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On Disjoint Homotopic Paths in the Plane

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ABSTRACT. A simple algorithmic proof is given for a directed version of a theorem of N. Robertson and P. Seymour on the existence of disjoint homotopic paths.

Introduction

Let $G = (V, E)$ be a directed planar graph (without loops and parallel edges) with two specified faces I and O. Let $S = \{s_1, s_2, \dots, s_k\}$ be a set of k distinct nodes of I and $T = \{t_1, t_2, \dots, t_k\}$ a set of k distinct nodes of O with $S \cap T = \emptyset$.

Assume that G is embedded in the plane in the following way. O is the infinite face. Every node and edge is in the annulus

$$F := \{(x, y): 1 \le x^2 + y^2 \le 4\}.$$

Furthermore $C_I := \{(x, y): x^2 + y^2 = 1\}$ contains precisely the nodes $s_1, s_2, \dots, s_k$ and $C_O := \{(x, y): x^2 + y^2 = 4\}$ contains precisely the nodes $t_1, t_2, \dots, t_k$. Namely, the coordinates of s_i are $(\cos\frac{i2\pi}{k}, \sin\frac{i2\pi}{k})$ and the coordinates of t_i are $(2\cos\frac{i2\pi}{k}, 2\sin\frac{i2\pi}{k})$ $(i = 1, 2, \dots, k)$. (Throughout we adopt the convention that the subscripts are meant modulo k.)

For $x_1 \in C_I$, $x_2 \in C_O$ we say that two curves K_1, K_2 in F connecting x_1 and x_2 are *homotopic* in F if K_1 can be moved continuously to K_2 in F without moving the endpoints.

We call a curve C in F going from s_i to t_i *of type* 0 if C is homotopic to L_i, the straight line segment connecting s_i and t_i. A curve C in F from s_i to t_{i+l} is said to be *of type* l $(l \in \mathbb{Z})$ if $C + C^0_{l-1} + C^0_{l-2} + \cdots + C^0_i$ forms a curve of type 0 where C^0_j stands for the arc of C_O between t_i and t_{i+1}. (See Figure 1.)

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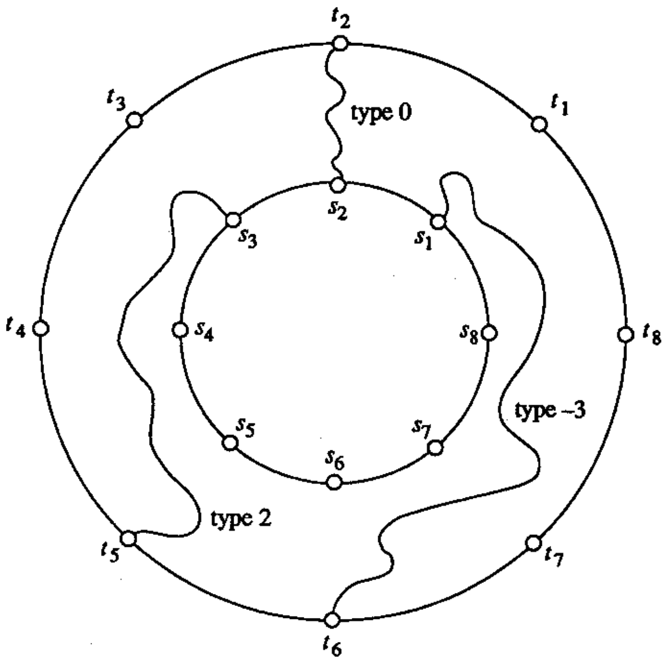

FIGURE 1

We call a set $\mathscr{P} = \{P_1, P_2, \dots, P_k\}$ of k disjoint directed paths in G going from S to T a *k-linkage*. Obviously, each P_i is of the same type j and j is said to be the *type of* $\mathscr{P}$.

The problem we investigate is as follows. *When does there exist a k-linkage of a given type j?*

To answer this question we need some further notation. Restrict the given embedding of G to the annulus F. Then $F \cap O$ decomposes into k distinct areas $O_1, O_2, \dots, O_k$, called *outer faces*. Likewise, $F \cap I$ decomposes into k distinct areas $I_1, I_2, \dots, I_k$, called *inner faces*. We choose the subscripts so that

$$t_i' := (2\cos(\tfrac{2i+1}{k}\pi),\ 2\sin(\tfrac{2i+1}{k}\pi))$$

belongs to O_i and

$$s_i' := (\cos(\tfrac{2i+1}{k}\pi),\ \sin(\tfrac{2i+1}{k}\pi))$$

belongs to I_i $(i = 1, 2, \dots, k)$. (See Figure 2.)

