair $\{i, j\}$ such that $X = C_i \cap C_j$.

By the definition of a convex set, for every $X \in \mathcal{X}$ and every $v \in X$, intermediate vertices of an (r, v)-walk are in X, i.e., there are two internally disjoint (r, v)-walks in $D[X]$. By this fact and the result of [10], there are two independent spanning r-arborescences $T_i^X (i \in I_X)$ in $D[X]$. Notice that r-arborescences $T_i^X (X \in \mathcal{X}; i \in I_X)$ are independent.

For $i = 1, 2, \ldots, k$, let D_i be a graph obtained from $D[C_i]$ by shrinking $C_i \setminus V_1$ into a new vertex c_i. Since C_i is a convex set, there is an (r, v)-walk in $D[C_i]$ for every $v \in C_i$. So, for every $v \in C_i \cap V_1$, there is a (c_i, v)-walk in D_i, and thus there is a spanning c_i-arborescence T_i' in D_i. We can construct desired arborescences by combining $T_i^X (X \in \mathcal{X}$ such that $i \in I_X)$ and T_i' for each $i = 1, 2, \ldots, k$. □

By using Theorem 2, we can obtain the following algorithmic result.

Theorem 3. *Let D be a directed graph with a specified vertex r and non-singleton convex subsets $C_1, C_2, \ldots, C_k \subseteq V(D)$ such that $r \in C_i$ and every vertex of $V(D) \setminus \{r\}$ is contained in at most two of $C_1, C_2, \ldots, C_k$. We can discern the existence of independent r-arborescences $T_1, T_2, \ldots, T_k$ such that $V(T_i) = C_i$ and find such arborescences if they exist in $O(n^3 + m)$ time.*

Proof. We first transform D so that $m = O(n^2)$ by removing unnecessary parallel arcs in $O(m)$ time. By using a well-known technique (described, for example, in [3]), for $v \in V(D)$ we can discern whether there are $|I(v)|$ internally disjoint (r, v)-walks in $O(n^2)$ time, where $I(v)$ is the set of i such that $v \in C_i$. So, we can discern the existence of desired arborescences in $O(n^3)$ time. Next we consider the time required for finding desired arborescences. Since we can construct $\mathcal{X}$ in $O(n^3)$ time, we evaluate the time required for finding $T_i^X (X \in \mathcal{X}; i \in I_X)$. It is known [5,7] that we can find arborescences $T_i^X (i \in I_X)$ in $O(|X|^3)$ time for each $X \in \mathcal{X}$. Since $\sum_{X \in \mathcal{X}} |X| \leq 2n + |\mathcal{X}|$, what remains is to evaluate $|\mathcal{X}|$. Since every vertex of $V \setminus \{r\}$ is contained in at most one element in $\mathcal{X}$ and $X \setminus \{r\} \neq \emptyset$ for every $X \in \mathcal{X}$, we have $|\mathcal{X}| \leq n$. So, the time required for finding $T_i^X (X \in \mathcal{X}; i \in I_X)$ is $O(n^3)$. □

3. Case 2

In this section, we prove the following theorem.

Theorem 4. *Let D be an acyclic directed graph with a specified vertex r and convex subsets $C_1, C_2, \ldots, C_k \subseteq V(D)$ such that $r \in C_i$. There are independent r-arborescences $T_1, T_2, \ldots, T_k$ such that $V(T_i) = C_i$ if and only if for every $v \in V(D)$ there are $|I(v)|$ internally disjoint (r, v)-walks, where $I(v)$ is the set of i such that $v \in C_i$.*

Proof. Since the *only if*-part is immediate, we prove the other direction. Our proof is based on the proof of Huck [7] for Question 1 in the acyclic case. Let V_0 be the set of $v \in V(D)$ such that $I(v) = \emptyset$. For $v \notin V_0$, an (r, v)-walk contains no vertex of V_0 by the definition of a convex set. So, removing V_0 does not affect the existence of $|I(v)|$ internally disjoint (r, v)-walks for $v \notin V_0$. So, without loss of generality, we can make the following assumption.

