COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
25. ALGEBRAIC METHODS IN GRAPH THEORY, SZEGED (HUNGARY), 1978.

ON DISJOINT TREES AND ARBORESCENCES

A. FRANK

An important min-max theorem of matroid theory gives a formula for the maximum number of pairwise disjoint bases in a matroid [1]. The theorem was originally proved by Tutte for the circuit matroid of a graph. [8].

Theorem (W.T. Tutte). *A graph contains k edge disjoint spanning trees iff for every partitions $V = V_1 \cup V_2 \cup \ldots \cup V_t$ of the vertex set, the number of edges connecting different V_i's is at least $k(t-1)$.*

A directed analogue of Tutte's theorem is due to Edmonds.

Theorem (J. Edmonds [2]). *A directed graph contains k edge disjoint spanning arborescences rooted at a fixed vertex iff the indegree of every subset of vertices, not containing the root, is at least k.*

One prupose of this paper is to give a common generalization of Tutte's and Edmonds' theorems for edge disjoint mixed trees in a mixed graph. This will be a consequence of an "orientation" theorem, which is (I hope) interesting for its own sake.

As a second purpose, we show an extension of Edmonds' theorem, in which the roots of arborescences may be different.

However Edmonds' theorem will be used in the proofs of both theorems.

NOTATIONS, DEFINITIONS

By a *mixed graph* $G = (V, E)$, we mean a graph in which some edges are directed, others are not.

A *mixed arborescence* rooted at r is a mixed tree such that the unique path leading from r to any vertex does not use reverse directed edge.

A directed edge *enters* a subset X of V if its head is in X, but the tail is not. A directed edge *leaves* X if it enters $V \setminus X$. An undirected edge *enters* X if X contains just one of its end vertices.

For $X \subseteq V$, the *indegree* $\rho(X)$ is the number of directed edges entering X. For $X, Y \subseteq V$, $d(X, Y)$ denotes the number of edges (directed or not), one end vertex of which is in $X \setminus Y$, the other is in $Y \setminus X$.

For $X \subseteq V$, we put $\bar{X} = V \setminus X$.

For $x, y \in V$, a subset P of V is called an *$x\bar{y}$-set* if $x \in P$ and $y \notin P$.

A pair X, Y of subsets is called *intersecting* if $X \cap Y, X \setminus Y, Y \setminus X$ are non-empty. An intersecting pair is *crossing* if $\overline{X \cup Y}$ is non-empty.

At this point we mention a simple and well-known formula for the indegree function of a directed graph, which will be crucial in our proofs:

(1) $$\rho(X) + \rho(Y) = \rho(X \cup Y) + \rho(X \cap Y) + d(X, Y)$$

for $X, Y \subseteq V$.

GRAPH ORIENTATIONS

Let $G = (V, E)$ be an undirected graph with vertex set V, and $r(X)$ be an integer-valued function on a family H of subsets of V.

Orientation problem. When does exist an orientation of the edges of G such that $\rho(X) \geqslant r(X)$, whenever $X \in H$?

Such an orientation is said to be *good* with respect to the function r.

There is no hope to give a good characterization for this problem since it is *NP*-hard: the problem of two-coloring of a hypergraph can be formulated as an orientation problem.

However the problem can be solved for some special classes of function r.

A function r is said to be *convex* (in G) on a pair X, Y of subsets of V if

(2) $$r(X) + r(Y) \leqslant r(X \cup Y) + r(X \cap Y) + d(X, Y).$$

In [4] the following result was proved.

Theorem 1. *Let $H = 2^V$, $r(\phi) = r(V) = 0$ and r be non-negative and convex on crossing pairs. G has a good orientation iff for every partition $V = V_1 \cup V_2 \cup \ldots \cup V_t$ of the vertex set, the number of edges connecting different V_i's is at least*

$$\max\left\{\sum_{i=1}^{t} r(V_i), \sum_{i=1}^{t} r(\bar{V}_i)\right\}.$$

Here we solve the orientation problem for another function class.