The faces of G different from O and I will be called *ordinary*. Henceforth by the *faces* of G we mean the ordinary, outer, and inner faces.

Let A and B be the faces on the two sides of a (directed) edge uv of G. We call A the *positive side* of uv and B the *negative side* of uv if uv is clockwise on B and counterclockwise on A. (See Figure 3.)

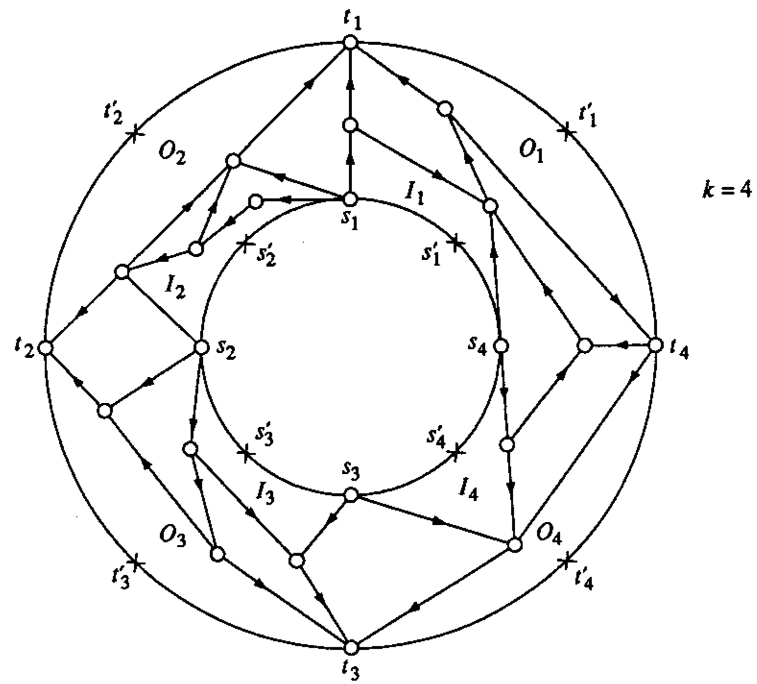

FIGURE 2

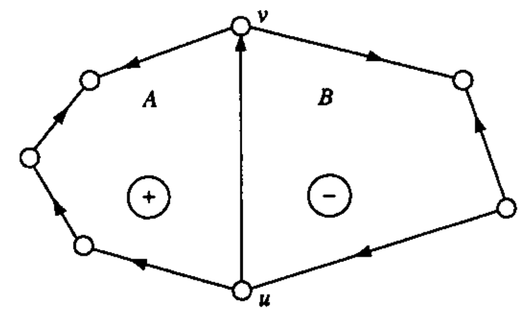

FIGURE 3

Let h be a directed curve of type l $(l \in \mathbb{Z})$ in F starting at s_i' and ending at t_{i+l}' $(i = 1, 2, \dots, k)$. We call h a *helix* if h may cross an edge of G only from the positive side to the negative side when $l > 0$, and from the negative side to the positive side when $l < 0$ (h may go through nodes of G arbitrarily). h is called *simple* if no nodes and faces of G are used by h more than once.

Obviously a k-linkage of type 0 and a helix of type l $(l \in \mathbb{Z})$ must have at least $|l|$ nodes of G in common. (See Figure 4.)

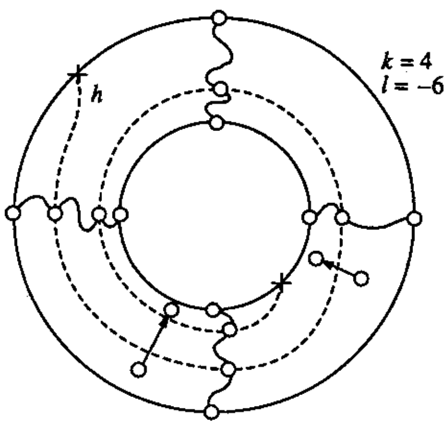

FIGURE 4

Therefore the necessity of the following characterization is clear.