Assumption 1. For every $v \in V(D)$, $I(v) \neq \emptyset$.

Furthermore, by the definition of internal disjointness, we can make the following assumption.

Assumption 2. All the parallel arcs of D are in $\Gamma^+(r)$.

Since there are $|I(v)|$ internally disjoint (r, v)-walks for every $v \in V(D) \setminus \{r\}$, we have

$$|\Gamma^-(v)| \geq |I(v)| \quad (v \in V(D) \setminus \{r\}). \tag{1}$$

So, it suffices to prove that if (1) holds, then there are desired arborescences. In the sequel, we assume that (1) holds. Notice if (1) holds, then every vertex of $V(D)$ is reachable from r by acyclicity of D and Assumption 1.

Let T be an r-arborescence T. A topological ordering π of $T - r$ is said to be (D, T)-*feasible*, if $\hat{\pi}$ is a topological ordering of $D \setminus A(T)[V(T) \setminus \{r\}]$. Moreover, T is said to be *D-eligible*, if there is a (D, T)-feasible ordering.

Claim 1. *There is a D-eligible r-arborescence T_k such that $V(T_k) = C_k$.*

Proof. We prove the claim by induction on n. For $n = 1$, the claim clearly holds. Assuming that the claim holds for $n = N \geq 1$, we consider the case of $n = N + 1$. Since D is acyclic, there is $s \in V(D)$ such that $\Gamma^+(s) = \emptyset$. Since every vertex of D is reachable from r in D and $n \geq 2$, we have $s \neq r$. Define $D' := D - s$, $V' := V(D) \setminus \{s\}$ and $C_i' := C_i \setminus \{s\}$. For $v \in V'$, let $I'(v)$ be the set of i such that $v \in C_i'$. For every $v \in V'$, $I(v) = I'(v)$ and $\Gamma^-(v) = \Gamma^-_{D'}(v)$ by $\Gamma^+(s) = \emptyset$. So,

$$|\Gamma^+_{D'}(v)| \geq |I'(v)| \quad (v \in V' \setminus \{r\}).$$

Thus, by the induction hypothesis, there is a D'-eligible r-arborescence T_k' such that $V(T_k') = C_k'$. Let π' be a (D', T_k')-feasible topological ordering. We will prove that an r-arborescence T_k such that $V(T_k) = C_k$ and a (D, T_k)-feasible ordering π can be constructed from T_k' and π'. If $s \notin C_k$, then the proof is done by setting $T_k := T_k'$ and $\pi := \pi'$.

If $s \in C_k$, then we need to add s to T_k' as well as an appropriate arc a' of $\Gamma^-(s)$ such that $\partial^+ a' \in C_k$. Define a vertex v' as follows. Let S be the set of vertices v of V such that there is an arc from v to s. If $r \in S$, then set $v' := r$. Otherwise, set v'

to be the unique element of $\arg\max_{v\in S} \pi'(v)$, and a' by the unique arc from v' to s. Notice that $S \neq \emptyset$ by $\Gamma^-(s) \neq \emptyset$. Since every vertex of D is reachable from r in D, there is an (r, s)-walk containing v', which implies $v' \in C_k$ by the definition of a convex set and $s \in C_k$. Thus, we can obtain T_k by adding a' to T'_k.