Theorem 2. *Let H be a collection of subsets of V, the intersection and the union of any two intersecting member of H are also in H, furthermore $\phi \notin H$, $V \in H$. Let r be convex on intersecting pairs of sets from H and $r(V) = 0$. G has a good orientation iff*

(3) $$e_t \geqslant \sum_{i=1}^{t} r(V_i)$$

for disjoint sets V_i $(i = 1, 2, \ldots, t)$ from H, where e_t denotes the *number of edges entering some* V_i.

It is worth to mention that $r(X)$ may be negative, and we shall really need this case in the application.

Proof. Henceforth, if we speak about a subset of V then a member of H is meant.

Necessity. If $\vec{G}$ is any orientation of G, then $e_t \geqslant \sum \rho(V_i)$. If G is good then $\rho(V_i) \geqslant r(V_i)$, thus (3) follows.

Sufficiency. By induction on $m(r) = \sum \{r(X)\colon X \in H, r(X) \geqslant 0\}$. Any orientation is good when $m = 0$.

$m > 0$. Let $r(X) > 0$. From (3), there exists an edge $e(ab)$, for which $a \notin X$, $b \in X$. Define $k(Y)$:

$$(4) \qquad k(Y) = \begin{cases} r(Y) - 1 & \text{if } Y \text{ is a } b\bar{a}\text{-set} \\ r(Y) & \text{otherwise.} \end{cases}$$

One can see that k satisfies the premisses and $m(k) < m(r)$, therefore, by the induction hypothesis, G has a good orientation $\vec{G}$ with respect to k. If $\vec{G}$ is good with respect to r as well, then we are ready with the proof. Otherwise there exists a $b\bar{a}$-set X with $\rho(X) = k(X)$. (In the proof $\rho(X)$ concen (X) concerns $\vec{G}$).

Definition. A set X is said to be *strict* (with respect to k) if $\rho(X) = k(X)$.

We are going to modify the present orientation so that it will be good with respect to r. Some simple propositions are needed.

Proposition 1. *The union and the intersection of any two intersecting strict sets are strict.*

Proof. The first and the last member of the inequality sequence below are equal, from which the statement follows:

$$\rho(X) + \rho(Y) = k(X) + k(Y) \leqslant k(X \cup Y) + k(X \cap Y) + d(X, Y) \leqslant$$
$$\leqslant \rho(X \cup Y) + \rho(X \cap Y) + d(X, Y) = \rho(X) + \rho(Y). \blacksquare$$

Proposition 2. *The intersection of some strict sets containing a fixed vertex is strict. If a collection of strict sets forms a connected hypergraph then their union is strict.*

Proof. Repeated application of Proposition 1 leads us to both statements. ∎

Proposition 3. *A strict* $a\bar{b}$*-set* A *and a strict* $b\bar{a}$*-set* B *are disjoint.*

Proof. Otherwise $r(A \cap B) = k(A \cap B)$. Then

$$\rho(A) + \rho(B) = k(A) + k(B) =$$
$$= r(A) + r(B) - 1 < r(A \cap B) + r(A \cup B) + d(A, B) =$$
$$= k(A \cap B) + k(A \cup B) + d(A, B) \leqslant$$
$$\leqslant \rho(A \cap B) + \rho(A \cup B) + d(A, B),$$

which contradicts to (1). ∎

Let $P(x)$ denote the intersection of all strict sets containing a vertex x. (Since V is strict, $P(x)$ is well-defined). By Proposition 2, $P(x)$ is strict.

Now we extend $\vec{G}$ by new directed edges which we call red edges. (The original edges are blue). Lead a red edge from x to every other vertex of $P(x)$ for $x \in V$. Obviously there is no red edge leaving a strict set.

Let C denote the set of vertices which can be reached from b by a directed path in the extended graph. Then there is no edge (whether red or blue) leaving C and C is the union of strict sets $P(x)$, $x \in C$.

Proposition 4. *There exists a vertex* x *in* C *for which* $a, b \in P(x)$.