THEOREM. *There exists a k-linkage of type j if and only if the paths from S to T cannot be covered by less than k nodes and every simple helix of type $j+l$ ($l \in \mathbb{Z}$) contains at least $|l|$ nodes of G.*

REMARK 1. N. Robertson and P. Seymour [1] proved this result for undirected graphs. In their characterization a helix is not allowed to cross edges. From the theorem above we can easily derive the theorem of Robertson and Seymour: replace each undirected edge uv by a pair of oppositely directed edges uv and vu.

REMARK 2. Recently P. Seymour found a characterization for a problem more general than ours: G is directed, and for each pair s_i, t_i one may prescribe whether path P_i goes from s_i to t_i or from t_i to s_i. For this general case Seymour's characterization and proof are much more complicated than those in our case (and this may be an excuse for publishing the present paper). For general results on disjoint homotopic paths, see the survey paper of A. Schrijver [2].

REMARK 3. From the theorem it follows that those j's for which a k-linkage of type j exists form an interval of integers.

The core of the proof is the following lemma.

LEMMA. *Let $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$ be a k-linkage of type i. One of the following two alternatives holds:*

(a) *there is a linkage of type $i+1$ (resp. $i-1$);*

(b) *there is a simple helix of type $i+l'$ for some $l' \le 0$ (resp. $l' \ge 0$) that has precisely $|l'|$ nodes of G.*

Obviously, it suffices to prove the main theorem for $j = 0$. By Menger's theorem there is a k-linkage $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$. Suppose its type is i. If $i = 0$, we are done. Let $i < 0$ (case $i > 0$ is analogous) and apply the lemma. If there is a linkage of type $i+1$, iterate. Otherwise, there is a helix h of type $i+l'$ ($l' \le 0$) containing $|l'|$ nodes of G. By choosing $l := l'+1$ we get $|l'| < |l|$, that is, h violates the condition of the theorem.

REMARK. The approach so far is the same as the one used by Robertson and Seymour. The novelty is in the proof of the lemma.

PROOF. It suffices to prove the lemma for $i = -1$ when it reads as follows.

Let $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$ be a linkage of type -1. One of the following alternatives holds:

(a) *there is a linkage of type 0;*

(b) *there is a helix of type l for some $l > 0$ that has precisely $|l|-1$ nodes of G.*

Assume that path P_i starts at s_i and ends at t_{i-1} ($i = 1, 2, \dots, k$). These k paths divide F into k (closed) regions called *sectors*. The sector between P_i and P_{i+1} is denoted by S_i. (See Figure 5.)

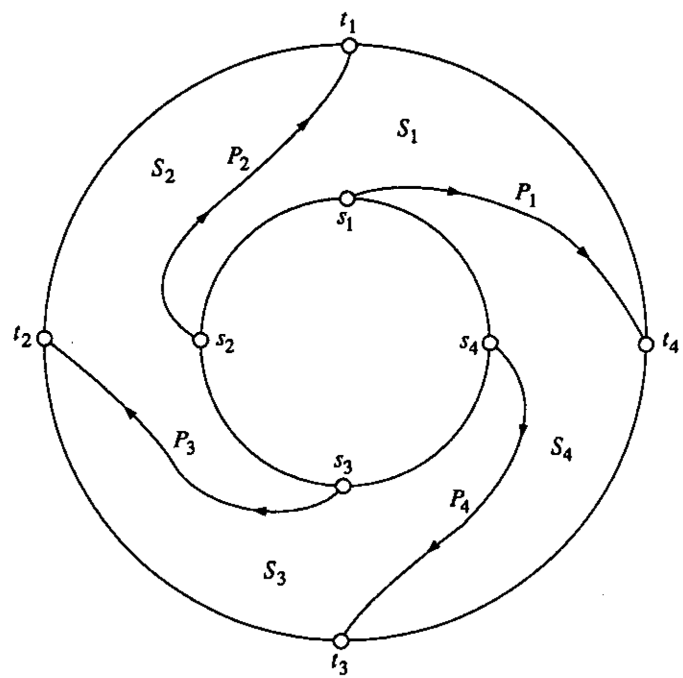

FIGURE 5

With each ordinary face A of G we associate an inner point $x(A)$ of A. The outer face O_i has an associated point $x(O_i) := t'_i$, and the inner face I_i has an associated point $x(I_i) = s'_i$.

