VARIATIONS FOR LOVÁSZ' SUBMODULAR IDEAS

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October 29, 2008

Dedicated to Lovász Laci on the occasion of his 60'th birthday by his youngest and oldest mathematical descendants

Abstract

ABSTRACT In [18], L. Lovász provided simple and short proofs for two classic min-max theorems of graph theory by inventing basic techniques to handle sub- or supermodular functions. In this paper, we want to demonstrate that these ideas are alive after thirty years of their birth.

1 Introduction

Sub- and supermodular set functions play an important role in proving theorems in graph theory. L. Lovász [18] introduced a submodular technique to derive the disjoint arborescences theorem of J. Edmonds [2] and another one to prove a min-max result of C. Lucchesi and D. Younger [19] on minimum coverings of dicuts of a digraph. It appears that this paper is the first occurence of the so called uncrossing procedure (apart from a Hungarian report reviewing Lovász' solution to a problem of a math student competition, see [16]). Uncrossing became later a particularly efficient proof techniqe in submodular optimization

In the last fifteen years it turned out that several results and techniques developed for sub- or supermodular set functions can be extended to functions defined on pairs of sets or on bi-sets. Given a ground-set V, we call a pair $X = (X_O, X_I)$ of subsets a **bi-set** if $X_I \subseteq X_O \subseteq V$ where X_O is the outer member and X_I is the inner member of X. By a bi-set function we mean a function defined on the set of bi-sets of V. We will tacitly identify a bi-set $X = (X_O, X_I)$ for which $X_O = X_I$ with the set X_I and hence bi-set functions may be considered as straight generalizations of set functions.

While supermodular set functions are typically used for handling only edge-connectivity problems, supermodular bi-set functions can be applied for handling both node- and edge-connectivity problems. For example, the directed edge-connectivity augmentation problem was solved in [9] via crossing supermodular set functions while a solution to its node-connectivity counterpart was derived from a min-max theorem on covering crossing supermodular bi-set functions [11]. Similarly, an answer to the cheapest rooted k-edge-connected subgraph problem followed from a min-max result on covering intersecting supermodular functions by digraphs [5, 6] while the rooted k-node-connected version was derived from an analogous result on supermodular bi-set functions [10].

One goal of this work is to exhibit the evolution of Lovász' proof technique given for proving Edmonds' arborescences theorem. In particular, we extend a theorem of L. Szegő [22] on disjoint coverings of set systems to those of bi-set systems. This will imply a recent theorem of N. Kamiyama, N. Katoh, and A. Takizawa [14] which is a proper extension of Edmonds' disjoint arborescences theorem.

Second, by using the uncrossing technique, a new min-max theorem will be proved on minimal coverings of two fully supemodular bi-set functions by digraphs. This may be considered as a generalization of (the cardinality version of) Edmonds' (poly)matroid intersection theorem [1, 3]. It also provides an answer to a simultaneous connectivity augmentation problem where two given digraphs on the same node set is to be simultaneously augmented by adding a minimum number of new edges so that the resulting digraphs includes k_i g_i-independent paths from s_i to t_i $(i = 1, 2)$ where g_i-independence of paths is a notion including both edge-disjoint and node-disjoint paths.

In the sequel we use the following notions and notation. The set of all bi-sets on ground-set V is denoted by $\mathcal{P}_2(V) = \mathcal{P}_2$. The intersection $\cap$ and the union $\cup$ of bi-sets is defined in a staightforward manner: for $X, Y \in \mathcal{P}_2$ let $X \cap Y := (X_O \cap Y_O, X_I \cap Y_I)$, $X \cup Y := (X_O \cup Y_O, X_I \cup Y_I)$. We write $X \subseteq Y$ if $X_O \subseteq Y_O, X_I \subseteq Y_I$ and this relation is a partial order on $\mathcal{P}_2$. Accordingly, when $X \subseteq Y$ or $Y \subseteq X$, we call X and Y **comparable**. A

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family of pairwise comparable bi-sets is called a **chain**. Two bi-sets X and Y are **independent** if $X_I \cap Y_I = \emptyset$ or $V = X_O \cup Y_O$. A set of bi-sets is independent if its members are pairwise independent. We call a set of bi-sets a **ring-family** if it is closed under taking union and intersection. Two bi-sets are **intersecting** if $X_I \cap Y_I \neq \emptyset$ and **properly intersecting** if, in addition, they are not comparable. Note that $X_O \cup Y_O = V$ is allowed for two intersecting bi-sets. In particular, two sets X and Y are properly intersecting if none of $X \cap Y, X - Y, Y - X$ is empty. A family of bi-sets is called **laminar** if it has no two properly intersecting members. A family $\mathcal{F}$ of bi-sets is **intersecting** if both the union and the intersection of any two intersecting members of $\mathcal{F}$ belong to $\mathcal{F}$. (In particular, a family $\mathcal{L}$ of subsets is intersecting if $X \cap Y, X \cup Y \in \mathcal{L}$ whenever $X, Y \in \mathcal{L}$ and $X \cap Y \neq \emptyset$.) A laminar family of bi-sets is obviously intersecting. Two bi-sets are **crossing** if $X_I \cap Y_I \neq \emptyset$ and $X_O \cup Y_O \neq V$ and **properly crossing** if they are not comparable. A bi-set (X_O, X_I) is **trivial** if $X_I = \emptyset$ or $X_O = V$. We will assume throughout the paper that the bi-set functions in question are integer-valued and that their value on trivial bi-sets is always zero. In particular, set functions are also integer-valued and zero on the empty set.

A directed edge **enters** or **covers** X if its head is in X_I and its tail is outside X_O. An edge **covers** a family of bi-sets if it covers each member of the family. For a bi-set function p, a digraph $D = (V, A)$ is said to cover p if $\varrho_D(X) \geq p(X)$ for every $X \in \mathcal{P}_2(V)$ where $\varrho_D(X)$ denotes the number of edges of D covering X. For a vector $z : A \to \mathbf{R}$, let $\varrho_z(X) := \sum[z(a) : a \in A, a \text{ covers } X]$. A vector $z : A \to \mathbf{R}$ **covers** p if $\varrho_z(X) \geq p(X)$ for every $X \in \mathcal{P}_2(V)$.

A bi-set function p is said to satisfy the **supermodular inequality** on $X, Y \in \mathcal{P}_2$ if

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y). \tag{1}$$

If the reverse inequality holds, we speak of the **submodular** inequality. p is said to be **fully supermodular** or supermodular if it satisfies the supermodular inequality for every pair of bi-sets X, Y. If (1) holds for intersecting (crossing) pairs, we speak of **intersecting** (**crossing**) **supermodular** functions. Analogous notions can be introduced for submodular functions. Sometimes (1) is required only for pairs with $p(X) > 0$ and $p(Y) > 0$ in which case we speak of **positively supermodular** functions. Positively intersecting or crossing supermodular functions are defined analogously. A typical way to construct a positively supermodular function is replacing each negative value of a fully supermodular functions by zero.

Proposition 1.1 *The in-degree function ϱ_D on $\mathcal{P}_2$ is submodular.* •

2 Packing arborescences

2.1 Basic cases

An arborescence is defined to be a directed tree in which every node is reachable from a specified root-node r_0. The starting point is a classical result of J. Edmonds [2]. A digraph D is called **rooted** (more specifically, **r_0-rooted**) **k-edge-connected** with respect to a root-node $r_0 \in V$ if the in-degree of every non-empty subset of $V - r_0$ is at least k. By the directed edge-version of Menger's theorem this is equivalent to requiring that there are k edge-disjoint paths from r_0 to every node of D.

THEOREM 2.1 (Edmonds' disjoint arborescences: weak form) *Let $D = (V, A)$ be a directed graph with a designated root-node r_0. D has k disjoint spanning arborescences of root r_0 if and only if D is rooted k-edge-connected, that is,*

$$\varrho(X) \geq k \text{ whenever } X \subseteq V - r_0, X \neq \emptyset. \bullet \tag{2}$$

Edmonds actually proved his theorem in a stronger form where the goal was packing k edge-disjoint branchings of given root-sets. A branching is a directed forest in which the in-degree of each node is at most one. The set of nodes of in-degree 0 is called the root-set of the branching. Note that a branching with root-set R is the union of $|R|$ node-disjoint arborescences (where an arborescence may consist of a single node and no edge but we always assume that an arborescence has at least one node). For a digraph $D = (V, A)$ and root-set $\emptyset \subset R \subseteq V$ a branching (V, B) is called a **spanning R-branching** of D if its root-set is R. In particular, if R is a singleton consisting of an element s, then a spanning branching is a spanning arborescence of root s.