Proof. Suppose there is no such vertex x. Consider the hypergraph formed by the sets $P(x)$, $x \in C$. The components of this hypergraph partition C into strict sets $V_1, V_2, \ldots, V_t$ by Proposition 2. If a and b were in the same V_i then there would exist a sequence of hyperedges $X_1, X_2, \ldots, X_s$ such that $a \in X_1$, $b \in X_s$ and $X_i \cap X_{i+1} \neq \phi$. Let this sequence be of minimum length. The indirect premise means that

$s > 1$. Then $A = (\bigcup X_i\colon i = 1, 2, \dots, s-1)$ and $B = X_s$ violates Proposition 3 from which we can see that if b is in V_1 (say) then a is not. Thus $k(V_1) = r(V_1) - 1$. Since there is no blue edge leaving C $e_t = = \sum \rho(V_i) = \sum k(V_i) = \sum r(V_i) - 1$, which contradicts to (3).∎

Let a path U, leading from b to a vertex c, be chosen in the extended graph in such a way that $a, b \in P(c)$ and the length of U is as small as possible. We shall often use the propetty that a strict set containing c also contains both a and b.

Lemma. *Reversing the orientation of all blue edges of U in $\vec{G}$, we obtain another orientation of G being good with respect to r.*

Proof. *Let* $\delta(X)$ denote the number of red edges of U leaving X. Then for the indegree function $\rho'(X)$ of the modified orientation we have $\rho'(X) \geq \rho(X) + \epsilon(X) - \delta(X)$ where

$$\epsilon(X) = \begin{cases} -1 & \text{if } X \text{ is a } c\bar{b}\text{-set} \\ +1 & \text{if } X \text{ is a } b\bar{c}\text{-set} \\ 0 & \text{otherwise.} \end{cases}$$

We are going to prove in Proposition 8 that

(5) $\qquad \rho(X) + \epsilon(X) - \delta(X) \geq r(X).$

This will prove the lemma and the theorem.

Proposition 5. *If zy is the first red edge on U (starting from b), leaving X and $W = P(z) \cup X$, then $\delta(W) = \delta(X) - 1$.*

Proof. Since no red edge leaves $P(z)$, $\delta(W) \leq \delta(X) - 1$. On the other hand, by the minimal property of U, $P(z)$ does not contain the head of any red edge of U, leaving X thus $\delta(W) = \delta(X) - 1$.∎

One can easily check that $\epsilon(X) + \epsilon(Y) = \epsilon(X \cap Y) + \epsilon(X \cup Y)$. This, (1) and (2) show that function $\gamma(X) = \rho(X) - r(X) + \epsilon(X)$ is submodular on intersecting pairs, i.e. $\gamma(X + \gamma(Y) \geq \gamma(X \cap Y) + \gamma(X \cup Y)$.

Proposition 6. $\gamma(X) \geq 0$.

Proof. What we have to prove is that $\rho(X) \geq r(X) - \epsilon(X)$. If X is a $c\bar{b}$-set then X is not strict thus $\rho(X) \geq k(X) + 1 = r(X) - \epsilon(X)$. If X is a $b\bar{c}$-set then $\rho(X) \geq k(X) \geq r(X) - \epsilon(X)$. If X is a $b\bar{a}$-set containing c then X is not strict thus $\rho(X) \geq k(X) + 1 = r(X) - \epsilon(X)$. Finally, if X belongs to neither of these types then $\rho(X) \geq k(X) = r(X) - \epsilon(X)$.∎

The next statement is straightforward:

Proposition 7. *Provided that $c \notin X$ and a or $b \notin X$, X is strict iff* $\gamma(X) = 0$.∎

We have to prove (5) which is equivalent to

Proposition 8. $\delta(X) \leq \gamma(X)$.

Proof. By induction on $\delta(X)$. The case $\delta(X) = 0$ has been settled in Proposition 6. Let $\delta(X) > 0$ and let zy be the first red edge on U, leaving X. Denote $W = P(z) \cup X$. By the minimality of U, c is not in $P(z)$ and a or b is not in $P(z)$, i.e. Proposition 7 applies thus $\gamma(P(z)) = = 0$ and $\gamma(P(z) \cap X) \geq 1$. Hence $\gamma(X) = \gamma(X) + \gamma(P(z)) \geq \gamma(W) + + \gamma(P(z) \cap X) \geq \gamma(W) + 1$. By Prpposition 5 $\delta(W) = \delta(X) - 1$. By the induction hypothesis $\delta(W) \leq \gamma(W)$ and thus $\gamma(X) \geq \gamma(W) + 1 \geq \geq \delta(W) + 1 = \delta(X)$.∎

We are ready with the proof of the lemma and the theorem.