Now we explain how to construct π. If $v' = r$, then it suffices to set $\pi(s) := |C_k| - 1$ and $\pi(v) := \pi'(v)$ for $v \in C'_k \setminus \{r\}$. If $v' \neq r$, then define

$$\pi(v) := \begin{cases} \pi'(v') & \text{if } v = s, \\ \pi'(v) & \text{if } v \neq s \text{ and } \pi'(v) < \pi'(v'), \\ \pi'(v) + 1 & \text{if } v \neq s \text{ and } \pi'(v) \geq \pi'(v'). \end{cases}$$

By the induction hypothesis, π is a topological ordering in $T_k - r$. Furthermore, since $\pi(s) > \pi(v)$ for every $v \in S \setminus \{v'\}, \pi(\partial^- a) > \pi(\partial^+ a)$ for every arc a of $D \setminus A(T_k)[C_k \setminus \{r\}]$, which implies that $\hat{\pi}$ is a topological ordering in $D \setminus A(T_k)[C_k \setminus \{r\}]$. □

Claim 2. *There are independent r-arborescences $T_1, T_2, \ldots, T_k$ such that $V(T_i) = C_i$.*

Proof. We prove the claim by induction on k. For $k = 1$, since every vertex of D is reachable from r, there is a spanning r-arborescence.

Assuming that the claim holds for the case of $k = N \geq 1$, we consider the case of $k = N + 1$. By Claim 1, there is a D-eligible r-arborescence T_k such that $V(T_k) = C_k$. For $v \in V(D)$, let $I^\circ(v)$ be the set of $i = 1, 2, \ldots, k-1$ such that $v \in C_i$. Let D° be the graph obtained from $D \setminus A(T_k)$ by removing $v \in V$ such that $I^\circ(v) = \emptyset$ and arcs around such vertices. Since all the parallel arcs of D° are clearly in $\Gamma^+_{D^\circ}(r)$, it suffices to show that

$$|\Gamma^-_{D^\circ}(v)| \geq |I^\circ(v)| \quad (v \in V(D^\circ) \setminus \{r\}). \tag{2}$$

Obviously,

$$|\Gamma^-_{D\setminus A(T_k)}(v)| \geq |I^\circ(v)| \quad (v \in V(D^\circ) \setminus \{r\}). \tag{3}$$

Notice that $I^\circ(v) = \emptyset$ if and only if $I(v) = \{k\}$. So, if there is no arc $a \in A$ such that $I(\partial^+ a) = \{k\}$ and $I(\partial^- a) \neq \{k\}$, then (2) follows from (3). If there is such an arc a of A, then there is an $(r, \partial^- a)$-walk in D containing $\partial^+ a$ and $\partial^- a \in C_i$ for some $i = 1, 2, \ldots, k-1$. Since $\partial^+ a \notin C_i$, this contradicts the convexity of C_i. So, there can not be such an arc a.

By (2) and the induction hypothesis, there are independent r-arborescences $T_1, T_2, \ldots, T_{k-1}$ in D° such that $V(T_i) = C_i$. Now we prove that arborescences $T_1, T_2, \ldots, T_k$ are independent. Since $T_1, T_2, \ldots, T_{k-1}$ are independent by the induction hypothesis, we prove that T_k and T_i are independent for $i = 1, 2, \ldots, k-1$. For this, it suffices to prove that rT_kv and rT_iv are internally disjoint for every $v \in C_k \cap C_i$. Let π be a (D, T_k)-feasible topological ordering. By the definition of a convex set, vertices of rT_kv and rT_iv are contained in $C_i \cap C_k$. For an intermediate vertex w of $rT_kv, \pi(v) < \pi(w)$. Since $\hat{\pi}$ is a topological ordering in $D \setminus A(T_k)[C_k \setminus \{r\}], \pi(v) > \pi(w)$ for every intermediate vertex w of rT_iv. So, rT_kv and rT_iv are internally disjoint. □

Theorem 4 follows from Claim 2. □

By using Theorem 4, we can obtain the following algorithmic result.