THEOREM 2.2 (Edmonds' disjoint branchings) *In a digraph $D = (V, A)$, let $\mathcal{R} = \{R_1, \dots, R_k\}$ be a family of k non-empty* (not necessarily disjoint or distinct) *subsets of V. There are k edge-disjoint spanning branchings of D with root-sets $R_1, \dots, R_k$, respectively, if and only if*

$$\varrho(X) \geq p(X) \text{ whenever } \emptyset \subset X \subseteq V \tag{3}$$

where $p(X)$ denotes the number of root-sets R_i disjoint from X. •

Remark In the special case of Theorem 2.2 when each root-set R_i is a singleton consisting of the same node r_0, we are back at Theorem 2.1. Conversely, when the R_i's are singletons (which may or may not be distinct), then Theorem 2.2 easily follows from Theorem 2.1. However, for general R_i's no reduction is known.

The original proof of Edmonds is pretty complex and does not seem to transform into a polynomial time algorithm. However, R.E. Tarjan observed [23] that Theorem 2.2 itself gives rise to such an algorithm provided an MFMC subroutine is available. It should be emphasized that this approach does make use of the theorem and does not provide an alternative proof of it. On the other hand, L. Lovász [18] gave a simple proof of Edmonds' theorem and this proof is algorithmic. Although Lovász derived only the weak form of Edmonds' theorem, his proof carries over to the strong one almost word for word.

It is interesting to formulate Edmonds' Theorem 2.2 in the following equivalent form. Let A_0 denote the set of edges of $D = (V, A)$ leaving the root-node r_0 and let $A^* := A - A_0$.

THEOREM 2.3 (Edmonds' disjoint arborescences: strong form) *Let $D = (V, A)$ be a directed graph with a designated root-node r_0. Let A_0 denote the set of edges leaving r_0 and $A^* := A - A_0$. Let $\mathcal{A} = \{A_1, \ldots, A_k\}$ be a partition of A_0 into k sets. Then D has k disjoint spanning arborescences $F_1, \ldots, F_k$ of root r_0 so that $F_i \cap A_0 \subseteq A_i$ for $i = 1, \ldots, k$ if and only if $\varrho_{A^*}(X) \geq p(X)$ for every non-empty subset X of $V - r_0$ where $p(X)$ denotes the number of those members of $\mathcal{A}$ which contain no edges entering X.* •

Note that if the requested arborescences exist, then they can be chosen in such a way that $F_i \cap A_0 = A_i$. Yet another equivalent formulation of the strong theorem is as follows.

THEOREM 2.4 (Edmonds' disjoint arborescences: equivalent strong form) *Let $D = (V, A)$ be a digraph whose node set is partitioned into a root-set $R = \{r_1, \ldots, r_k\}$ and a terminal set T. Suppose that no edge of D enters any node of R. There are k disjoint arborescences $F_1, \ldots, F_k$ in D so that F_i is rooted at r_i and spans $T + r_i$ for each $i = 1, \ldots, k$ if and only if $\varrho_D(X) \geq |R - X|$ for every subset $X \subseteq V$ for which $X \cap T \neq \emptyset$.* •

This follows easily by applying Theorem 2.2 to the subgraph D' of D induced by T with the choice $R_i = \{v :$ there is an edge $r_i v \in A\}$ $(i = 1, \ldots, k)$. The same construction shows the reverse implication, too.

It has been tempting to find further extensions of the strong version of Edmonds' theorem but straightforward attempts failed. One natural conjecture, for example, was already disproved by Lovász in his original paper: even if there are $k(= 2)$ edge-disjoint paths from a root-node r_0 to every element of a specified terminal set $T \subseteq V - r_0$, the digraph not necessary includes k edge-disjoint arborescences of root r_0 so that each of them contains every node of T. Actually this problem can be shown to be NP-complete. In another possible variation, there are k specified subsets V_i of V each containing a root-node r_i. The problem consists of finding disjoint arborescences F_i $(i = 1, \ldots, k)$ so that each F_i is rooted at r_i, contains no node outside V_i, and spans V_i. This problem is NP-complete even in the very special case when $k = 2$, $V_1 = V$ and $V_2 = V - t$ for a specified node t. Indeed, it can be shown that a polynomial algorithm to this special case gives rise to a polynomial algorithm for the two edge-disjoint paths problem of a digraph, a well-known NP-complete problem.

However, we point out that the strong form of Edmonds' theorem implies its sharpening when the following result of Frank and Tardos [7] (which, incidentally, had been motivated by another old paper of Lovász [17]) is used.

THEOREM 2.5 *Let $G = (V, U; E)$ be a simple bipartite graph, $p : 2^V \rightarrow \mathbf{Z}_+$ a positively intersecting supermodular function, and $g : V \rightarrow \mathbf{Z}_+$ an upper bound function. There is a subset $F \subseteq E$ of the edges of G for which*

$$|\Gamma_F(X)| \geq p(X) \text{ for every } X \subseteq V \tag{4}$$

and

$$d_F(v) \leq g(v) \text{ for every node } v \in V \tag{5}$$

if and only if

$$|\Gamma_E(X)| + g(Z) \geq p(X \cup Z) \tag{6}$$

holds for every pair of disjoint subsets X and Z of V where $\Gamma_F(X)$ denotes the set of neighbours of X in the graph induced by $F \subseteq E$. •

Now the extension of Theorem 2.2 is as follows. Note that none of the equivalent Theorems 2.2, 2.3, 2.4 implies it immediately.

THEOREM 2.6 *Let $D = (V, A)$ be a directed graph and $g : V \rightarrow \mathbf{Z}_+$ an upper bound function. Let $\mathcal{U} = \{U_1, \ldots, U_k\}$ be a family of k subsets of V. There is a family $\mathcal{R} = \{R_1, \ldots, R_k\}$ of non-empty subsets of*

V and k disjoint spanning branchings of D with root sets $R_1, \dots, R_k$, respectively, in such a way that $R_i \subseteq U_i$ for $i = 1, \dots, k$ and each node $v \in V$ belongs to at most $g(v)$ members of $\mathcal{R}$ if and only if

$$u(X) + g(Z) \geq k - \varrho_D(X \cup Z) \tag{7}$$

for every pair X, Z of disjoint subsets of V with non-empty union where $u(X)$ denotes the number of U_i's intersecting X.

Proof. Suppose first that the requested family $\mathcal{R}$ and the k branchings exist. For disjoint subsets X and Z of V, at most $u(X)$ members of $\mathcal{R}$ intersect X due to $R_i \subseteq U_i$, and at most $g(Z)$ members of $\mathcal{R}$ intersect Z since each element z of Z belongs to at most $g(z)$ members of $\mathcal{R}$. Therefore there must be at least $k - u(X) - g(Z)$ members of $\mathcal{R}$ which are disjoint from $X \cup Z$ and hence the in-degree of $X \cup Z$ must be at least this number, that is, (7) is necesseary.

To see the sufficiency, construct a bipartite graph $G = (V, U; E)$ where $U = \{u_1, \dots, u_k\}$ and a node u_i is connected with $v \in V$ precisely if $v \in U_i$. Let a set function p on V be defined by $p(X) = k - \varrho_D(X)$ if $\emptyset \subset X \subseteq V$ and $p(\emptyset) = 0$. Then p is intersecting supermodular. Since $u(X) = |\Gamma_E(X)|$, (7) and (6) are equivalent. Hence Theorem 2.5 implies the existence of a subset F of E meeting (4) and (5). For each u_i, let R_i denote the neighbours of u_i in the subgraph induced by F. By the construction $R_i \subseteq U_i$, (4) is equivalent to (3), while (5) implies that each node $v \in V$ belongs to at most $g(v)$ members of $\mathcal{R}$. By Theorem 2.2 the reqested branchings exist. •

For the special case $g \equiv 1$, we formulate the result in the following equivalent version.

