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GENERALIZED POLYMATROIDS

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ABSTRACT

We extend the concept of polymatroids due to J. Edmonds and prove the intersection theorem for generalized polymatroids. As applications we derive a theorem of McDiarmid and show that the Edmonds–Giles polyhedron is the projection of the intersection of two generalized polymatroids.

1. INTRODUCTION

Let S be a finite set and b an integer-valued β_0-function on 2^S, i.e. $b(\phi) = 0$, $b(X) \geqslant b(Y)$ for $X \supset Y$ and $b(X) + b(Y) \geqslant b(X \cap Y) + b(X \cup Y)$. The polyhedron $P = \{x\colon x \geqslant 0,\ x \in \mathbf{R}^S,\ x(F) \leqslant b(F)$ for $F \subseteq S\}$ is called an (integral) *polymatroid*, where $x(F)$ stands for the sum of components of x in F.

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The concept of polymatroids was introduced by J. Edmonds [1], who discovered quite a few results on polymatroids. For example, he established the vertices of P, characterized the facets and, generalizing the matroid intersection theorem, obtained the

Polymatroid Intersection Theorem. *For any two polymatroids* P_1, P_2, *the linear program* $\max(cx\colon x \in P_1 \cap P_2)$ *has an integral optimal solution. If, in addition,* c *is integral, the dual linear program also has an integral optimal solution.*

McDiarmid [9] extended this theorem as follows. Let f and g be integral vectors, α, β integers.

Theorem ([9]). *The linear program*

$$\max\{cx\colon x \in P_1 \cap P_2,\ f \leqslant x \leqslant g,\ \alpha \geqslant x(S) \geqslant \beta\}$$

has an integral optimal solution (if it has an optimal solution at all). If, in addition, c *is integral, the dual linear program has an integral optimal solution.*

The following useful concept is due to Hoffman [8] and Edmonds – Giles [2]. A linear system $Ax \leqslant b,\ x \geqslant 0$, is called *totally dual integral* (TDI) if, for any integral c, the dual linear program $\min\{yb\colon y \geqslant 0,\ yA \geqslant c\}$ has an integral optimal solution if it has an optimal solution. The basic feature of TDI systems is given by the following

Theorem ([2], [8]). *A TDI linear system defines a polyhedron whose facets contain integer points (in particular, the vertices are integral).*

Note that the TDI property regards the linear system and not the polyhedron defined by this system. See also [7], [10], [12].

Therefore the preceding theorems state that the linear systems in question are TDI.

A further generalization, Edmonds–Giles' theorem, also states the TDI-ness of a certain linear system. The exact formulation is in Section 2.

The purpose of this paper is to extend the concept of polymatroids

and prove the intersection theorem in this general setting. It turns out that the polyhedron in McDiarmid's theorem is the intersection of two generalized polymatroids while the Edmonds–Giles polyhedron is the projection of such an intersection.

2. PRELIMINARIES

Subsets A, B of a finite set S are *intersecting* if none of $A \cap B$, $A - B$, $B - A$ is empty. If, in addition, $A \cup B \neq S$ then A, B are *crossing.* A family $\mathscr{F}$ of subsets is *intersecting* (*crossing*) if $A \cap B$, $A \cup B \in \mathscr{F}$ for all intersecting (crossing) members of $\mathscr{F}$ and $\phi \notin \mathscr{F}$ ($\phi, S \notin \mathscr{F}$). A family $\mathscr{F}$ is a *ring* family if $\mathscr{F}$ is closed under taking union and intersection. A family $\mathscr{F}$ is *laminar* if no two members of it are intersecting.

A set function b is *submodular* on A, B if $b(A) + b(B) \geqslant$ $\geqslant b(A \cup B) + b(A \cup B)$. A set function is *supermodular* if $-p$ is submodular. m is *modular* if it is both sub- and supermodular.

Let T be a subset of S. By the *projection* along T of a polyhedron $P = \{x\colon x \in \mathbf{R}^S,\ Ax \leqslant b\}$ we mean the polyhedron $P' = \{x_1\colon x_1 \in \mathbf{R}^{S-T},\ (x_1, x_2) \in P$ for some $x_2 \in \mathbf{R}^T\}$. Obviously, if the vertices of P are integral then so are those of P'.

Let $G = [V, E]$ be a directed graph with n vertices and m arrows. Multiple arrows are allowed but loops are not. An arrow uv *enters* (*leaves*) $B \subset V$ if $v \in B,\ u \notin B$ ($v \notin B,\ u \in B$). For a vector $x \in \mathbf{R}^E$ let

$$\lambda_x(B) = \sum (x(e)\colon e \text{ enters } B) - \sum (x(e)\colon e \text{ leaves } B).$$

Obviously $\lambda_x(B)$ is a modular set function.