Theorem 5. *Let D be a weakly connected acyclic directed graph with a specified vertex r and non-singleton convex subsets $C_1, C_2, \ldots, C_k \subseteq V(D)$ such that $r \in C_i$. We can discern the existence of independent r-arborescences $T_1, T_2, \ldots, T_k$ such that $V(T_i) = C_i$ and find such arborescences if they exist in $O(km)$ time.*

Proof. By Theorem 4, we can test the existence of desired arborescences by checking if (1) holds in $O(kn + m)$ time. If (1) holds, then following the proof of Claim 1 we can find in $O(m)$ time a D-eligible r-arborescence T_k such that $V(T_k) = C_k$. Take every topological ordering π^* of $D - r$. Then, start with T such that $V(T) = \{r\}, A(T) = \emptyset$, and the empty topological ordering π. Following the topological ordering π^* in decreasing order, we grow T and update the topological ordering π in $T - r$ as described in the proof of Claim 1. By using an appropriate data structure, we can execute each update in $O(1)$ time. Then, by recursively applying this operation for $D \setminus A(T_k)$, we can find desired independent arborescences in $O(km)$ time. □

4. Case 3

In this section, we prove the following theorem.

Theorem 6. *Let D be an acyclic directed graph with (possibly not distinct) specified vertices $r_1, r_2, \ldots, r_k$ and and convex subsets $C_1, C_2, \ldots, C_k \subseteq V(D)$ such that $r_i \in C_i$ and every vertex of D is contained in at most two of $C_1, C_2, \ldots, C_k$. there are independent r_i-arborescences T_i $(i = 1, 2, \ldots, k)$ such that $V(T_i) = C_i$ if and only if for every $v \in V(D)$ there are internally disjoint (r_i, v)-walks $P_i (i \in I(v))$, where $I(v)$ is the set of i such that $v \in C_i$.*

Proof. Since the *only if* part is immediate, we prove the other direction. For $v \in V(D)$, define $J(v)$ be the set of $i \in I(v)$ such that $v \neq r_i$. For $v \in V(D)$ such that $v \in C_i$, an (r_i, v)-walk consists of arcs $a \in A(D)$ such that $\partial^+ a, \partial^- a \in C_i$, due to the definition of a convex set. So, removing arcs for which Assumption 3 does not hold does not affect the existence of internally disjoint (r_i, v)-walks $P_i (i \in I(v))$. So, without loss of generality, we can make the following assumption.

Assumption 3. For every $a \in A(D)$, there is i such that $\partial^+ a, \partial^- a \in C_i$.

Furthermore, by the definition of internal disjointness, we can make the following assumption.

Assumption 4. Suppose that $a, b \in A(D)$ are parallel, and let v be the tail of a, b. Then, $|I(v) \setminus J(v)| = 2$.

By the definition of internal disjointness, if for every $v \in V(D)$ there are internally disjoint (r_i, v)-walks $P_i (i \in I(v))$, then

$$|\Gamma^-(v)| \geq |J(v)| \quad (v \in V(D)). \tag{4}$$

By the definition of a convex set, $\partial^+ a \in C_i$ for every $v \in V(D)$, every $i \in J(v)$ and every arc a of P_i entering v. So,

$$\text{there is } a \in \Gamma^-(v) \quad \text{such that } \partial^+ a \in C_i \ (v \in V(D); i \in J(v)). \tag{5}$$

If (5) holds, since D is acyclic, then every vertex of C_i is reachable from r_i in $D[C_i]$. We will prove that if (4) and (5) hold, then there are desired arborescences. In the sequel, we assume that (4) and (5) hold. For $v \in V(D)$, let $I^\circ(v)$ be the set of $i = 1, 2, \ldots, k-1$ such that $v \in C_i$, and let $J^\circ(v)$ be the set of $i \in I^\circ(v)$ such that $v \neq r_i$.

