On the structure of a generalization of weakly associative lattices

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Abstract

The concept of weakly associative lattices (i.e. relational systems with a reflexive and antisymmetric relation $\leq$, in which for each pair of elements there exist a least upper and a greatest lower bound) was introduced in [3] and [5]. In [4] WU-systems are defined, i.e. weakly associative lattices with the unique bound property, and their equivalence with projective planes is described. In this paper we introduce WU_λ -systems, and discuss their relation to symmetric $2-(v,k,\lambda)$ designs equipped with a special "loop-free" mapping.

The concept of weakly associative lattices (i.e. relational systems with a reflexive and antisymmetric relation $\leq$, in which for each pair of elements there exist a least upper and a greatest lower bound) was introduced in [3] and [5]. Ervin Fried and Vera T. Sós defined WU-systems in [4]; i.e. weakly associative lattices with the unique bound property; their equivalence with projective planes is described in [4]. In this paper we introduce WU_λ -systems (here 'U' stands for 'uniform'), and discuss their relation to symmetric $2-(v,k,\lambda)$ designs equipped with a special "loop-free" mapping.

Definition. Let $\mathcal{V} = \langle V, \leq \rangle$ be a system with a reflexive and antisymmetric relation $\leq$. Let $U(v) = \{w \in V : v \leq w\}$ and $L(v) = \{w \in V : w \leq v\}$ for each $v \in V$. Given a positive integer λ, we call $\mathcal{V}$ a WU_λ -system if for each $v_1 \neq v_2 \in V$ both of the sets $U(v_1) \cap U(v_2)$ and $L(v_1) \cap L(v_2)$ has size λ.

Note that for $\lambda = 1$ we get WU-systems of [4].

Remark. The Paley tournaments for (prime power) $q = 4k-1$ give a series of examples for WU_λ -systems with $\lambda = (q+1)/4$. ($V = GF(q)$, and for $a, b \in GF(q)$ define $a \leq b$ iff $(b-a)$ is a square element of $GF(q)$.)

Definition. A $2-(v,k,\lambda)$ design (sometimes they are denoted by $S_\lambda(2,k,v)$) is an incidence structure $\mathcal{D} = (\mathbf{P}, \mathbf{B})$, where the elements of $\mathbf{P}$ are called *point*s, the elements $B \in \mathbf{B}$ are subsets of $\mathbf{P}$ called *block*s, and $|\mathbf{P}| = v$, $|B| = k$ for all $B \in \mathbf{B}$, and for any pair of distinct points there are exactly λ blocks containing both of them. A design is called *symmetric*, if $|\mathbf{P}| = |\mathbf{B}|$; in this case $v = \frac{k(k-1)}{\lambda} + 1$ holds ([2]).

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We want to prove that in a WU$_\lambda$-system all the subsets $U(v)$ and $L(v)$ have the same size, under some slight conditions. First we prove

Lemma 1 *Let $\mathcal{V} = (V, \leq)$ be a WU$_\lambda$-system, and $v_1, v_2 \in V$. Suppose that $v_1 \not\leq v_2$. Then $|U(v_1)| = |L(v_2)|$.*

Proof: Consider the set $M = \{(u, w) \in U(v_1) \times L(v_2) : u \geq w\}$. For a fixed $u \in U(v_1)$ there are λ elements w satisfying $w \leq v_2,\ w \leq u$. So $|M| = \lambda|U(v_1)|$. Similarly for a $w \in L(v_2)$ there are λ elements u satisfying $u \geq v_1$ and $u \geq w$. So $|M| = \lambda|L(v_2)|$. $\square$

The following lemma is a generalization of Theorems 1 and 2 in [4]:

Lemma 2 *Let $\mathcal{V} = (V, \leq)$ be a WU$_\lambda$ -system. Define an incidence structure as follows: $\mathcal{D}_\mathcal{V} = (\mathbf{P}, \mathbf{B})$, where $\mathbf{P} = V,\ \mathbf{B} = \{U(v) : v \in V\}$. Then any two blocks $B_1 \neq B_2 \in \mathbf{B}$ intersect in exactly λ points, and any two points $p_1 \neq p_2 \in \mathbf{P}$ are contained in exactly λ blocks.*

Proof: Let $B_1 = U(v_1), B_2 = U(v_2)$. As $v_1 \neq v_2$, we have $|U(v_1) \cap U(v_2)| = \lambda$. A block $B = U(v)$ contains both p_1 and p_2 if and only if $v \leq p_1$ and $v \leq p_2$, so if $v \in L(p_1) \cap L(p_2)$. There are exactly λ such elements. $\square$

Now we are ready to prove the following

Theorem 3 *Let $\mathcal{V} = (V, \leq)$ be a WU$_\lambda$ -system. Define the incidence structure $\mathcal{D}_\mathcal{V}$ as in the previous theorem. Then either*

(i) $\lambda \geq 2$ and $\mathcal{D}_\mathcal{V}$ is a symmetric design; or

(ii) $\lambda = 1$ and $\mathcal{D}_\mathcal{V}$ is a (possibly degenerate) projective plane.