THEOREM 2.7 *Let $D = (V, A)$ be a directed graph with a designated root-node r_0. Let A_0 denote the set of edges leaving r_0 and let $\mathcal{A} = \{A_1, \dots, A_k\}$ be a family of k* (not-necessarily disjoint) *subsets of A_0. D has k disjoint spanning arborescences $F_1, \dots, F_k$ of root r_0 so that $F_j \cap A_0 \subseteq A_j$ if and only if*

$$\varrho_{A^*}(Z) + \varrho'(Z) \geq h \tag{8}$$

for every non-empty subset Z of $V^* := V - r_0$ and for every choice of h members $A_{i_1}, \dots, A_{i_h}$ of $\mathcal{A}$ where $\varrho'(X)$ denotes the number of edges in $A_{i_1} \cup \ldots \cup A_{i_h}$ entering X.

The following corollary is still a proper extension of Theorem 2.4.

THEOREM 2.8 *Let $D = (V, A)$ be a digraph whose node set is partitioned into a root-set $R = \{r_1, \dots, r_q\}$ and a terminal set T. Suppose that no edge of D enters any node of R. Let $m : R \to \mathbf{Z}_+$ be a function and let $k = m(R)$. There are k disjoint arborescences in D so that $m(r)$ of them are rooted at r and spanning $T + r$ for each $r \in R$ if and only if*

$$\varrho_D(X) \geq m(R - X) \text{ for every subset } X \subseteq V \text{ for which } X \cap T \neq \emptyset. \tag{9}$$

Proof. Contract R into a new node r_0 and define k subsets of the edge set A_0 leaving r_0 as follows. For each $r \in R$, take $m(r)$ copies of the subset of A_0 corresponding to the set of edges of D leaving r. Then (9) is equivalent to (8) and the result follows from Theorem 2.7. •

2.2 Extensions

One may be wondering whether there is a direct proof of Theorem 2.7 which follows the original lines of Lovász' proof without relying on Theorem 2.5. To understand better its nature, it has been tempting to extend Lovász' technique to more abstract settings. For example, the following 'abstract form' of the weak Edmonds theorem was derived in [5].

THEOREM 2.9 *Let $D = (V, A)$ be a digraph and $\mathcal{F}$ an intersecting family of subsets of V. It is possible to partition A into k coverings of $\mathcal{F}$ if and only if the in-degree of every member of $\mathcal{F}$ is at least k.* •

Obviously, when $\mathcal{F}$ consists of every non-empty subset of $V - r_0$, we obtain the weak form of Edmonds' theorem. A disadvantage of Theorem 2.9 is that it does not imply the strong version of Edmonds' theorem. The following result of L. Szegő [22], however, overcame this difficulty.

THEOREM 2.10 (Szegő) *Let $\mathcal{F}_1, \dots, \mathcal{F}_k$ be intersecting families of subsets of nodes of a digraph $D = (V, A)$ with the following mixed intersection property:*

$$X \in \mathcal{F}_i, \ Y \in \mathcal{F}_j, \ X \cap Y \neq \emptyset \ \Rightarrow \ X \cap Y \in \mathcal{F}_i \cap \mathcal{F}_j.$$

Then A can be partitioned into k subsets $A_1, \dots, A_k$ such that A_i covers $\mathcal{F}_i$ for each $i = 1, \dots, k$ if and only if $\varrho_D(X) \geq p_1(X)$ for all non-empty $X \subseteq V$ where $p_1(X)$ denotes the number of $\mathcal{F}_i$'s containing X. •

When the k families are identical, we are back at Theorem 2.9. When $\mathcal{F}_i = 2^{V-R_i} - \{\emptyset\}$, we obtain Edmonds' Theorem 2.2. The proof of Szegő is based on the observation that the mixed intersection property implies that p_1 is positively intersecting supermodular and this is why Lovász' approach works again. But Szegő's theorem is still not general enough to imply Theorem 2.7.

As a new contribution of the present work, we extend Szegő's theorem to k families of bi-sets and this will immeadiately yield Theorem 2.7. The proof uses again the same technique. We say that the bi-set families $\mathcal{F}_1, \ldots, \mathcal{F}_k$ satisfy the **mixed intersection** property if

$$X \in \mathcal{F}_i,\ Y \in \mathcal{F}_j,\ X_I \cap Y_I \neq \emptyset \ \Rightarrow\ X \cap Y \in \mathcal{F}_i \cap \mathcal{F}_j.$$

For a bi-set X, let $p_2(X)$ denote the number of indices i for which $\mathcal{F}_i$ contains X. For $X \in \mathcal{F}_i,\ Y \in \mathcal{F}_j$, the inclusion $X \subseteq Y$ implies $X = X \cap Y \in \mathcal{F}_j$ and hence p_2 is monotone non-increasing in the sense that $X \subseteq Y$, $p_2(X) > 0$ and $p_2(Y) > 0$ imply $p_2(X) \geq p_2(Y)$. We will need the following preparatory lemma.

Lemma 2.11 *If $p_2(X) > 0$, $p_2(Y) > 0$ and $X_I \cap Y_I \neq \emptyset$, then $p_2(X) + p_2(Y) \leq p_2(X \cap Y) + p_2(X \cup Y)$. Moreover, if there is an $\mathcal{F}_i$ for which $X \cap Y \in \mathcal{F}_i$ and $X, Y \notin \mathcal{F}_i$, then strict inequality holds.*

Proof. Consider the contribution of one family $\mathcal{F}_i$ to the two sides of the claimed inequality. If this contribution to the left hand side is two, that is, if both X and Y are in $\mathcal{F}_i$, then so are $X \cap Y$ and $X \cup Y$ and hence the contribution to the right hand side is also two. Suppose now that X belongs to $\mathcal{F}_i$ but Y does not. Since $p_2(Y) > 0$ is assumed, Y belongs to an $\mathcal{F}_j$. But then $X \cap Y$ belongs to $\mathcal{F}_i$ due to the mixed intersection property, that is, in this case the contribution of $\mathcal{F}_i$ to the right hand side is at least one. An $\mathcal{F}_i$ with the properties in the second part contributes only to the right hand side ensuring this way the strict inequality. •

THEOREM 2.12 *Let $D = (V, A)$ be a digraph and $\mathcal{F}_1, \ldots, \mathcal{F}_k$ intersecting families of bi-sets on ground set V satisfying the mixed intersection property. The edges of D can be partitioned into k parts $F_1, \ldots, F_k$ in such a way that F_i covers $\mathcal{F}_i$ for each $i = 1, \ldots, k$ if and only if*

$$\varrho_D(X) \geq p_2(X) \text{ for every bi-set } X. \tag{10}$$

Proof. The condition is clearly necessary. We prove the sufficiency by induction on $\sum_i |\mathcal{F}_i|$. There is nothing to prove if this sum is zero so we may assume that $\mathcal{F}_1$, say, is non-empty. Let U be a maximal member of $\mathcal{F}_1$. Call a bi-set **tight** if $\varrho(X) = p_2(X) > 0$.

Claim 2.13 *There is an edge e entering U in such a way that each tight bi-set covered by e is in $\mathcal{F}_1$.*

Proof. Suppose indirectly that no such an edge exists. Then each edge e entering U enters some tight bi-set $M \notin \mathcal{F}_1$. By the mixed intersection property, we cannot have $M \subseteq U$. Select a minimal tight bi-set $M \notin \mathcal{F}_1$ which intersects U. Since p_2 is monotone non-increasing, we know that $p_2(U \cap M) \geq p_2(M)$. Here, in fact, strict inequality must hold since $U \cap M \in \mathcal{F}_1$ and $M \notin \mathcal{F}_1$. The inequality $p_2(U \cap M) > p_2(M)$ implies that D has an edge $f = uv$ for which $u \in M - U$, $v \in U \cap M$. By the indirect assumption, f enters some tight bi-set $Z \notin \mathcal{F}_1$. Lemma 2.11 implies that the intersection of M and Z is tight. Since neither of M and Z is in $\mathcal{F}_1$, the second part of the lemma implies that $M \cap Z$ is not in $\mathcal{F}_1$ either, contradicting the minimal choice of M. •

Let e be an edge ensured by the Claim. Let $\mathcal{F}_1' := \{X \in \mathcal{F}_1 : e \text{ does not enter } X\}$. Then $\mathcal{F}_1'$ is an intersecting family of bi-sets. We claim that the mixed intersection property holds for the families $\mathcal{F}_1', \mathcal{F}_2, \ldots, \mathcal{F}_k$. Indeed, let $X \in \mathcal{F}_1'$ and $Y \in \mathcal{F}_i$ be two intersecting bi-sets for some $i = 2, \ldots, k$. Since $\mathcal{F}_1' \subseteq \mathcal{F}_1$, one has $X \cap Y \in \mathcal{F}_i$. If indirectly $X \cap Y$ is not in $\mathcal{F}_1'$, then e enters $X \cap Y$. Since e enters U and U was selected to be maximal in $\mathcal{F}_1$, it follows that $X \subseteq U$. But then e must enter X as well, contradicting the assumption $X \in \mathcal{F}_1'$.