Let $\mathscr{B}'$ be a crossing family of subsets of V and b' a real-valued function on $\mathscr{B}'$ submodular on crossing sets. Let $f, g \in \mathbf{R}^E$ be two real vectors with $f \leqslant g$. (f, g may include infinite components). The following theorem is due to Edmonds and Giles.

Theorem ([2]). *The linear system* $f \leqslant x \leqslant g,\ \lambda_x(B) \leqslant b'(B)$ *for* $B \in \mathscr{B}'$ *is totally dual integral.*

(The solution set of this linear system is an *Edmonds–Giles polyhedron*.)

3. GENERALIZED POLYMATROIDS

Edmonds [1] proved that, given an intersecting family $\mathcal{B}$ and an integer-valued function b on $\mathcal{B}$ submodular on intersecting sets, the polyhedron

$$P = \{x:\ x \geq 0,\ x(B) \leq b(B)\ \text{ for }\ B \in \mathcal{B}\}$$

defined a polymatroid. We will generalize this description of polymatroids.

Definition. Let $\mathcal{B}$ and $\mathcal{P}$ be intersecting families and b and p integer-valued functions on $\mathcal{B}$ and $\mathcal{P}$, respectively, which are sub- and supermodular on intersecting sets, respectively. Assume furthermore that $B \in \mathcal{B}$, $P \in \mathcal{P}$, $B - P \neq \phi$, $P - B \neq \phi$ imply that $B - P \in \mathcal{B}$, $P - B \in \mathcal{P}$ and $b(B) - p(P) \geq b(B-P) - p(P-B)$. Then the polyhedron $Q = \{x:\ x(B) \leq b(B)$ for $B \in \mathcal{B}$, $x(P) \geq p(P)$ for $P \in \mathcal{P}\}$ is called a *generalized polymatroid* or briefly a *g-polymatroid.*

Remark. If $\mathcal{P}$ consists of the singletons and $p \equiv 0$ then we obtain an (ordinary) polymatroid.

Proposition 1. *Let Q be a g-polymatroid and f and g integral vectors $(f \leq g)$. Let α, β be integers $(\alpha \leq \beta)$. The polyhedron $\{x:\ x \in Q,\ f \leq x \leq g,\ \alpha \leq x(S) \leq \beta\}$ is a g-polymatroid.*

Proof. Join the singletons and $\{S\}$ to both $\mathcal{B}$ and $\mathcal{P}$ and extend b and p so that $b(s) = g(s)$, $p(s) = f(s)$ for $s \in S$ and $b(S) = \beta$ and $p(S) = \alpha$. These new families and functions define a g-polymatroid which is obviously the polyhedron in question. ∎

Proposition 2. *Let Q be a g-polymatroid and k an integer. Let s be a new element and $S' = S + s$. The polyhedron in $\mathbf{R}^{S'}$ defined by $\{(x, x_s):\ x \in Q,\ x(S) + x_s = k\}$ is a g-polymatroid.* ∎

Proposition 3. *Let $\mathcal{B}'$ be a crossing family of subsets of S' and b' an integer-valued function on $\mathcal{B}'$ submodular on crossing sets. Let k be an integer. The polyhedron in $\mathbf{R}^{S'}$ defined by $\{x:\ x(B) \leq b'(B)$ for $B \in \mathcal{B}'$, $x(S') = k\}$ is a g-polymatroid.*

Proof. Let s be an arbitrary element of S' and $S = S' - s$. Let $\mathcal{B}$ consist of the members of $\mathcal{B}'$ not containing s and let $\mathcal{P}$ consist of the complements of those members of $\mathcal{B}'$ which contain s. Let $b(B) = b'(B)$ for $B \in \mathcal{B}$ and $p(P) = k - b'(S' - P)$ for $P \in \mathcal{P}$. Now apply Proposition 2. ∎

Let us formulate two properties of g-polymatroids interesting for their own sake. It was proved in [3] (and independently by S. Fujishige [13]) that, given a constant k, a crossing family $\mathcal{B}'$ and a submodular function b' on it, there exists a ring family $\mathcal{R}$ and a function r on $\mathcal{R}$ submodular on every pair such that $\{x:\ 1x = k,\ x(B) \leq b'(B)$ for $B \in \mathcal{B}'\} = \{x:\ 1x = k,\ x(R) \leq r(R)$ for $R \in \mathcal{R}\}$.