Proof: Case *(ii)* comes from the previous theorem. Note that when $\mathcal{D}_\mathcal{V}$ is a degenerate projective plane then $\mathcal{V}$ has the following shape: $V = \{0\} \cup \{1\} \cup A$ for some set A not containing 0 and 1; and $1 < 0$ and $0 < a < 1$ holds for all $a \in A$. Such a WU-system is called *singular* in [4].

If $\lambda \geq 2$ then for any two $v_1 \neq v_2 \in V$ there is at least one element $w \in V$ such that $w < v_1$ and $w < v_2$. (If $\lambda = 1$ then if e.g. $v_1 < v_2$ there is no such element w.) Now, using Lemma 1 with $v_1 \not\leq w$ and $v_2 \not\leq w$ we have $|U(v_1)| = |L(w)| = |U(v_2)|$. So all the blocks have the same size and $\mathcal{D}_\mathcal{V}$ is a $2 - (|V|, |U(v)|, \lambda)$ design for any $v \in V$. It is symmetric as the number of points and the number of blocks coincide. $\square$

We remark that $\mathcal{D}_\mathcal{V}$ has a very special property: there is a natural bijective mapping $\varphi : \mathbf{P} \to \mathbf{B}$ satisfying $p \in \varphi(p)$ for every $p \in \mathbf{P}$; indeed, define $\varphi(v) = U(v)$ for $v \in V$.

Given an arbitrary symmetric design $\mathcal{D} = (\mathbf{P}, \mathbf{B})$ and a bijective mapping $\varphi : \mathbf{P} \to \mathbf{B}$ satisfying $p \in \varphi(p)$ for every $p \in \mathbf{P}$; we will say that a pair of distinct points (p_1, p_2) is a *φ-loop*, if $p_2 \in \varphi(p_1)$ and $p_1 \in \varphi(p_2)$. If there are no φ-loops, we call φ a *loop-free* mapping.

In $\mathcal{D}_\mathcal{V}$, as the relation $\leq$ is antisymmetric, the natural bijective mapping $\varphi(v) = U(v)$ for $v \in V$ is a loop-free mapping. This property characterizes WU$_\lambda$ -systems as Theorem 3 and the following statement show:

Theorem 4 *Given a symmetric $2-(v,k,\lambda)$ design $\mathcal{D}=(\mathbf{P},\mathbf{B})$ with a bijective mapping $\varphi:\mathbf{P}\to\mathbf{B}$ satisfying $p\in\varphi(p)$ for every $p\in\mathbf{P}$, and φ is loop-free, then there exists a WU_λ-system $\mathcal{V}=(V,\leq)$ such that $\mathcal{D}=\mathcal{D}_\mathcal{V}$ and φ is the natural bijective mapping $\varphi(v)=U(v)$ for $v\in V$.*

Proof: Let $V=\mathbf{P}$, and define $\leq$ such that $p_1\leq p_2$ iff $p_2\in\varphi(p_1)$. Now $p\in\varphi(p)$ implies reflexivity and antisymmetry follows from the fact that φ is loop-free. The rest of the statement is obvious. □

Now it remains a question how many different loop-free φ mappings can be defined for a given symmetric design (where *different* means that they result in non-isomorphic WU_λ-systems). For $\lambda=1$ there exists at least one for any projective plane (note that in this case the condition of forbidden loops holds automatically and König's theorem on matchings of regular bipartite graphs shows the existence of such a mapping); they result in isomorphic WU_λ - systems if the plane is degenerate; in [4] some examples of different mappings were shown for the same non-degenerate plane of order three. (We remark that there are at least $(q+1)!$ such mappings in a projective plane of order q; to determine the number of non-isomorphic ones seems to be a hard problem however.)

Here we present an example where $\lambda=2$: a $2-(7,4,2)$ design in fact. Its points are

$$\mathbf{P}=\{0,1,2,3,4,5,6\},$$

the blocks are

$$\mathbf{B}=\{B_0(0234),B_1(0125),B_2(2456),B_3(1236),B_4(1345),B_5(0356),B_6(0146)\},$$

and put

$$\varphi(p)=B_p \text{ for all } p\in\mathbf{P}.$$

It can be checked that it is a loop-free bijective mapping. In fact this design, on the one hand, is the example coming from the Paley tournament for $q=7$, on the other hand it is the complement of the Fano-plane, i.e. its points are

the points of the projective plane of order 2, its blocks are the complements of the lines. One can show easily that this example is unique in the sense that