Let $p_2'(X)$ denote the number of these families containing X (that is, $p_2'(X) = p_2(X) - 1$ if $X \in \mathcal{F}_1$ and e enters X and $p_2'(X) = p_2(X)$ otherwise). Let ϱ' denote the in-degree function on bi-sets with respect to $D' := D - e$. The choice of e implies $\varrho' \geq p_2'$. By induction, the edge set of D' can be partitioned into k parts $F_1', \ldots, F_k$ in such a way that F_1' covers $\mathcal{F}_1$ and F_i covers $\mathcal{F}_i$ for $i = 2, \ldots, k$. By letting $F_1 := F_1' + e$, we obtain a partition of A requested by the theorem. • •

Though not needed in the sequel, we point out that Theorem 2.12 can be reformulated in terms of set families. For a subset $T \subseteq V$, we say that a family $\mathcal{F}$ of subsets of V is **T-intersecting** if $X, Y \in \mathcal{F}$ and $X \cap Y \cap T \neq \emptyset$ imply $X \cap Y, X \cup Y \in \mathcal{F}$.

THEOREM 2.14 *Let $D = (V, A)$ be a digraph with a specified subset T of nodes containing the head of every edge of D. Let $\mathcal{F}_1, \ldots, \mathcal{F}_k$ be T-intersecting families of subsets of nodes of a digraph D with the following mixed intersection property: $X \in \mathcal{F}_i,\ Y \in \mathcal{F}_j,\ X \cap Y \cap T \neq \emptyset \Rightarrow\ X \cap Y \in \mathcal{F}_i \cap \mathcal{F}_j$. Then A can be partitioned into k subsets $A_1, \ldots, A_k$ such that A_i covers $\mathcal{F}_i$ for each $i = 1, \ldots, k$ if and only if $\varrho_D(X) \geq p_1(X)$ for all non-empty $X \subseteq V$ where $p_1(X)$ denotes the number of $\mathcal{F}_i$'s containing X.*

Proof. The necessity is evident again. For the sufficiency, define a family $\mathcal{F}'_i$ of bi-sets as follows. For each set $X \in \mathcal{F}_i$ let the bi-set $(X, X \cap T)$ be a member of $\mathcal{F}'_i$. Then each $\mathcal{F}'_i$ is intersecting and they meet the mixed intersection property. Since the head of every edge is in T, an edge enters a subset X precisely when it enters the bi-set $(X, X \cap T)$. Hence the partition of A into k sets ensured by Theorem 2.12 meets the requirement of the theorem. •

The reverse implication is equally simple and is left to the reader.

Alternative proof of the sufficiency in Theorem 2.7 . Let D' be a digraph arising from D by subdividing first each edge $e \in A_0$ by a node v_e and deleting then r_0. Let V_0 denote the set of the subdividing nodes and let V_i denote the subset of V_0 corresponding to the set A_i $(i = 1, \ldots, k)$.

For each $j = 1, \ldots, k$, let $\mathcal{F}_j$ be a family of bi-sets (X_O, X_I) for which $\emptyset \neq X_I \subseteq V^*$, $X_I = X_O \cap V^*$, $X_O \subseteq V_0 \cup V^*$ and $X_O \cap V_j = \emptyset$. Then $\mathcal{F}_j$ is an intersecting family of bi-sets and it follows from the definition that these k families meet the mixed intersecting property. It is also straightforward that (8) is equivalent to requiring that the number of edges entering a bi-set X is at least $p_2(X)$, the number of $\mathcal{F}_j$'s containing X. By Theorem 2.12, there are disjoint subsets $F'_1, \ldots, F'_k$ of the edge set of D' so that F'_j covers $\mathcal{F}_j$. We may assume that each $\mathcal{F}_j$ is a minimal covering of $\mathcal{F}_j$ (with respect to inclusion) which implies that an edge uv with $u \in V_0, v \in V^*$ can belong to F'_j only if $u \in V_j$. By the construction, the edges set F_j of D corresponding to F'_j is a spanning arborescence of D rooted at r_0 so that $F_j \cap A_0 \subseteq A_j$. •

Recently, N. Kamiyama, N. Katoh, and A. Takizawa [14] were able to find a surprising new proper extension of Theorem 2.7 (and hence the strong Edmonds theorem). We are going to show that their result can also be derived from Theorem 2.12. This is, however, a bit trickier due to the fact that the corresponding set function p_1 in their theorem is no more supermodular (and for the same reason their original proof is rather complicated). Similarly to Edmonds' theorem, this new result has also several equivalent formulations. One of them is as follows.

THEOREM 2.15 (Kamiyama, Katoh, and Takizawa [14]) *Let $D = (V, A)$ be a directed graph and let $R = \{r_1, r_2, \ldots, r_k\} \subseteq V$ be a list of k* possibly not distinct *root nodes. Let S_i denote the set of nodes reachable from r_i. There are edge-disjoint r_i-arborescences A_i spanning S_i for $i = 1, \ldots, k$ if and only if*

$$\varrho_D(Z) \geq p_1(Z) \text{ for every subset } Z \subseteq V \tag{11}$$

where $p_1(Z)$ denotes the number of sets S_i's for which $S_i \cap Z \neq \emptyset$ and $r_i \notin Z$.

Proof. The necessity of the condition is evident.

For brevity, we call a strongly connected component of D an **atom**. It is known that the atoms form a partition of the node set of D and that there is a so-called topological ordering of the atoms so that there is no edge from a later atom to an earlier one. By a subatom we mean a subset of an atom. Clearly, a subset $X \subseteq V$ is a subatom if and only if any two elements of X are reachable in D from each other. Note that any atom is disjoint from or included in S_i for each $i = 1, \ldots, k$.

Define k bi-set families $\mathcal{F}_i$ for $i = 1, \ldots, k$ as follows. For each non-empty subset $X_O \subseteq V - r_i$, let $\mathcal{F}_i := \{(X_O, X_I) : X_I = X_O \cap S_i \neq \emptyset \text{ a non-empty subatom}\}$. For each bi-set X, let $p_2(X)$ denote again the number of $\mathcal{F}_i$'s containing X. It follows immediately that $\mathcal{F}_i$ is an intersecting bi-set family.

Proposition 2.16 *The bi-set families $\mathcal{F}_i$ meet the mixed intersecting property.*

Proof. Let $X = (X_O, X_I)$ and $Y = (Y_O, Y_I)$ be members of $\mathcal{F}_i$ and $\mathcal{F}_j$, respectively, and suppose that X and Y are intersecting, that is, $X_I \cap Y_I \neq \emptyset$. Since a subatom and a subset with no leaving edges are never properly intersecting, we obtain that $X_O \cap S_i \subseteq S_i \cap S_j$ and $Y_O \cap S_j \subseteq S_i \cap S_j$. This implies for the sets $Z_O := X_O \cap Y_O$ and $Z_I := X_I \cap Y_I$ that $Z_O \cap S_i = Z_I = Z_O \cap S_j$ and hence $X \cap Y = (Z_O, Z_I) \in \mathcal{F}_i \cap \mathcal{F}_j$, as required. •

Proposition 2.17 $\varrho(X) \geq p_2(X)$ *for each bi-set X.*

Proof. Let $q := p_2(X)$ and suppose that X belongs to $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_q$. Let $Z := (V - (S_1 \cup S_2 \cup \ldots \cup S_q)) \cup X_I$. Since no edge leaves any S_i, every edge entering Z must enter X_I and hence also the bi-set X. Therefore $\varrho(X) \geq \varrho(Z)$. By (11), $\varrho(Z) \geq p_1(Z)$. It follows from the definition of Z that $p_1(Z) \geq q = p_2(X)$, and hence $\varrho(X) \geq p_2(X)$ •

Therefore Theorem 2.12 applies and hence the edges of D can be partitioned into sets $F_1, \ldots, F_k$ so that F_i covers $\mathcal{F}_i$ for $i = 1, \ldots, k$.

Proposition 2.18 *Each F_i includes an r_i-arborescence A_i which spans S_i.*

Proof. If the requested arborescence does not exist for some i, then there is a non-empty subset Z of $S_i - r_i$ so that F_i contains no edge from $S_i - Z$ to Z. Consider a topological ordering of the atoms and let Q be the earliest one intersecting Z. Since no edge leaving a later atom can enter Q, no edge with tail in Z enters Q.