Let $S' = \{s'_1, s'_2, \dots, s'_k\}$, $T' = \{t'_1, t'_2, \dots, t'_k\}$ and let V' consist of all the associated points and the node set of the paths P_i ($i = 1, 2, \dots, k$). We define an auxiliary digraph $G' = (V', E')$ as follows. E' has three kinds of edges. First, if $uv \in E$ is an edge not used by any P_i with positive side A and negative side B, we assign $x(B)x(A)$ to E'. Second, if a face A in sector S_i is incident to a node $v \in V(P_{i+1})$, we assign $vx(A)$ to E'. Third, if a face in sector S_i is incident to a node $v \in V(P_i)$, we assign $x(A)v$ into E'. (See Figure 6.) (In particular, $s_i's_{i-1}$ and t_it_i' belong to E' for all $i = 1, 2, \dots, k$.)

There may be two cases.

CASE 1. In G' there is a path from S' to T'.

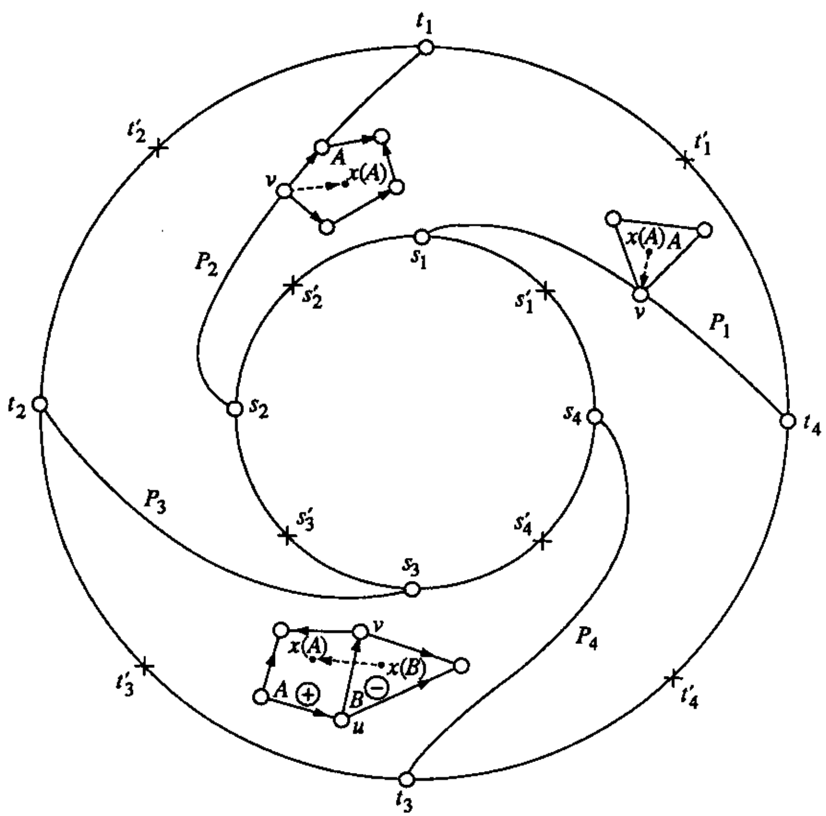

FIGURE 6

Let P' be such a simple path with starting point $s_j' \in S'$ and endpoint $t_{i+l}' \in T'$. By the definition of G', l is negative, P' may enter a sector S_i only through a node of P_{i+1}, and P' may leave S_i only through a node of P_i. Therefore P' determines a simple helix h of type $l < 0$ that has precisely $|l| - 1$ nodes of G and we are at alternative (b). (See Figure 7.)

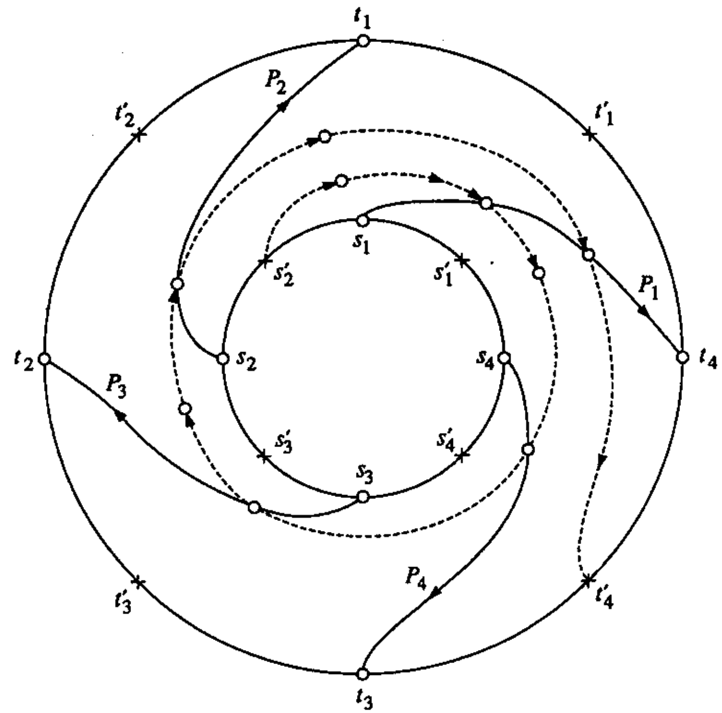

FIGURE 7

CASE 2. In G' there is no path from S' to T'.