Note that the proof described above is, in fact, an algorithm provided that we can determine quickly the minimal strict set containing a fixed vertex in a given orientation of G.

DISJOINT ARBORSENCES

Theorem 3. *A mixed graph $F = (V, E)$ has k edge disjoint spanning mixed arborescences rooted at r iff for disjoint subsets $V_1, V_2, \dots, V_t$ of $V \setminus \{r\}$, $f_t \geq kt$, where f_t is the number of edges (directed or not) entering some V_i.*

Proof. F contains k edge disjoint spanning mixed arborescences rooted at r just if its undirected edges can be oriented so that the obtained directed graph contains k edge disjoint arborescences. This latter is equivalent to Edmonds' condition: the indegree of every subset of $V \setminus \{r\}$

is at least k. Thus our task is to find a good orientation of the graph $G = (V, E')$ of undirected edges of F with respect to function $r(X) = k - \rho(X)$ $(\phi \neq X \subset V \setminus \{r\})$ and $r(V \setminus \{r\}) = 0$. Theorem 2 can be applied with $H = \{X: X \subseteq V \setminus \{r\}\}$ and (3) is equivalent to the present condition since

$$e_t = f_t - \sum_{i=1}^{t} \rho(V_i). \blacksquare$$

In [5] the following result was proved.

Theorem 4. *A directed graph* $G = (V, E)$ *has* k *edge disjoint spanning arborescences (possibly rooted at different vertices) iff* $\sum_{i=1}^{t} \rho(V_i) \geq k(t-1)$ *for every collection of disjoint non-empty subsets* V_i *of* V $(i = 1, 2, \dots, t)$.

Let an integer $u(x)$ $(0 \leq u(x) \leq k)$ be assigned to every vertex x of a directed graph $G = (V, E)$.

Theorem 5. G *has* k *edge disjoint spanning arborescences, containing any vertex* x *as a root, not more than* $u(x)$ *times iff*

(a) $\rho(X) + \sum_{x \in X} u(x) \geq k$ *for* $\phi \neq X \subseteq V$ *and*

(b) $\sum_{i=1}^{t} \rho(V_i) \geq k(t-1)$ *for every collection of disjoint non-empty subsets* V_i *of* V $(i = 1, 2, \dots, t)$.

Proof.

Necessity. Let k arborescences exist with the required property. k_1 of them have root in X. Then $k - k_1$ arborescences are rooted outside X, thus $\rho(X) \geq k - k_1$. Since $k_1 \leq \sum_{x \in X} u(x)$, we obtain condition (a). Condition (b) is equally straightforward because an arborescence "enters" (along an edge) all but one V_i, therefore k edge disjoint arborescences use at least $k(t-1)$ entering edges, i.e. $\sum_{i=1}^{t} \rho(V_i) \geq k(t-1)$.

Sufficiency. Extend G by a new vertex r and some new edges

starting from r. More exactly, lead $u(x)$ new parallel edges from r to x, for every vertex x of G. The extended graph is $G' = (V', E')$ with indegree function ρ'. By the construction of G', we have $\rho'(X) = \rho(X) + \sum_{x \in X} u(x) \geq k$, i.e.

$$\text{(6)} \qquad \rho'(X) \geq k \quad \text{for} \quad X \subseteq V.$$

Delete as many new edges as possible without destroying (6). Denote the arising graph by $G_1 = (V', E_1)$ and its indegree function by ρ_1. Now $\rho_1(V) \geq k$ however we state

Proposition. $\rho_1(V) = k$.

Proof. Suppose, indirectly, G_1 contains $k+1$ new edges $e_1, e_2, \dots, e_{k+1}$ starting from r. Deleting e_i, (6) becomes false, i.e. e_i enters a subset X_i of V with $\rho_1(X_i) = k$. The components of the hypergraph formed by X_i's partition X into disjoint subsets V_j such that $\rho_1(V_j) = k$. (See Proposition 2 in the previous paragraph).