Let $X_O := (V - S_i) \cup (Z \cap Q)$ and $X_I := X_O \cap S_i$. Then $X_I = Z \cap Q$ is a subatom and $X = (X_O, X_I)$ belongs to $\mathcal{F}_i$. Therefore there is an edge $e = uv$ in F_i which enters X. It follows that $v \in X_I \subseteq Z$ and that $u \in S_i - X_I$. Since u is not in Z and not in $V - S_i$, it must be in $S_i - Z$, that is, e is an edge from $S_i - Z$ to $X_I \subseteq Z$, contradicting the assumption that no such an edge exists. • •

Note that Theorem 2.8 can immediately be obtained from Theorem 2.15. To this end, add $m(i)$ new root-nodes to D and add an edge from each of them to r_i for $i = 1, \dots, q$. This way we will get k distinct (new nodes) and each node of V is reachable from every new root. In this setting the necesseary conditions in Theorems 2.15 and 2.7 coincide and each of the k maximal arborescences ensured by Theorem 2.15 will span the whole V.

To describe the original form of the theorem of Kamiyama et al., we call a branching B of D maximal if no edge of D leaves the node set of B.

THEOREM 2.19 (Kamiyama, Katoh, Takizawa [14]) *In a digraph $D = (V, A)$, let $\mathcal{R} = \{R_1, \dots, R_q\}$ be a family of non-empty* (not necessarily disjoint or distinct) *subsets of V and let S_i denote the set of nodes of D reachable from R_i. Let $m_1, \dots, m_q$ be positive integers whose sum is k. There are k edge-disjoint maximal branchings of D so that R_i is the root-set of m_i of them for $i = 1, \dots, q$ if and only if*

$$\varrho(X) \geq \sum [m_i : R_i \cap X = \emptyset \text{ and } X \text{ is reachable from } R_i] \quad \text{for every } X \subseteq V. \tag{12}$$

Proof. For each root-set R_i, let $r_i^1, \dots, r_i^{m_i}$ be new nodes and extend the digraph by adding k new parallel edges from r_i^j to every element of R_i for $i = 1, \dots, q$. An easy calculation shows that (11) is equivalent to (12) and the k disjoint arborescences ensured by Theorem 2.15 when restricted to V provide the requested maximal branchings of D. •

In [12], Frank, Király, and Kriesell observed that Edmonds' disjoint arborescences theorem can be extended to dypergraphs. A subset F of a ground-set V with a specified head-node in F is called a directed hyperedge, or briefly a dyperedge. F is said to enter a subset $X \subseteq V$ if its head is in X but $F \not\subseteq X$. A dypergraph $D = (V, \mathcal{D})$ is a hypergraph consisting of dyperedges in which $\varrho_D(X)$ denotes the number of dyperedges entering a subset X. We call D **rooted k-edge-connected** with respect to a root-node r_0 if the in-degree of every non-empty subset of $V - r_0$ is at least k. In the special case $k = 1$, the dypergraph is **root-connected**. In [12], with a rather easy reduction to Edmonds' disjoint arborescences theorem, it was shown that the dyperedges of a rooted k-edge-connected dypergraph can always be decomposed into k root-connected dypergraphs. With a similar approach, we can derive the following result.

THEOREM 2.20 *Let $D = (V, \mathcal{D})$ be a dypergraph and $R = \{r_1, \dots r_k\}$ a root-set. Let S_i denote the set of nodes reachable from r_0 in D. Then D includes k disjoint dypergraphs $D_1 = (S_1, \mathcal{D}_1), \dots, D_k = (S_k, \mathcal{D}_k)$ so that each D_i is root-connected at r_i if and only if $\varrho_D(X) \geq p(X)$ for every $X \subseteq V$ where $p(X)$ denotes the number of roots r_i for which $r_i \notin X$ and $S_i \cap X \neq \emptyset$.* •

3 Covering supermodular bi-set functions by digraphs

As mentioned in the introduction, the uncrossing technique was invented by Lovász [18] in order to obtain a short proof of the Lucchesi-Younger theorem. Later the method has become an indispensible tool for deriving combinatorial min-max theorems concerning sub- or supermodular set functions.

As a new application of the uncrossing procedure, we derive a result on covering simultaneously two supermodular bi-set functions by a digraph. (Recall that these functions were assumed to have positive values only on non-trivial bi-sets and they are integer-valued.) There have been two earlier results of this kind. Frank and Jordán [11] proved (in an equivalent form) the following result on minimum coverings of crossing supermodular bi-set functions (whose special case for set-functions appeared in [9]).

THEOREM 3.1 *Let p be a positively crossing supermodular bi-set function. The minimum number of directed edges covering p is equal to* $\max\{\sum[p(X) : X \in \mathcal{F}] : \mathcal{F}$ *an independent set of bi-sets*$\}$. •

The other result of similar vein concerns cheapest coverings of intersecting supermodular bi-set functions (generalizing its set-function version from [5, 6]).

THEOREM 3.2 ([10]) *Let $D = (V, A)$ be a digraph. Let $p : \mathcal{P}_2 \to \mathbf{Z}$ be a positively intersecting supermodular bi-set function and $g : A \to \mathbf{Z}_+ \cup \{\infty\}$ a non-negative upper bound on the edges of D that covers p. The linear system*

$$\varrho_x(Z) \geq p(Z) \text{ for every bi-set } Z \in \mathcal{P}_2,\ 0 \leq x \leq g \tag{13}$$

is totally dual integral.

3.1 Simultaneous coverings

Both theorems were motivated by and have several applications in network design. Our new contribution is a min-max theorem on smallest simultaneous coverings of two fully supermodular bi-set functions. It is neither a special case nor a generalization of the two previous results and has no special set function version known earlier. In what follows, we work throughout with a ground-set V of cardinality n. Let $D^* = (V, A^*)$ denote the complete digraph on V where $A^* := \{uv : u, v \in V\}$ denotes the set of all the $n(n-1)$ directed edges on V. Recall that $\mathcal{P}_2(V) = \mathcal{P}_2$ denoted the set of all bi-sets. A bi-set function p is positively supermodular if the supermodular inequality holds for every pair $\{X, Y\}$ of bi-sets for which $p(X) > 0, p(Y) > 0$. For example, if p is supermodular on a ring-family, and its value is zero otherwise, then p is positively supermodular.

THEOREM 3.3 *Let p_1 and p_2 be two positively supermodular bi-set functions which may be positive only on non-trivial bi-sets. Let $p := \max\{p_1, p_2\}$ where p is defined by $p(X) := \max\{p_1(X), p_2(X)\}$. Then p can be covered by γ* (possibly parallel) *directed edges if and only if*

$$p_1(X) + p_2(Y) \leq \gamma \tag{14}$$

for every pair of independent bi-sets X, Y.

Note that, due to $p(\emptyset, \emptyset) = 0$, (14) includes the necessary conditions $p_1(X) \leq \gamma$ and $p_2(Y) \leq \gamma$ so they need not be mentioned explicitly and a similar statement holds for later variations of the theorem.

It is more convenient to prove this result in a slightly more general form. We call a bi-set function p positively **2/3-supermodular** if for any choice of three bi-sets with positive p-value there are two of them that satisfy the supermodular inequality. Clearly, the maximum of two supermodular functions is 2/3-supermodular, but it turns out that there are 2/3-supermodular functions not arising this way.

THEOREM 3.4 *A positively 2/3-supermodular bi-set function p can be covered by γ* (possibly parallel) *directed edges if and only if $p(X) + p(Y) \leq \gamma$ for every pair of independent bi-sets X, Y. Equivalently, the minimum number $\tau(p)$ of edges covering p is equal to $\nu(p) := \max\{p(X) + p(Y) : \{X, Y\}$ independent bi-sets$\}$.*

Proof. The necessity of the condition is obvious since an edge can cover at most one of two independent bi-sets.

Lemma 3.5 *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two chains of nontrivial bi-sets and let $\mathcal{F} = \mathcal{C}_1 \cup \mathcal{C}_2$* (in the sense that a bi-set belonging to both chains occurs in two copies in $\mathcal{F}$)*. Suppose that no edge of D^* covers more than h members of $\mathcal{F}$. Then the members of $\mathcal{F}$ can be coloured by h colours so that each edge enters at most one member of each colour class. Furthermore, each colour class consists of at most two bi-sets.*

Proof. Since two comparable bi-sets are not independent, the second statement of the lemma is immediate.

