Az alábbi leírás a "Frank: Connection in Combinatorial Optimization" c. könyv 497-dik oldalán kezdődő leírás kicsit javított változata.

1 Supermodular colourings

As an application of g-polymatroids defined by intersecting paramodular pairs, we derive an interesting theorem of Schrijver on colourings that cover supermodular functions. Let h be an integer-valued, intersecting supermodular function on a ground-set S and k a positive integer. We say that a k-colouring of S covers h if each subset $X \subseteq S$ contains at least $h(X)$ distinct colours. The condition

$$h(X) \le \min\{k, |X|\} \text{ for every } X \subseteq S \tag{1}$$

is clearly necessary for the existence of a k-colouring covering h. We can (and shall) assume that

$$h(v) = 1 \text{ for every } v \in V \tag{2}$$

since (1) requires $h(v) \le 1$ and in the case when $h(v) = 0$ we can increase $h(v)$ to 1 without changing the colourability or intersecting supermodularity. We can also assume that $k = \max\{h(X) : X \subseteq S\}$.

Schrijver proved that (1) is not only sufficient but there always exists a common k-colouring covering two such functions when both meet (1). We are going to prove this result by using the approach of Tardos who studied first the properties of one colour class in a k-colouring covering a single intersecting supermodular function h and proved the following result. The present proof is a variant of that of Schrijver, appeared in his book.

THEOREM 1.1 (Tardos) *Let h be an intersecting supermodular function satisfying (1) and (2), and let $k = \max\{h(X) : X \subseteq S\}$. There exists a subset $C \subseteq S$ such that*

$$C \textit{ intersects each subset } Y \subseteq S \textit{ for which } h(Y) = k, \tag{3}$$

and

$$|X - C| \ge h(X) - 1 \textit{ for every } X \subseteq S \textit{ intersecting } C. \tag{4}$$

Moreover, the characteristic vectors of such subsets C are the integral elements of a non-empty g-polymatroid.

Proof. Let $\mathcal{T} = \{T_1, \dots, T_q\}$ be the set of minimal sets X for which $h(X) = k$. Standard supermodular technique shows that $\mathcal{T}$ is a subpartition of S. Indeed, if X and Y were two properly intersecting members of $\mathcal{T}$, then $k + k = h(X) + h(Y) \le h(X \cap Y) + h(X \cup Y) \le k + k$ would imply $h(X \cap Y) = k$ contradicting the minimality of X. Let $T_0 := S - (T_1 \cup \dots \cup T_q)$. Define a set-function b for $X \ne \emptyset$ by

$$b(X) := |X| - h(X) + 1$$

and define b' as follows. Let $b'(\emptyset) = 0$ and for a non-empty X, let

$$b'(X) := \begin{cases} b(X) & \text{if } X \subseteq T_i \text{ for some } i = 0, \dots, q \\ \infty & \text{for otherwise.} \end{cases} \tag{5}$$

Note that $b(X)$ (and thus $b'(X)$) is positive on every non-empty set X, and $b(X) = 1$ for every singleton $X = \{v\}$. Define p as follows.

$$p(Y) := \begin{cases} 1 & \text{if } Y \in \mathcal{T} \\ 0 & \text{if } |Y| \le 1 \text{ and } Y \notin \mathcal{T} \\ -\infty & \text{for otherwise.} \end{cases} \tag{6}$$

We claim that (p, b') is an intersecting paramodular pair. Indeed, since h is intersecting supermodular, b is intersecting submodular and hence so is b'. Also, p is clearly intersecting supermodular. The cross-inequality for two properly intersecting sets X and Y holds automatically since in this case $b(X) = \infty$ or $p(Y) = -\infty$.

Since a weak paramodular pair is known to define a g-polymatroid, $Q = Q(p, b')$ is an integral g-polymatroid. Q is non-empty since the characteristic vector $\chi(Z)$ of a subset $Z \subseteq S$ for which $|Z \cap T_i| = 1$ for each $i = 1, \dots, q$ belongs to Q. Furthermore, as $p(v) \ge 0$ and $b'(v) = 1$ for each $v \in S$, every integral element of Q is a $(0,1)$-vector.