Definition 1. Let T_k be r_k-arborescence such that $V(T_k) = C_k$. For $i = 1, 2, \ldots, k-1$, define

$$X_i := \begin{cases} C_i \cap C_k & \text{if } r_i \neq r_k, \\ (C_i \cap C_k) \setminus \{r_k\} & \text{if } r_i = r_k. \end{cases}$$

Then, $\pi_1, \pi_2, \ldots, \pi_{k-1}$ such that π_i is a topological ordering in $T_k[X_i]$ are said to be (D, T_k)-*feasible*, if $\hat{\pi}_i$ is a topological ordering in $D \setminus A(T_k)[X_i]$.

Definition 2. An r_k-arborescence T_k such that $V(T_k) = C_k$ is said to be *D-eligible*, if

(i) for every $v \in V(D)$ and every $i \in J^\circ(v)$, there is $a \in \Gamma^-_{D \setminus A(T_k)}(v)$ such that $\partial^+ a \in C_i$, and
(ii) there are (D, T_k)-feasible topological orderings $\pi_1, \pi_2, \ldots, \pi_{k-1}$.

Claim 3. *There is a D-eligible r_k-arborescence T_k such that $V(T_k) = C_k$.*

Proof. We prove the claim by induction on n. For $n = 1$, the claim clearly holds. Assuming that the claim holds for $n = N \geq 1$, we consider the case of $n = N + 1$. If $|C_i| = 1$ for some $i = 1, 2, \ldots, k$, then the proof is done. So, we assume that $|C_i| \geq 2$ for every $i = 1, 2, \ldots, k$. Since D is acyclic, there is $s \in V(D)$ such that $\Gamma^+(s) = \emptyset$. Since every $v \in C_i$ is reachable from r_i in $D[C_i]$ and $|C_i| \geq 2$, we have $s \neq r_i$. Define D', V', C'_i and $I'(v)$ in the same manner as in Claim 1. For $v \in V'$, let $J'(v)$ be the set of $i \in I'(v)$ such that $v \neq r_i$. For every $v \in V'$, $J(v) = J'(v)$ and $\Gamma^-(v) = \Gamma^-_{D'}(v)$ by $\Gamma^+(s) = \emptyset$. So, by the induction hypothesis, there is a D'-eligible r_k-arborescence T'_k such that $V(T'_k) = C'_k$. Define $X'_1, X'_2, \ldots, X'_{k-1}$ for $C'_1, C'_2, \ldots, C'_k$ in the same manner as X_i in Definition 1. Let $\pi'_1, \pi'_2, \ldots, \pi'_{k-1}$ be (D', T'_k)-feasible topological orderings. We will prove that an r_k-arborescence T_k such that $V(T_k) = C_k$ and (D, T_k)-feasible topological orderings $\pi_1, \pi_2, \ldots, \pi_{k-1}$ can be constructed from T'_k and $\pi'_1, \pi'_2, \ldots, \pi'_{k-1}$. If $s \notin C_k$, then we can obtain T_k and $\pi_1, \pi_2, \ldots, \pi_k$ by setting $T_k := T'_k$ and $\pi_i := \pi'_i$.

If $s \in C_k$, then we need to add s to T'_k as well as an appropriate arc $a' \in \Gamma^-(s)$ such that $\partial^+ a \in C'_k$. Since by (5) there is $a' \in \Gamma^-(s)$ such that $\partial^+ a' \in C_k$, if $I(s) = \{k\}$, then we can obtain T_k by adding a' to T'_k. By $I'(s) = \emptyset$, Condition (i) of Definition 2 is satisfied. Moreover, $s \notin X_i$ for every $i = 1, 2, \ldots, k$, which implies $X_i = X'_i$ for every $i = 1, 2, \ldots, k$. So, we can obtain $\pi_1, \pi_2, \ldots, \pi_{k-1}$ by setting $\pi_i := \pi'_i$.