Construct an undirected graph $B = (U, A)$ whose nodes correspond to the elements of $\mathcal{F}$ and two nodes are connected by an undirected edge if the corresponding members X, Y of $\mathcal{F}$ can be covered by an edge of D^*, that is, if they are not independent. Since $\mathcal{F}$ consists of two chains, B is the complement of a bipartite graph, and hence B is perfect.

Claim 3.6 *Let $Q \subseteq U$ be the node-set of a clique of graph B and let $\mathcal{F}_Q$ denote the members of $\mathcal{F}$ corresponding to the elements of Q. Then there is an edge of D^* covering all members of $\mathcal{F}_Q$.*

Proof. Assume first that $\mathcal{F}_Q$ is a chain. Let t be any node in the inner set of the smallest member of $\mathcal{F}_Q$ while s any node outside the outer set of the largest member of $\mathcal{F}_Q$. Then st covers all members of $\mathcal{F}_Q$. Therefore $\mathcal{F}_Q$ may be assumed to be the union of two non-empty chains $\mathcal{C}'_1$ and $\mathcal{C}'_2$. Let X_1 and X_2 be the smallest members of $\mathcal{C}'_1$ and $\mathcal{C}'_2$.

As Q is a clique, X_1 and X_2 are not independent, so there is a node $t \in V$ in the intersection of their inner sets. Similarly, let Y_1 and Y_2 be the largest members of $\mathcal{C}'_1$ and $\mathcal{C}'_2$. They are not independent either so there is a node $s \in V$ outside the union of their outer sets. Then st covers all the members of $\mathcal{C}'_1$ and $\mathcal{C}'_2$. •

The claim and the hypothesis of the lemma imply that the largest clique of B has at most h elements. Since B is perfect, its node set can be partitioned into h stable sets. Therefore $\mathcal{F}$ can be partitioned into h independent sets of bi-sets families. • •

Let us turn to the proof of the non-trivial inequality $\tau(p) \leq \nu(p)$ in the theorem. We proceed by induction on $\sum[p(X) : X \in \mathcal{P}_2]$. If this sum is zero, then the digraph $(V, \emptyset)$ with no edge will cover p. Suppose now that this sum is positive. For an edge $e \in A^*$, let $p_e(X) := (p(X) - 1)^+$ if e enters X and $p_e(X) := p(X)$ otherwise. Since the in-degree function (on bi-sets) of a digraph is fully submodular, $p_e(X)$ is 2/3-supermodular.

Lemma 3.7 *If $p(Z) > 0$ for a bi-set Z, then there is an edge $e \in A^*$ entering Z such that $\nu(p_e) < \nu(p)$.*

Proof. Let A denote the set of edges entering Z and suppose on the contrary that $\nu(p_e) = \nu(p)$ for each element e of A. That is, there is an independent pair $\mathcal{F}_e := \{X, Y\}$ of bi-sets for which e enters neither X nor Y and $p(X) + p(Y) = \nu(p)$

Let $\mathcal{F}'$ consist of bi-set Z plus all of the bi-sets which are members of some $\mathcal{F}_e$ in the sense that each bi-set X is taken into $\mathcal{F}'$ in as many copies as the number of pairs $\mathcal{F}_e$ containing X. Note that $(*)$ every edge of D^* enters at most $h := |A|$ members of $\mathcal{F}'$. The uncrossing procedure consists of finding two non-comparable elements X, Y of $\mathcal{F}'$ for which the supermodular inequality holds and replacing them by their intersection and union. Apply the uncrossing procedure as long as possible. Because the sum $\sum[|X_I|^2 + |X_O|^2 : X \in \mathcal{F}']$ strictly increases at each uncrossing step, the procedure terminates after a finite number of steps. Discard all members with p-value zero and let $\mathcal{F}$ denote the resulting family. Clearly $|\mathcal{F}| \leq |\mathcal{F}'|$, $p(Z) + h\nu(p) = p(\mathcal{F}') \leq p(\mathcal{F})$, and $(*)$ holds for $\mathcal{F}$, too. $\mathcal{F}$ cannot contain three pairwise non-comparable bi-sets for otherwise, by the 2/3-supermodularity of p, two of them would satisfy the supermodular inequality, and then they could have been uncrossed. If a partially ordered set contains no three pairwise uncomparable elements, then, by Dilworth's theorem, there are two disjoint chains covering the ground-set of the poset. Therefore the members of $\mathcal{F}$ can be partitioned into two chains $\mathcal{C}_1$ and $\mathcal{C}_2$. By Lemma 3.5 the members of $\mathcal{F}$ can be partitioned into h independent parts $\mathcal{I}_i$ $(i = 1, \ldots, h)$. So for one of these we must have $p(\mathcal{I}_i) \geq \lfloor p(\mathcal{F})/h \rfloor > \nu(p)$, a contradiction. •

For the edge e provided by Lemma 3.7, we have by induction $\tau(p_e) - 1 \leq \tau(p) \leq \nu(p) \leq \nu(p_e) - 1 \leq \tau(p_e) - 1$ from which equality holds throughout, and in particular $\tau(p) = \nu(p)$. • • •

Theorem 3.4 has a self-refining nature as it gives rise to its own extension. Let S and T be two non-empty subsets of V. We call a directed edge st an **ST-edge** if $s \in S$ and $t \in T$. Two bi-sets X and Y are **ST-independent** if there is no ST-edge covering both (or, equivalently, at least one of the sets $T \cap X_I \cap Y_I$ and $S - (X_O \cup Y_O)$ is empty).

THEOREM 3.8 *Let q be a positively 2/3-supermodular bi-set function so that $q(X)$ can be positive only if there is an ST-edge covering X. Then q can be covered by γ* (possibly parallel) *ST-edges if and only if $q(X) + q(Y) \leq \gamma$ for every pair of ST-independent bi-sets X, Y.*

Proof. The necessity of the condition is evident since an ST-edge cannot cover two ST-independent bi-sets. For the sufficiency, define a bi-set function p on $\mathcal{P}_2$, as follows.

$$p(X) := \begin{cases} \max\{q(X') : X' \in \mathcal{P}_2(V), X_I = X'_I \cap T,\ X_O = (X'_O \cup (V - S))\} & \text{if } X_I \subseteq T \text{ and } V = X_O \cup S, \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

Proposition 3.9 *p is positively 2/3-supermodular.*

Proof. Let X, Y, and Z be bi-sets for which $p(X) > 0$, $p(Y) > 0$, and $p(Z) > 0$. By the definition of p, there is a biset X' for which $p(X) = q(X')$ and $X_I = X'_I \cap T$, $X_O = (X'_O \cup (V - S))$, and similarly there are bi-sets Y' and Z' with analogous properties.

It follows that

$$X_I \cap Y_I = (X'_I \cap Y'_I) \cap T \text{ and } X_O \cap Y_O = (X'_O \cap Y'_O) \cup (V - S)$$

from which $q(X' \cap Y') \leq p(X \cap Y)$, and analogously,

$$X_I \cup Y_I = (X'_I \cup Y'_I) \cap T \text{ and } X_O \cup Y_O = (X'_O \cup Y'_O) \cup (V - S)$$

from which $q(X' \cup Y') \leq p(X \cup Y)$.

Since q is positively 2/3-supermodular, among the three bi-sets X', Y', Z', there are two, say X' and Y' satisfying the supermodular inequality. Hence $p(X) + p(Y) = q(X') + q(Y') \leq q(X' \cap Y') + q(X' \cup Y') \leq p(X \cap Y) + p(X \cup Y)$, as required. •

If $p(X) > 0$, then $p(X) = q(X')$ for some X' and hence $p(X) = q(X') \leq \gamma$. If $p(X) > 0$ and $p(Y) > 0$ for independent X and Y, then there are bi-sets X' and Y' for which $p(X) = q(X')$ and $p(X) = q(X')$. The definition of p implies that X' and Y' are ST-independent and hence $p(X) + p(Y) = q(X') + q(Y') \leq \gamma$. Therefore Theorem 3.4 implies the existence of a set of γ edges covering p. The definition of p implies that every edge covering a bi-set X with $p(X) > 0$ is necessarily an ST-edge, moreover any set covering p also covers q, and hence the theorem follows. • •

As a corollary, we have the following extension of Theorem 3.3.