Proposition 1.2 *$\widetilde{x}(X) \leq b(X)$ for each $x \in Q$ and $X \subseteq S$, that is, $Q(p,b') = Q(p,b)$.*

Proof. There is nothing to prove when $b'(X) = b(X)$ so suppose that $b'(X) = \infty$, that is, there is a member T of $\mathcal{T}$ such that both $X - T$ and $X \cap T$ are non-empty. Since $h(T) = k \geq h(X \cup T)$ and $h(X) + h(T) \leq h(X \cap T) + h(X \cup T)$, we have $h(X) \leq h(X \cap T)$ from which

$$\widetilde{x}(X) = \widetilde{x}(X \cap T) + \sum[x(v) : v \in (X - T)] \leq b'(X \cap T) + \sum[b'(v) : v \in (X - T)] \leq$$

$$[|X \cap T| - h(X \cap T) + 1] + |X - T| \leq |X| - h(X) + 1 = b(X). \bullet$$

The theorem follows by observing that a subset C satisfies (3) and (4) if and only if $\underline{\chi}_C$ belongs to $Q(p,b)$. $\bullet \bullet$

Lemma 1.3 *The vector $m_k := (1/k, \ldots, 1/k) \in \mathbf{R}^S$ belongs to Q.*

Proof. Since $p(X)$ is positive only when $X \in \mathcal{T}$, we have $|X| \geq h(X) = k$ and hence $\widetilde{m}_k(X) = |X|/k \geq 1 = p(X)$.

To see the other inequality $\widetilde{m}_k(X) \leq b'(X)$, we distinguish two cases. If $|X| \leq k$, then $h(X) \leq |X|$ implies

$$\widetilde{m}_k(X) = |X|/k \leq 1 \leq |X| - h(X) + 1 = b'(X).$$

If $|X| \geq k$, then

$$h(X) \leq k \leq |X| \leq |X| - |X|/k + 1 = |X| - \widetilde{m}_k(X) + 1,$$

that is, $\widetilde{m}_k(X) \leq b'(X)$ $\bullet$

A k-colouring of S is **equitable** if the size of each colour class is $\lfloor |S|/k \rfloor$ or $\lceil |S|/k \rceil$.

Exercise 1.1 *Prove that a k-colouring $\mathcal{S}' := \{S_1, \ldots, S_k\}$ of S is equitable if $\lfloor |S|/k \rfloor \leq |S_1| \leq \lceil |S|/k \rceil$ and $\{S_2, \ldots, S_k\}$ is an equitable $(k-1)$-colouring of $S - S_1$.*

THEOREM 1.4 (Schrijver) *Let k be a positive integer, p_1 and p_2 two positively intersecting supermodular functions on a ground-set S for which $p(X) \leq \min\{k, |X|\}$ for every $X \subseteq S$ where*

$$p(X) := \max\{p_1(X), p_2(X)\}.$$

Then there is a k-colouring of S covering p. Moreover, the colouring can be chosen to be equitable.

Proof. Induction on k. The theorem is trivial for $k = 1$ so we assume that $k \geq 2$.

Lemma 1.5 *There is a subset $S_1 \subset S$ such that* (A) *S_1 intersects every X with $p(X) = k$,* (B) *$|X - S_1| \geq p(X) - 1$ for every $X \subseteq S$, and* (C) *$\lfloor |S|/k \rfloor \leq |S_1| \leq \lceil |S|/k \rceil$.*

Proof. Let Q_i denote the intersection of the g-polymatroid in Theorem 1.1 assigned to p_i $(i = 1, 2)$ and the plank $K(\alpha, \beta) := \{x : \alpha \leq \widetilde{x}(S) \leq \beta\}$ (where $\alpha = \lfloor |S|/k \rfloor$ and $\beta = \lceil |S|/k \rceil$. Then Q_i is an integral g-polymatroid.