Proposition 4. *Any g-polymatroid can be obtained in the form $\{x:\ x(B) \leq b_1(B)$ for $B \in \mathcal{B}_1$, $x(P) \geq p_1(P)$ for $P \in \mathcal{P}_1\}$ where $\mathcal{B}_1$ and $\mathcal{P}_1$ are ring families with sub- and supermodular functions b_1, p_1 on them, respectively, and for any $B \in \mathcal{B}_1$, $P \in \mathcal{P}_1$, we have $B - P \in \mathcal{B}_1$, $P - B \in \mathcal{P}_1$ and $b_1(B) - p_1(P) \geq b_1(B-P) - p_1(P-B)$.*

Proof. Let $S' = S + s$. The g-polymatroid Q is the projection of $Q' = \{(x, x_s):\ x \in Q,\ x(S) + x_s = k\}$. Let $\mathcal{B}' = \{X:\ X \in \mathcal{B}$ or $s \in X$ and $S' - X \in \mathcal{P}\}$ and

$$b'(X) = \begin{cases} b(X) & \text{if } X \in \mathcal{B} \\ k - p(S - X) & \text{if } s \in X \text{ and } S - X \in \mathcal{P}. \end{cases}$$

Then $\mathcal{B}'$ is a crossing family and b' is submodular on crossing members; furthermore, $Q' = \{x':\ x'(B) \leq b'(B)$ for $B \in \mathcal{B}'$ and $1x' = k\}$. By the remark mentioned above there exists a ring family $\mathcal{R}$ and a submodular function r on $\mathcal{R}$ for which $Q' = \{x':\ x'(R) \leq r(R)$ for $R \in \mathcal{R}$ and $1x' = k\}$. Let $\mathcal{B}_1 = \{X:\ s \notin X \in \mathcal{R}\}$, $b_1(X) = r(X)$ for $X \in \mathcal{B}_1$ and $\mathcal{P}_1 = \{X:\ s \notin X$ and $S - X \in \mathcal{R}\}$, $p_1(X) = k - r(S - X)$ for $X \in \mathcal{P}_1$. One can see that $\mathcal{B}_1, b_1, \mathcal{P}_1, p_1$ satisfy the requirements. ∎

We say that a g-polymatroid is given in a *strong form* if the defining families and functions satisfy the requirements of Proposition 4. For the next proposition let us assume that Q is given in a strong form.

Proposition 5. *For $T \subset S$, the projection Q_T of a g-polymatroid Q along $S-T$ is a g-polymatroid, namely $Q_T = \{x_T\colon x_T \in \mathbf{R}^T,\ x_T(B) \leq b(B)$ for $B \in \mathcal{B},\ B \subseteq T,\ x_T(P) \geq p(P)$ for $P \in \mathcal{P},\ P \subseteq T\}$.*

Proof. We prove the statement when $|S-T| = 1$. The general case follows by induction. Let $\{s\} = S - T$. It is well-known from polyhedral theory that Q_T can be described by two types of inequalities. One of them consists of those inequalities in which the coefficient corresponding to s is 0. Namely, these inequalities are:

(a)
$$x_T(B) = x(B) \leq b(B) \quad \text{for } B \in \mathcal{B},\ B \subseteq T,$$
$$x_T(P) = x(P) \geq p(P) \quad \text{for } P \in \mathcal{P},\ P \subseteq T.$$

For any pair B, P with $B \in \mathcal{B}$, $P \in \mathcal{P}$ and $s \in B \cap P$, the two inequalities $x(B) \leq b(B)$, $x(P) \geq p(P)$ generate an inequality of second type as follows: $p(P) - x_T(P) \leq x(s) \leq b(B) - x_T(B)$, i.e. $x_T(B) - x_T(P) \leq b(B) - p(P)$. However this is equivalent to

(b)
$$x_T(B-P) - x_T(P-B) \leq b(B) - p(P)$$

and we claim that (a) implies (b), that is, (a) itself determines Q_T, as stated in the proposition. Indeed, $B - P \in \mathcal{B}$, $P - B \in \mathcal{P}$ and $s \notin B - P$, $s \notin P - B$. Thus the inequalities $x_T(B-P) \leq b(B-P)$ and $x_T(P-B) \geq p(P-B)$ occur in (a) since Q was given in a strong form. But then we have $b(B-P) - p(P-B) \leq b(B) - p(P)$ and (b) follows. ▮

Henceforth we allow the g-polymatroids in question to be defined in the form given in the definition and not necessarily in a strong form. Before formulating the main result we need the following.