THEOREM 3.10 *Let q_1 and q_2 be two positively supermodular bi-set functions for which $q_i(X)$ can be positive only if there is an ST-edge covering X. Let $q := \max\{q_1, q_2\}$. Then q can be covered by γ ST-edges if and only if $q_1(X) + q_2(Y) \leq \gamma$ for every pair of ST-independent bi-sets X, Y.* •

We will point out in Subsection 3.3 that positively supermodular functions do not behave well from an algorithmic point of view. In typical applications, however, one encounters with fully supermodular functions defined on a ring-family of bi-sets that may take negative values. For this case, Theorem 3.10 specializes as follows.

THEOREM 3.11 *For $i = 1, 2$ let p_i be a supermodular function on a ring-family $\mathcal{R}_i$ of bi-sets and assume that $p_i(X)$ may be positive only if there is an ST-edge covering X. There is a set of γ ST-edges covering both p_1 and p_2 if and only if*

$$p_i(X) \leq \gamma \text{ for every } X \in \mathcal{R}_i \quad (i = 1, 2) \tag{16}$$

and

$$p_1(X') + p_2(X'') \leq \gamma \text{ for every } ST\text{-independent } X' \in \mathcal{R}_1, X'' \in \mathcal{R}_2. \tag{17}$$

Equivalently, the minimum number of (possibly parallel) *ST-edges covering p_1 and p_2 is equal to $\nu := \max\{\nu_1, \nu_2, \nu_3, \nu_4\}$ where*
$\nu_1 = \max\{p_1(X') : X' \in \mathcal{R}_1\}$
$\nu_2 = \max\{p_2(X'') : X'' \in \mathcal{R}_2\}$,
$\nu_3 = \max\{p_1(X') + p_2(X'') : X' \in \mathcal{R}_1, X'' \in \mathcal{R}_2, X'_I \cap X''_I \cap T = \emptyset\}$,
$\nu_4 = \max\{p_1(X') + p_2(X') : X' \in \mathcal{R}_1, X'' \in \mathcal{R}_2, S - (X'_O \cup X''_O) = \emptyset\}$,
where the maximum on the empty set is defined to be zero. •

This implies the following equivalent version of Edmonds' polymatroid intersection theorem [1] (which was originally formulated for submodular functions).

THEOREM 3.12 (Edmonds) *Let p_1 and p_2 be supermodular functions on a common ground-set T. Then* $\min\{z(T) : z : T \to \mathbf{Z}_+, z(X) \geq \max\{p_1(X), p_2(X)\}$ *for every* $X \subseteq T\} = \max\{p_1(X) + p_2(Y) : X \cap Y = \emptyset\}$.

Proof. Let s be a new element, $V := T + s$ and $S := \{s\}$. Apply Theorem 3.11 for the special case when $\mathcal{R}_i := \{(X_O, X_I) : X_O = X_I \subseteq T\}$ and observe that in this case $\nu = \nu_3$ and the ST-edges can be identified with the elements of T. •

It should be noted that Edmonds extended the theorem for the more general case as well when p_i is intersecting supermodular only (that is, the supermodular inequality is required only for intersecting sets). In this case the maximum formula in the theorem is more complicated as it includes partitions rather then sets only. No extension of Theorem 3.11 is known to cover this form. The difficulty is indicated by the fact that the following natural-looking statement is **false**: *For $i = 1, 2$ let p_i be a non-negative integer-valued intersecting supermodular function on an intersecting family $\mathcal{R}_i$ of sets so that $p_i(X)$ may be positive only if there is an ST-edge covering X. The minimum number of (possibly parallel) ST-edges covering p_1 and p_2 is equal to* $\max\{\sum[p_1(X) : X \in \mathcal{F}_1] + \sum[p_2(X) : X \in \mathcal{F}_2]\} : \mathcal{F}_i \subseteq \mathcal{R}_i$ *is laminar, $\mathcal{F}_1 \cup \mathcal{F}_2$ is ST-independent.*$\}$ (A family $\mathcal{R}$ of sets is intersecting if it contains $X \cap Y$ and $X \cup Y$ whenever $X, Y \in \mathcal{R}, X \cap Y \neq \emptyset$. $\mathcal{R}$ is laminar if one of the sets $X - Y, Y - X, X \cap Y$ is empty for every two members X, Y.) Let $V = \{v_1, v_2, v_3, v_4\}, \mathcal{R}_1 := \{\{v_1\}, \{v_2, v_3\}, \{v_1, v_2, v_3\}\}, \mathcal{R}_2 := \{\{v_3, v_4\}, \{v_1, v_3, v_4\}\}$, let p_i be identically one on $\mathcal{R}_i$, and let $S := T := V$. Then the minimum value in the statement is 3 while the maximum is only 2.

Edmonds' intersection theorem extends to the weighted case, as well, asserting, in a concise form, that the linear system $\{z \geq 0, z(X) \geq \max\{p_1(X), p_2(X)\}$ for every $X \subseteq T\}$ is totally dual integral (TDI). The min-cost version of Theorem 3.10 includes NP-complete connectivity augmentation problems so it is unlikely to have a TDI-ness result concerning Theorem 3.10. In other connectivity augmentation problems however [8] the special case of node-induced costs were nicely solvable (where node-induced means that the cost of an edge st arises as the sum of the given node-costs of s and t). The corresponding problem in the enviroment of Theorem 3.10 remains open.

3.2 Applications to bipartite graphs and digraphs

Let us derive a graphical consequence concerning bipartite matchings.

THEOREM 3.13 *For $i = 1, 2$, let $G_i = (S, T; E_i)$ be a bitpartite graph with $n = |S| = |T|$. There is a set F of at most γ* (undirected) *ST-edges so that both $G_1 + F$ and $G_2 + F$ has a perfect matching if and only if*

$$q_1(Z') + q_2(Z'') \leq \gamma \tag{18}$$

holds for every two disjoint subsets Z', Z'' of S and for every two disjoint subsets Z', Z'' of T. Here $q_i(Z) := |Z| - |\Gamma_i(Z)|$ where $\Gamma_i(Z)$ (for $Z \subseteq S$ or $Z \subseteq T$) denotes the set of nodes having at least one neighbour in Z in the graph G_i $(i = 1, 2)$.

Proof. The necessity of (18) is evident, we prove its sufficiency. For $i = 1, 2$, define ring-families $\mathcal{R}_i$ of bi-sets as follows. $\mathcal{R}_i := \{X = (X_O, X_I) : X_I = X_O \cap T,\ X_O \supseteq (X_I \cup \Gamma_i(X_I)\}$. For simplicity we will not distinguish between the directed ST-edges and the undirected edges connecting S and T. Since an edge of G_i, when considered to be oriented toward T, cannot cover any member of $\mathcal{R}_i$, it follows that $\mathcal{R}_i$ is indeed a ring-family. For $X \in \mathcal{R}_i$, let

$$p_i(X) := 2|X_I| - |X_O|.$$

Clearly, p_i is supermodular on $\mathcal{R}_i$. For $X \in \mathcal{R}_i$ let $Z := S - X_O$. Since $\Gamma_i(X_I) \subseteq X_O - X_I$ we have $p_i(X) = |X_I| - (|X_O| - |X_I|) \leq |X_I| - |\Gamma_i(X_I)| = q_i(X_I)$. Since $\Gamma_i(Z) \subseteq T - X_I$, we have $|X_I| + |\Gamma_i(Z)| \leq |T| = |S| = |Z| + |X_O - X_I| = q_i(Z)$. Based on these, (18) implies (17). Since the bi-set $(\emptyset, \emptyset)$ belongs to $\mathcal{R}_i$ and $p_i(\emptyset, \emptyset) = 0$, we conclude that (17) implies (16).

By Theorem 3.11, there is a set F of γ ST-edges covering both p_1 and p_2. we claim that $G_i^+ := G_i + F$ satisfies the Hall condition. Indeed, if the set Y' of neighbours of a subset $Y \subseteq T$ in G_i^+ had fewer elements than $|Y|$, then $p_i(Y \cup Y', Y) > 0$ and F would not cover the bi-set $(Y \cup Y', Y)$. By Hall's theorem, G_i^+ has a perfect matching, as required. •

It should be noted that requiring (18) only for the subsets of T is not sufficient (unlike the situation in Hall's theorem on perfect matching in bipartite graphs where Hall's criterion $|X| \leq |\Gamma(X)|$ is violated by a subset of S if and only if it is violated by a subset of T). Given the simple condition (18) in Theorem 3.13, one may feel tempted to derive the result from classical matching theory or matroid intersection, and, indeed, J. Pap [20] found a short, elegant way to derive Theorem 3.13 directly from Edmonds' matroid intersection theorem [3].