We proved earlier that the intersection of two integral g-polymatroids is an integral polyhedron. Since in our case the intersection is non-empty as m_k belongs to both Q_1 and to Q_2 by Lemma 1.3, there is an integral element of $Q_1 \cap Q_2$, and this element is, by Theorem 1.1, the incidence vector of a subset $S_1 \subseteq S$ satisfying properties (A), (B), and (C). $\bullet$

The existence of set S_1 ensured by Lemma 1.5 implies the theorem as follows. Define set-functions p'_i $(i = 1, 2)$ on $S' = S - S_1$ by $p'_i(Z) := \max\{p_i(Z), \max\{p_i(Z \cup X) - 1 : \emptyset \subset X \subseteq S_1\}\}$ $(i = 1, 2)$. These are clearly positively intersecting supermodular for which $p'_i(Z) \leq k - 1$ holds by (A), and $p'_i(Z) \leq |Z|$ holds for every $Z \subseteq S'$ by (B). By the inductive hypothesis, S' has an equitable $(k-1)$-colouring $\{S_2, \ldots, S_k\}$ covering both p'_1 and p'_2. But then $\{S_1, \ldots, S_k\}$ is an equitable k-colouring of S covering both p_1 and p_2. $\bullet \bullet$

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2 Applications to branchings and augmentations

2.1 Disjoint bibranchings

Schrijver used his supermodular colouring theorem to derive an extensions of the weak form of Edmonds' theorem on disjoint arborescences Let $D = (V, A)$ be a directed graph with a bipartition $\{S, T\}$ of its node-set. A **bibranching** is a digraph that includes a directed path from S to every $t \in T$ and includes a directed path to T from every $s \in S$. For example, when $|S| = 1$, the notion of a bibranching is equivalent to root-connectivity.

THEOREM 2.1 (Schrijver's disjoint bibranchings theorem) *A digraph D includes k edge-disjoint bibranchings if and only if $\varrho_D(X) \geq k$ holds for every $\emptyset \subset X \subseteq T$ and $\delta_D(X) \geq k$ holds for every $\emptyset \subset X \subseteq S$.*

Proof. The proof of necessity is straightforward. For proving the sufficiency, consider the set F of edges of D entering T. For a non-empty subset $X \subseteq F$, define

$$p_1(X) := \max\{k - \varrho_{D-X}(Z) : Z \subseteq T,\ Z \text{ includes the head of each element of } X\}.$$

It is easy to check that p_1 is an intersecting supermodular function on ground-set F.

Let D' denote the digraph arising from D by shrinking S into a single node s and consider a given k-colouration $\{F_1, \ldots, F_k\}$ of F. The strong form of Edmonds' disjoint arborescences theorem implies that $\{F_1, \ldots, F_k\}$ can be extended to disjoint spanning arborescences of D' if and only if $\varrho_{D-X}(Z)$ plus the number of colour-classes entering Z is at least k for every $\emptyset \subset Z \subseteq T$. This is equivalent to requiring that the number of colour classes intersecting X is at least $p_1(X)$ for every non-empty subset X of F.

Analogous arguments show that $\{F_1, \ldots, F_k\}$ can be extended to disjoint spanning reverse arborescences of D'' if and only if the number of colour-classes intersecting X is at least $p_2(X)$ for every non-empty subset X of F where D'' arises from D by shrinking T into a single node and

$$p_2(X) := \max\{k - \varrho_{D-X}(Z) : Z \subseteq S,\ Z \text{ includes the tail of each element of } X\}.$$

Therefore all we need to show is that there is a k-colouring of F covering both p_1 and p_2. Theorem 1.4 just ensures the existence of such a colouring. •

Note that the same proof shows that the k disjoint bibranchings can be chosen in such a way that they define an equitable colouring of the set of edges entering T. This result is interesting even in the special case when $|S| = 1$ when it asserts that if a digraph includes k disjoint spanning arborescences of a given root r_0, then the k arborescences can be chosen in such a way that their out-degrees at r_0 differ (pairwise) by at most one.

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