Proposition 6. *The linear programming problem*

$$\Big\{\min \textstyle\sum y(B)b(B) - \sum y(P)p(P)\colon\ y \geq 0,$$
$$\textstyle\sum (y(B)\colon e \in B) - \sum (y(P)\colon e \in P) = c_e,\ e \in S\Big\}$$

has an optimal solution (if it has one at all) for which the family $\{X\colon y(X) > 0,\ X \in \mathcal{B} \cup \mathcal{P}\}$ is laminar.

Sketch of proof. Let us consider that optimal solution y for which the sum $\sum (y(X)\,|X|\,|S-X|\colon X \in \mathcal{B} \cup \mathcal{P})$ is minimal. This y satisfies the requirements. ▮

Our main result is the intersection theorem for g-polymatroids.

Theorem. *Given two g-polymatroids, the linear system*

$$\{x(B) \leq b_i(B) \text{ for } B \in \mathcal{B}_i,\ x(P) \geq p_i(P) \text{ for } P \in \mathcal{P}_i\ (i = 1, 2)\}$$

is totally dual integral.

Sketch of proof. The proof proceeds just along the same line as that of the original intersection theorem given by R. Giles [6]. For the use of this machinery, see also [11]. The dual linear programming problem is

$$\Big\{\min \sum_{i=1,2} \big[\textstyle\sum (y_i(B)b_i(B)\colon B \in \mathcal{B}_i) -$$
$$- \textstyle\sum (y_i(P)p_i(P)\colon P \in \mathcal{P}_i)\big]\colon\ y_1, y_2 \geq 0,$$
$$\sum_{i=1,2} \big[\textstyle\sum (y_i(B)\colon B \in \mathcal{B}_i) - \sum (y_i(P)\colon P \in \mathcal{P}_i)\big] = c\Big\}.$$

What we have to show is that this linear programming problem has an integral optimal solution for any integral c for which there exists an optimal solution. By Proposition 6 there is an optimal solution for which the families $\{X\colon X \in \mathcal{B}_i \cup \mathcal{P}_i,\ y_i(X) > 0\}$ $(i = 1, 2)$ are laminar. It is well known that a matrix $\binom{M_1}{M_2}$ is totally unimodular if M_i $(i = 1, 2)$ is the incidence matrix of a laminar family. Consequently the optimal solution is integral if c is integral. ▮

By Proposition 1 McDiarmid's theorem follows at once. In order to get the Edmonds–Giles polyhedron as the projection of an intersection of two g-polymatroids, replace each vertex v of $G = (V, E)$ by as many new vertices as there are arrows incident to v. Denote by $\varphi(v)$ the set of new copies of v. For a subset X of V put $\varphi(X) = \bigcup (\varphi(v)\colon v \in X)$ and $\varphi(\mathcal{B}') = \{\varphi(B)\colon B \in \mathcal{B}'\}$.

Therefore we obtain a set S of $2|E|$ elements. The arrows of G determine a partition of S into 2-element subsets. Denote by e_u and e_v the elements in S corresponding to the arrow $e = uv$. Let $S_0 = \{e_u : e = uv \in E\}$. Now define two g-polymatroids on S. Let $Q_1 = \{z: f(e) \leq z(e_v) \leq g(e),\ z(e_u) + z(e_v) = 0$ for each $e = uv \in E\}$ and $Q_2 = \{z: z(\varphi(B)) \leq b(B)$ for $B \in \mathcal{B}',\ 1z = 0\}$.

Now consider the optimization problem $\max cx$ over the Edmonds–Giles polyhedron. Define the objective function for the optimization problem over $Q_1 \cap Q_2$ so that $c(e_v) = c(uv)$ and $c(e_u) = 0$ for $e = uv \in E$. For a vector $x \in \mathbf{R}^E$ let $h(x)$ denote the vector $z \in \mathbf{R}^S$ for which $z(e_v) = x(uv)$, $z(e_u) = -x(uv)$.

Proposition 7. *A vector x is in the Edmonds–Giles polyhedron if and only if $h(x)$ is in $Q_1 \cap Q_2$. Moreover $cx = ch(x)$ and the Edmonds–Giles polyhedron is the projection of $Q_1 \cap Q_2$ along S_0.*

Proof. The proposition is a straightforward consequence of the definition of Q_1 and Q_2. ∎

Remark. It can be shown that if neither f nor g contains infinite components, the translation of the Edmonds–Giles polyhedron by $-f$ is the projection of the intersection of two ordinary polymatroids and the hyperplane $1x = |E|$.

It would be wortwhile to investigate how other nice properties of polymatroids are reflected in g-polymatroids. For instance, what are the vertices or the facets of a g-polymatroid? For a relationship between g-polymatroids and matroids, see [5].

In a forthcoming paper [14] we analyze such questions in detail.

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