Proposition 5 *If $q>2$ then the design coming from the complement of a projective plane of order q does not admit a loop-free bijective mapping from its points onto its blocks.* □

The "prototype" of symmetric designs is $PG_{n-1}(n,q)$, the projective space of dimension n over the finite field $GF(q)$, with its hyperplanes as blocks. They do not admit any loop-free bijective mapping for $n\geq 3$ as the following theorem states:

Theorem 6 *There is no loop-free bijective mapping in $PG_{n-1}(n,q)$ for $n\geq 3$.*

Proof: Suppose there exists a loop-free bijective mapping φ between points and hyperplanes; let $\varphi(P) = H$. For any other point $Q \in H \setminus \{P\}$ consider $\varphi'(Q) = \varphi(Q) \cap H$, an $(n-2)$ dimensional subspace of H containing Q. For different $Q_1 \neq Q_2$ they should have $\varphi'(Q_1) \neq \varphi'(Q_2)$ (otherwise $\varphi(Q_1)$ and $\varphi(Q_2)$ would form a loop). As H has as many points as $(n-2)$ dimensional subspaces and P does not have a pair $\varphi'(P)$, there is exactly one $(n-2)$ dimensional subspace of H not appearing in $\Sigma = \{\varphi'(Q) : Q \in H, Q \neq P\}$. Hence union of the subspaces in Σ covers P, so there exists a point $Q_0 \in H$ such that both $P \in \varphi(Q_0)$ and $Q_0 \in H = \varphi(P)$ hold, contradiction. □

Note that the argument above works also for designs where it is true that fixing a block $B = \varphi(P)$, the traces $B \cap \varphi(Q)$ of the other blocks associated to the points Q of $B \setminus \{P\}$ will surely cover P.

A construction. Here we give a general construction for a series of WU_λ -systems. Let $(G, +)$ be a group, and $D \subset G$ a λ-difference set in it, i.e. in the multiset of differences $\{d_1 - d_2 : d_1, d_2 \in D, d_1 \neq d_2\}$ each of the non-zero elements of G occurs exactly λ times ([1]). Without loss of generality we may assume that $0 \in D$. If, moreover,

$$D \cap -D = \{0\}, \qquad (*)$$

i.e. D does not contain any nonzero element and its inverse simultanously, then the following $2-(|G|, |D|, \lambda)$-design can be constructed:

$$\mathbf{P} = G, \quad \mathbf{B} = \{B_g = g + D : g \in G\}.$$

Now $(*)$ implies that $\varphi(g) = B_g$ is a loop-free bijective mapping satisfying $g \in \varphi(g)$.

Note that this mapping is preserved under a large automorphism group, which is isomorphic to G. We also remark, that the Paley-tournament, defined in the Remark at the beginning of this paper, or, its $2-(p, \frac{p+1}{2}, \frac{p+1}{4})$ design equivalent, being the complement of an Hadamard design, can be constructed this way: $G = (GF(p), +)$ and $D = \{x^2 : x \in GF(p)\}$.

There are several examples of such difference sets, see [1] for some cyclic ones. We cite here one of them, due to M. Hall Jr.: let $p = 4x^2 + 9$ be a prime, x an odd integer, then $G = (GF(p), +), D = \{a^4 : a \in GF(p)\}$ is a difference set satisfying $D \cap -D = \{0\}$ as -1 is not a quartic power in $GF(p)$. The arising design has parameters $(v, k, \lambda) = (p = 4x^2 + 9, x^2 + 3, \frac{x^2+3}{4})$. Here $x \in \{1, 5, 19, ...\}$.

Questions. There are several open problems arising at this point. **1.** The complement of the Fano plane described after Theorem 4 is a biplane (i.e. a symmetric design with $\lambda = 2$). We conjecture that this is the only biplane admitting a loop-free bijective mapping; for some biplanes it can be checked easily. **2.** It would be interesting to prove that there exists a function $1 << f(v)$ such that if a symmetric $2-(v, k, \lambda)$-design admits a loop-free bijective mapping then $\lambda = 1$ or $\lambda \geq f(v)$. Each of the examples known to us have quite a big λ value.

References

[1] L.D. Baumert, *Cyclic difference sets*, Lecture Notes in Mathematics **182** (1971), Springer-Verlag.

[2] P.J. Cameron and J.H. van Lint, *Designs, graphs, codes and their links*, London Mathematical Society Student Texts **22** (1991), Cambridge University Press, Cambridge.

[3] E. Fried, Tournaments and nonassociative lattices, *Ann. Univ. Sci. Budapest Sect. Math.* **13** (1970), 151-164.

[4] E. Fried and V. T. Sós, Weakly associative lattices and projective planes, *Algebra Universalis* **5** (1975), 114-119.

[5] H. Skala, Trellis theory, *Algebra Universalis* **1** (1971), 218-233.

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