Since at least $k+1$ new edges enter X, we have $\sum \rho(V_i) \leq \sum \rho_1(V_j) - (k+1) = kt - (k+1) = k(t-1) - 1$ contradicting condition b. $\blacksquare$

Finally, applying Edmonds' theorem for G_1 we find k edge disjoint spanning arborescences rooted at r. All of them contains exactly one new edge, consequently, deleting the k new edges, they form k arborescences of G. These satisfy the requirements of the theorem. $\blacksquare$

Remark. The proof above provides an algorithm. First apply $|V|$ flow algorithms to control (6), then at most $k|V|$ flow algorithms is required to carry out the deletions. Finally, Lovász' [6] algorithm is used to find k edge disjoint arborescences of G. Since Lovász' algorithm needs $O(|V|^4 k)$ steps, while there exists a flow algorithm of $O(|V|^3)$, our proposed algorithm is of $O(|V|^4 k)$.

Remark. When $u(x) \equiv k$, Condition a is automatically satisfied, thus we get Theorem 4. When $u(x)$ is identically 0 except one vertex r wherein $u(r) = k$, then Condition a transforms into Edmonds' condition, furthermore it implies b.

COVERING BRANCHINGS

Definition. A *(mixed) branching* is a forest consisting of disjoint (mixed) arborescences.

In [5] I proved a directed analogue of Nash-Williams's famous theorem on covering forests [7].

Theorem 6. *The edges of a directed graph can be covered by k branchings iff*

(a) *the indegree of every vertex is at most k and*

(b) *the underlying undirected graph can be covered by k forests.*

Using this, we prove here

Theorem 7. *For a mixed graph $G = (V, E)$, E can be covered by k mixed branchings iff*

(c) $\rho(X) + b(X) \leq k|X|$ *for* $X \subset V$ *and*

(d) $b(X) \leq k(|X| - 1)$ *for* $X \subseteq V$,

where $b(X)$ denotes the number of edges (directed or not) spanned by X.

Note that Nash-Williams' theorem states the equivalence of Conditions b and d.

Proof.

Necessity. One mixed branching can cover $|X| - 1$ edges spanned by X, thus Condition d follows. If a mixed branching contains k_1 directed edges entering X then it covers at most $|X| - k_1$ edges spanned by X, whence Condition c follows.

Sufficiency. For $X = \{x\}$; c means that $\rho(x) \leq k$. Let $u(x) = k - \rho(x)$ for $x \in V$. The graph G' of undirected edges of G can be oriented so that the indegree of every vertex x is at most $u(x)$. To see this, apply Theorem 1 of [3] which states that an undirected graph has an orientation the indegree of every vertex x is at most $u(x)$ iff $b'(X) \leq \sum_{x \in X} u(x)$ for $X \subseteq V$. But we have $k|X| \geq \rho(X) + b(X) = \sum_{x \in X} \rho(x) + b'(X)$ thus $b'(X) \leq k|X| - \sum_{x \in X} \rho(x) = \sum_{x \in X} u(x)$, as required.

Orienting the undirected edges of G in this way, G becomes a directed graph satisfying a and b. Apply Theorem 6. ∎

REFERENCES

[1] J. Edmonds, Lehman's switching game and a theorem of Tutte and Nash-Williams, *J. Res. Nat. Bur. St., B,* 69 (1965) 73-77.

[2] J. Edmonds, Edge disjoint branchings, in *Combinatorial Algorithms*, 91-96, Academic Press, 1973.

[3] A. Frank – A. Gyárfás, How to orient the edges of a graph, *Combinatorics (eds. A. Hajnal, V.T. Sós) North-Holland,* 1976, 353-362.

[4] A. Frank, On the orientation of graphs, *J. Comb. Th. Ser. B.*, (28), 3 (1980), 251-261.

[5] A. Frank, Covering branchings, *Acta Sci. Math.* (Szeged), 41 (1-2) (1979), 77-82.

[6] L. Lovász, On two minimax theorems in graph theory, *J. Comb. Th. Ser. B,* 2 (1976).

[7] C.St.J.A. Nash-Williams, Decompositions of finite graphs into forests, *J. London Math. Soc.*, 39, 12 (1964).

[8] W.T. Tutte, On the problem of decomposing a graph into n connected factor, *J. London Math. Soc.*, 36 (1961), 221-230.

A. Frank
Research Institute for Telecommunication, Gábor Á. u. 65, Budapest, H-1026, Hungary.