Let Z denote the set of nodes of G' reachable from S'. We will say that a face of G is *reachable* if its associated point in G' belongs to Z. (In particular, the inner faces are reachable while the outer faces are not; see Figure 8.)

For each sector S_i $(i = 1, 2, \dots, k)$ we define a subgraph $G_i = (V_i, E_i)$ of G. V_i is the set of nodes of G in S_i. E_i has three kinds of edges. First, let $e \in E$ be an edge belonging to S_i for which $e \notin E(P_i) \cup E(P_{i+1})$. We put e into E_i if exactly one of the two sides of e is reachable (namely, the positive side). Second, for $e \in E(P_i)$ we put e into E_i if the positive side of e is reachable. Third, for $e \in E(P_{i+1})$ we put e into E_i if the negative side of e is not reachable.

CLAIM 1. No edge of G_i and G_{i+1} may share a node v in common.

PROOF. Such a v must be on P_{i+1}. Recall that $V(P_i) \subset V'$. If $v \in Z$, then each face in S_i incident to v is reachable, and therefore no edge of G_i can be incident to v. If $v \notin Z$, then no face in S_{i+1} incident to v is reachable; therefore no edge of G_{i+1} can be incident to v. □

Let $\rho_i(v)$ $(\delta_i(v))$ denote the in-degree (out-degree) of a node v in G_i.

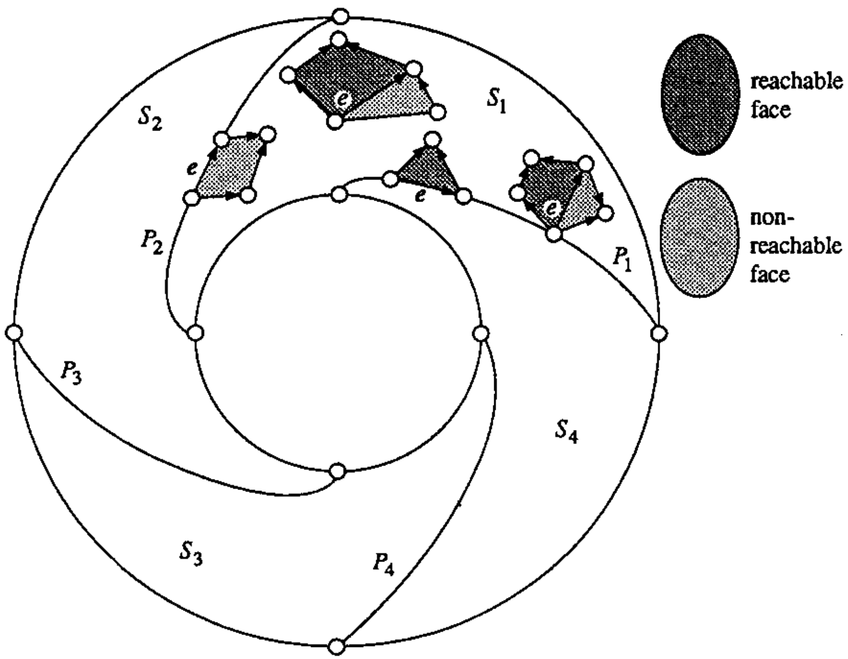

FIGURE 8

CLAIM 2. In G_i, $\rho_i(s_i) + 1 = \delta_i(s_i)$, $\rho_i(t_i) = \delta_i(t_i) + 1$ and, for each $v \in V_i - \{s_i t_i\}$, $\rho_i(v) = \delta_i(v)$.

PROOF. The proof immediately follows from the definition of G_i. □

By Claim 2 there is a path R_i in G_i from s_i to t_i and the type of R_i is 0. By Claim 1 the paths R_i $(i = 1, 2, \ldots, k)$ are pairwise disjoint and we are at alternative (a). □

Note that the above proof provides a simple polynomial-time algorithm.

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