Assume that s is contained in C_k and (without loss of generality) C_{k-1}. We first consider the case where there is $a' \in \Gamma^-(s)$ such that $\partial^+ a' \notin C_{k-1}$. Since $\partial^+ a' \in C_k$ by Assumption 3, we can obtain T_k by adding a' to T'_k. By (5), there is $b \in \Gamma^-(s)$ such that $\partial^+ b \in C_{k-1}$. By $\partial^+ a' \notin C_{k-1}$, b is an arc of $\Gamma^-(s) \setminus \{a'\} = \Gamma^-_{D \setminus A(T_k)}(s)$. So, Condition (i) of Definition 2 holds. Next we consider Condition (ii) of Definition 2. For every $i = 1, 2, \ldots, k-2$, it suffices to set $\pi_i := \pi'_i$ by $s \notin X_i$. Define $\pi_{k-1}(s) := |X_{k-1}|$ and $\pi_{k-1}(v) := \pi'_{k-1}(v)$ for $v \in X_k \setminus \{s\}$. By $\partial^+ a' \notin C_{k-1}$, we can easily prove that π_{k-1} is a desired topological ordering.

Now we consider the case where $\partial^+ a \in C_{k-1}$ for every $a \in \Gamma^-(s)$. By (5), there is $a \in \Gamma^-(s)$ such that $\partial^+ a \in C_k \cap C_{k-1}$, and at least one of the following two statements holds.

(a) There are $a, b \in \Gamma^-(s)$ such that $a \neq b$ and $\partial^+ a, \partial^+ b \in C_k \cap C_{k-1}$.
(b) There is $a \in \Gamma^-(s)$ such that $\partial^+ a \notin C_k$.

So, even if we add any arc $a \in \Gamma^-(s)$ such that $\partial^+ a \in C_k \cap C_{k-1}$ to T'_k, Condition (i) of Definition 2 holds. What remains to show is how to choose $a \in \Gamma^-(s)$ such that $\partial^+ a \in C_k \cap C_{k-1}$ so that Condition (ii) of Definition 2 is satisfied. For every $i = 1, 2, \ldots, k-2$, whichever arc $a \in \Gamma^-(s)$ we add to T'_k, it suffices to set $\pi_i := \pi'_i$ by $s \notin X_i$. So, we consider π_{k-1}. Let S be the set of $v \in V(D)$ such that there is an arc from v to s. Define v' as follows. If $r_k = r_{k-1}$ and $r_k \in S \cap C_k$, set $v' = r_k$. Otherwise, set v' to be the unique element of $\arg\max_{v \in S \cap C_k} \pi'_{k-1}(v)$, and let a' be the unique arc from v' to s. We can obtain T_k adding a' to T'_k. Define $\pi_{k-1}: X_{k-1} \to \{1, \ldots, |X_{k-1}|\}$ by

$$\pi_{k-1}(v) := \begin{cases} \pi'_{k-1}(v'), & \text{if } v = s, \\ \pi'_{k-1}(v), & \text{if } v \neq s \text{ and } \pi'_{k-1}(v) < \pi'_{k-1}(v'), \\ \pi'_{k-1}(v) + 1, & \text{if } v \neq s \text{ and } \pi'_{k-1}(v) \geq \pi'_{k-1}(v'). \end{cases}$$

Then, we can prove that $\pi_1, \pi_2, \ldots, \pi_{k-1}$ are (D, T_k)-feasible topological orderings in the same manner as in the last part of the proof of Claim 1. □

Claim 4. *There are independent r_i-arborescences $T_i (i = 1, 2, \ldots, k)$ such that $V(T_i) = C_i$.*

Proof. We prove the claim by induction on k. For $k = 1$, since every $v \in C_1$ is reachable from r_1 in $D[C_1]$, the claim holds for $k = 1$. Assuming that the claim holds for $k = N \geq 1$, we consider the case of $k = N + 1$.

By Claim 3, there is a D-eligible r_k-arborescence T_k such that $V(T_k) = C_k$. In order to apply the induction hypothesis, we need to transform $D \setminus A(T_k)$ so that Assumptions 3 and 4 are satisfied. Let H_1 be the graph by transforming $D \setminus A(T_k)$ so that Assumption 3 is satisfied. Moreover, let H_2 be be the graph by transforming H_1 so that Assumption 4 is satisfied.

