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NOTE

A NOTE ON k-STRONGLY CONNECTED ORIENTATIONS OF AN UNDIRECTED GRAPH

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Each k-strongly connected orientation of an undirected graph can be obtained from any other k-strongly connected orientation by reversing consecutively directed paths or circuits without destroying the k-strong connectivity.

A directed graph is called *k-strongly connected* if the indegree $\rho(X)$, the number of edges with head in X but tail not, is at least k for each nonempty proper subset X of vertices. Since Nash-Williams' work [2] we know the necessary and sufficient condition for an undirected graph to have a k-strongly connected orientation, namely, it must not contain cuts consisting of less than $2k$ edges. For generalizations, see [1]. In this note we prove that different k-strongly connected orientations of an undirected graph are connected in a certain sense.

Theorem. *If G' and G'' are two k-strongly connected orientations of an undirected graph $G=(V,E)$, then there is a sequence $G'=G_0, G_1, \ldots, G_k=G''$ of k-strongly connected orientations of G such that each G_i $(i=1,2,\ldots,k)$ arises from G_{i-1} by reversing one directed circuit or path.*

Proof. Let ρ' and ρ'' denote the indegree functions of G' and G'', respectively.

Case 1. For each vertex v, $\rho'(v)=\rho''(v)$. (We write $\rho(v)$ instead of $\rho(\{v\})$.) Call red those edges of G' which are oppositely directed in G''. The set of red edges forms a directed graph in which the indegree and outdegree coincide at each vertex. Such a graph is the union of edge disjoint directed circuits. Reversing the orientations of these circuits in any order we obtain G'' while the indegree function, and so the k-strong connectivity, is unchanged.

Case 2. For a vertex v, $\rho'(v)<\rho''(v)$.

Subcase 2.1. There is a vertex u with $\rho'(u)>\rho''(u)$ and $\rho'(X)>k$ whenever $u\in X$ and $v\notin X$.

Then there is a directed path from v to u in G'. Reversing its edges we have a new k-strongly connected orientation of G whose indegrees at the vertices are

nearer to that of G'' than the indegrees of G'. Repeating this procedure, eventually either Case 1 or Subcase 2.2 will occur.

Subcase 2.2. Each vertex u, for which $\rho'(u)>\rho''(u)$, is contained in a set of indegree k in G' which does not contain v. Let us consider the maximal sets V_i for which $\rho'(V_i)=k$ and $v\notin V_i$, $i=1,2,\ldots,t$. Set $V_0=V-\bigcup V_i$.

Now V_i and V_j are disjoint $(1\le i<j\le t)$. For otherwise

$$k+k=\rho'(V_i)+\rho'(V_j)\ge\rho'(V_i\cup V_j)+\rho'(V_i\cap V_j)\ge k+k$$

whence $\rho'(V_i\cup V_j)=k$, contradicting the maximality of V_i and V_j.

Both in G' and G'' let us count the number of edges having at most one endvertex in V_i $(i>0)$. These numbers c' and c'' are, of course, equal. However the next inequality shows that $c'<c''$:

$$c'=kt+\sum(\rho'(z)\colon z\in V_0)<kt+\sum(\rho''(z)\colon z\in V_0)\le c''.$$

(We exploited that $\rho'(z)\le\rho''(z)$ for $z\in V_0$ and $\rho'(v)<\rho''(v)$.) By this contradiction Subcase 2.1 is impossible.

References

[1] A. Frank, On the orientations of graphs, J. Combin. Theory (B) 28 (3) (June 1980) 251–261.
[2] C.St.J.A. Nash-Williams, Well-balanced orientations of finite graphs and unobstrusive odd-vertex pairings, in: Recent Progress in Combinatorics (Academic Press, New York, 1969) 133.