As another corollary, we exhibit a connectivity augmentation result concerning simultaneous augmentations of two digraphs. In order to handle edge-disjoint and node-disjoint paths uniformly, the following common generalization was introduced in [10].

Let $D = (V, F)$ be a digraph and $g : V \to \mathbf{Z}_+$ a function. A set of edge-disjoint st-paths is said to be ***g*-bounded** if each node $v \in V - \{s, t\}$ is used by at most $g(v)$ of these paths. We stress that g-boundedness automatically means that the paths are edge-disjoint. Let $\lambda_g(s, t; D)$ denote the maximum number of g-bounded st-paths. Note that for large g (say, $g \equiv |F|$) $\lambda_g(s, t; D)$ is the maximum number of edge-disjoint st-paths, while for $g \equiv 1$, $\lambda_g(s, t; D)$ is the maximum number of openly disjoint st-paths.

We will need the bi-set function μ_g defined by

$$\mu_g(X) := \sum [g(v) : v \in X_O - X_I] \quad (= \mu_g(X_O) - \mu_g(X_I)). \tag{19}$$

It is easily seen that for bi-sets X and Y

$$\mu_g(X) + \mu_g(Y) = \mu_g(X \cap Y) + \mu_g(X \cup Y). \tag{20}$$

The following characterization can be easily derived from the edge-version of Menger's theorem (and was done in [10]).

Proposition 3.14 (Variation of Menger's theorem) *In a digraph $D = (V, F)$ there are k g-bounded st-paths if and only if*

$$\varrho_F(X) \geq k - \mu_g(X) \text{ holds for every bi-set } X = (X_O, X_I) \text{ with } t \in X_I \text{ and } X_O \subseteq V - s. \ \bullet \tag{21}$$

We say that D is **(k, g)-connected** from s to t if there are k g-bounded paths from s to t.

Suppose now that $D_i = (V, A_i)$ are digraphs for $i = 1, 2$ on the same node set V in which s_i and t_i are designated source and sink nodes. Moreover, let $g_i : V \to \mathbf{Z}_+$ be a function and k_i positive integer. Consider the ring-family $\mathcal{R}_i := \{X \in \mathcal{P}_2(V) : t \in X_I, X_O \subseteq V - s\}$ of bi-sets and define a bi-set function p_i on $\mathcal{R}_i$ by

$$p_i(X) := k_i - \varrho_{D_i}(X) - \mu_{g_i}(X).$$

Since ϱ_{D_i} is submodular and μ_{g_i} is modular, p_i is supermodular on $\mathcal{R}_i$. Let S and T be two non-empty subsets of V so that there is an ST-edge covering each bi-set with positive p_1- or p_2-value. By Theorem 3.11, we get the following.

THEOREM 3.15 *Given D_i, $s_i, t_i, \mathcal{R}_i$ g_i, k_i, p_i S, T for $i = 1, 2$ as above, there is a set F of γ ST-edges whose addition to D_i results in a digraph which is (k_i, g_i)-connected from s_i to t_i if and only if*

$$p_i(X) \leq \gamma \text{ for every } X \in \mathcal{R}_i, \ (i = 1, 2), \text{ and}$$

$$p_1(X') + p_2(X'') \leq \gamma \text{ for every } ST\text{-independent } X' \in \mathcal{R}_1 \text{ and } X'' \in \mathcal{R}_2. \ \bullet$$

3.3 Algorithmic aspects

Before sketching an algorithmic approach, we make some observations on classes of supermodular functions.

Claim 3.16 *If a positively 2/3-supermodular function is given by an evaluation oracle, then its maximum cannot be computed in polynomial time.*

Proof. Let p be a set function which takes positive value on exactly one subset and zero otherwise. This is positively 2/3-supermodular and to find out its maximum one must, in worst case, call for the value of all subsets. $\bullet$

Therefore there is no polynomial algorithm for computing the extrema in Theorem 3.4 if the 2/3-submodular function is given by an evaluation oracle. The question arises whether the problem in Theorem 3.4 is more general at all than the one in Theorem 3.11. The next claim shows that the answer is yes.

Claim 3.17 *Not every 2/3-supermodular function arises from two positively supermodular functions as their maximum.*

Proof. Let the ground-set $V = \{v_1, v_2, \ldots, v_5, s\}$ have six elements so that the first five elements are arranged around a circle according to their subscripts. Define $p(X)$ to be 1 if $s \in X$ and the elements of $X - s$ are consecutive around the circle (in particular, if $X = V$ or $X = \{s\}$), otherwise let $p(X) = 0$. Then easy case-checking shows that p is 2/3-supermodular but there cannot be two positively supermodular functions p_1 and p_2 so that $p(X) = \max\{p_1(X), p_2(X)\}$. Indeed, $p(V_i) = 1$ for $V_i := V - v_i$, $1 \leq i \leq 5$. Since the non-consecutive pairs form a five-gon, one of p_1 and p_2, say p_1, must take value one on two sets V_i, V_j with non-consecutive v_i, v_j. But then p_1 cannot be positively supermodular since $p_1(V_i \cap V_j) \leq p(V_i \cap V_j) = 0$ by definition and $p_1(V_i \cup V_j) \leq p(V_i \cup V_j) \leq 1$. $\bullet$

An analogous question concerning positively supermodular functions was answered by T. Király [15]:

Claim 3.18 *Not all positively supermodular functions arise as the non-negative part of a fully supermodular function.*

Proof. Let X_1, X_2, X_3 be three subsets of a ground-set V in general position. Let $p(X_i) = 1, p(X_i \cup X_j) = 2$ $(i \neq j), p(X_1 \cup X_2 \cup X_3) = 4$ and $p(X) = 0$ on the remaining sets. Then p is positively supermodular and a simple argument shows that it cannot be the nonnegative part of a supermodular function. $\bullet$

The only general construction we know for positively supermodular functions is taking the non-negative part of a supermodular function on a ring-family, and likewise, we do not know any general class, let alone applications, of 2/3-supermodular functions which are not the maximum of two supermodular ones. On one hand, these function classes gave rise to formally more general results and their use made the proofs technically simpler, on the other hand they are not convenient for algorithmic handling. This is why we formulated separately Theorem 3.11: there is a strongly polynomial algorithm for computing the extrema in that theorem.

The very nature of the theorem makes it possible to compute a digraph H covering p_1 and p_2 with a minimum number of edges, provided that a subroutine is available for computing ν given in the theorem. With some work, such a subroutine can indeed be constructed by making use of an existing algorithm for maximizing supermodular functions [13, 21] (and in the special case of Theorem 3.15 even a Max-flow Min-cut subroutine suffices). So suppose that such a subroutine is available. The digraph H with a minimum number of edges that covers p_1 and p_2 will be defined with the help of a function $z : A^* \to \mathbf{Z}_+$ which tells us the number $z(a) \geq 0$ of parallel copies of every possible ST-edge a to be taken into H. The digraph defined by z covers p_i if $\varrho_z \geq p_i$ for $i = 1, 2$.

For a given z, let $\nu(z)$ denote the optimum in Theorem 3.11 with respect to the revised bi-set functions $p_1 - \varrho_z$ and $p_2 - \varrho_z$. Call a function $z : A^* \to \mathbf{Z}_+$ **good** if

$$\nu = \nu(z) + z(A^*). \tag{22}$$

By definition $z \equiv 0$ is good and the problem of finding a minimum z is equivalent to construct a good z covering of p.

Consider the elements of A^* in an arbitrary order $a_1, \ldots, a_m$. At the beginning $z \equiv 0$. At a general step, suppose that the values of $z(a_1), \ldots, z(a_{i-1})$ have already been computed in such a way that the vector $z = (z(a_1), \ldots, z(a_{i-1}), 0, \ldots, 0)$ is good. Compute $\nu(z)$. If this number is zero, then z is a covering of p and the algorithm terminates by returning z. Suppose now that $\nu(z) > 0$. Let z' be a vector arising from z by setting $z(a_i)$ to be a big enough number M and compute $\nu(z')$. It follows from Theorem 3.11 that by setting $z(a_i)$ to be $\nu(z) - \nu(z')$ the revised vector keeps to be good and the algorithm may proceed to the next index $i + 1$.

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