We first prove that

$$|\Gamma^-_{D\setminus A(T_k)}(v)| \geq |J^\circ(v)| \quad (v \in V(D)), \tag{6}$$

$$\text{there is } a \in \Gamma^-_{D\setminus A(T_k)}(v) \quad \text{such that } \partial^+ a \in C_i \ (v \in V(D);\, i \in J^\circ(v)). \tag{7}$$

For every $V \setminus C_k$ and every $v \in C_k$ such that $|I(v)| = 1$, (6) and (7) clearly hold. Let v be a vertex of C_k such that $|I(v)| = 2$ (say, given by $I(v) = \{k, i\}$). If $v = r_i$, then (6) and (7) clearly hold. If $v \neq r_i$, then (6) and (7) hold since there is $a \in \Gamma^-_{D\setminus A(T_k)}(v)$ such that $\partial^+ a \in C_i$ by Definition 2.

Next we prove that

$$|\Gamma^-_{H_1}(v)| \geq |J^\circ(v)| \quad (v \in V(D)), \tag{8}$$

$$\text{there is } a \in \Gamma^-_{H_1}(v) \quad \text{such that } \partial^+ a \in C_i \ (v \in V(D);\, i \in J^\circ(v)). \tag{9}$$

We say that $a \in A \setminus A(T_k)$ is *illegal*, if there is no $i = 1, 2, \ldots, k-1$ such that $\partial^+ a, \partial^- a \in C_i$, i.e., H_1 is obtained by removing all the illegal arcs from $D \setminus A(T_k)$. By Assumption 3, $\partial^+ a, \partial^- a \in C_k$ for every illegal $a \in A \setminus A(T_k)$. So, for every $v \notin C_k$, there is no illegal arc of $\Gamma^-_{D\setminus A(T_k)}(v)$, which implies that $\Gamma^-_{H_1}(v) = \Gamma^-_{D\setminus A(T_k)}(v)$ for every $v \in V \setminus C_k$. Thus, since (6) and (7) hold, (8) and (9) hold for every $v \in V \setminus C_k$. Since (8) and (9) clearly hold for every vertex v contained in only C_k, we consider $v \in C_k$ such that $|I(v)| = 2$ (say, given by $I(v) = \{k, i\}$). If $r_i = v$, then (8) and (9) clearly hold. If $r_i \neq v$, then there is $a \in \Gamma^-_{D\setminus A(T_k)}(v)$ such that $\partial^+ a \in C_i$ by (7). Since a is not illegal, $a \in \Gamma^-_{H_1}(v)$.

We are now ready to prove that

$$|\Gamma^-_{H_2}(v)| \geq |J^\circ(v)| \quad (v \in V(D)), \tag{10}$$

$$\text{there is } a \in \Gamma^-_{H_2}(v) \quad \text{such that } \partial^+ a \in C_i \ (v \in V(D);\, i \in J^\circ(v)). \tag{11}$$

Assume that there are parallel arcs of $A(H_1)$ whose tail is $t \in V(D)$ such that $|I^\circ(t) \setminus J^\circ(t)| < 2$. Let h be the head of these parallel arcs. By Assumption 4, $r_k = t$ and there is $i = 1, 2, \ldots, k-1$ such that $r_i = t$. Since there is no illegal arc of $A(H_1)$, we have $h \in C_i$. If $|I^\circ(h)| = 1$, then (10) and (11) clearly hold even if we remove all but one parallel arc between t and h, So, we assume that $h \in C_j$ for some $j = 1, 2, \ldots, k-1$ such that $j \neq i$. If $r_j = h$, then (10) and (11) clearly hold. If $r_j \neq h$, then by (9) there is $a \in \Gamma^-_{H_1}(h)$ such that $\partial^+ a \in C_j$. Since $j \neq k, i$, we have $t \notin C_j$, which implies that $\partial^+ a \neq t$. So, (10) and (11) hold.

By (10), (11) and the induction hypothesis, there are independent r_i-arborescences T_i $(i = 1, 2, \ldots, k-1)$ in H_2 such that $V(T_i) = C_i$. In order to prove the claim, it is sufficient to prove that $r_k T_k v$ and $r_i T_i v$ are internally disjoint for every $i = 1, 2, \ldots, k-1$ and every $v \in C_k \cap C_i$. Let $\pi_1, \pi_2, \ldots, \pi_{k-1}$ be (D, T_k)-feasible topological orderings. Vertices of $r_k T_k v$ (resp., $r_i T_i v$) contained in X_i form a subwalk of $r_k T_k v$ (resp., $r_i T_i v$) whose terminal vertex is v by the definition of a convex set. Also, π_i (resp., $\hat{\pi}_i$) is a topological ordering in $T_k[X_i]$ (resp., $D \setminus A(T_k)[X_i]$). Hence we have $\pi_i(v) < \pi_i(w)$ (resp., $\pi_i(v) > \pi_i(w)$) for every vertex w of $r_k T_k v$ (resp., $r_i T_i v$) contained in $X_i \setminus \{v\}$. So, $r_k T_k v$ and $r_i T_i v$ are internally disjoint. □

Theorem 6 follows from Claim 4. □

By using Theorem 6, we can obtain the following algorithmic result.

Theorem 7. *Let D be a weakly connected acyclic directed graph with (possibly not distinct) specified vertices $r_1, r_2, \ldots, r_k$ and and non-singleton convex subsets $C_1, C_2, \ldots, C_k \subseteq V(D)$ such that $r_i \in C_i$ and every vertex of D is contained in at most two of $C_1, C_2, \ldots, C_k$. We can discern the existence of independent r_i-arborescences $T_i (i = 1, 2, \ldots, k)$ such that $V(T_i) = C_i$ and find such arborescences if they exist in $O(km)$ time.*

Proof. By Theorem 6, we can test the existence of desired arborescences by checking if for every $v \in V(D)$ there are internally disjoint (r_i, v)-walks $P_i(i \in I(v))$. As said in the beginning of this section, even if we transform the input graph so that Assumptions 3 and 4 are satisfied, the existence (or non-existence) of such walks does not change. Note that we can complete such a transformation in $O(kn + m)$ time. By Theorem 6, we can discern the existence of desired arborescences by checking if (4) and (5) hold. We can carry out this in $O(m)$ time. Furthermore, we assume without loss of generality that every vertex is contained in at least one convex set. Note that we can transform the input graph so that this condition is satisfied, in $O(m)$ time.

Now, we assume that (4) and (5) hold. Following the proof of Claim 3, we can develop an $O(m)$ algorithm for finding a D-eligible r_k-arborescences T_k such that $V(T_k) = C_k$. Take every topological ordering π^* in D. Assume that $\pi^*(t) = n$ for some $t \in V(D)$. Here we prove that there is i such that $r_i = t$. Assume that $r_i \neq t$ for every i. By the definition of a topological ordering, $\Gamma^-(t) = \emptyset$, which contradicts the fact that every vertex is contained in at least one convex set and (4) holds. Without loss of generality, we assume that $t = r_k$. Then start with T such that $V(T) = \{t\}$, $A(T) = \emptyset$, and empty topological orderings $\pi_1, \pi_2, \ldots, \pi_{k-1}$. Following the topological ordering π^* in decreasing order, we grow T and update $\pi_1, \pi_2, \ldots, \pi_{k-1}$ described as in the proof of Claim 3. We can execute each update in $O(1)$ time. Hence, we can find desired arborescences in $O(km)$ time. □

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