ON THE GRAPH OF A FUNCTION OVER A PRIME FIELD WHOSE SMALL POWERS HAVE BOUNDED DEGREE

SIMEON BALL AND ANDRÁS GÁCS

Abstract. Let f be a function from a finite field $\mathbb{F}_p$ with a prime number p of elements, to $\mathbb{F}_p$. In this article we consider those functions $f(X)$ for which there is a positive integer $n > 2\sqrt{p-1} - \frac{11}{4}$ with the property that $f(X)^i$, when considered as an element of $\mathbb{F}_p[X]/(X^p - X)$, has degree at most $p - 2 - n + i$, for all $i = 1, \ldots, n$. We prove that every line is incident with at most $t - 1$ points of the graph of f, or at least $n + 4 - t$ points, where t is a positive integer satisfying $n > (p-1)/t + t - 3$ if n is even and $n > (p-3)/t + t - 2$ if n is odd. With the additional hypothesis that there are $t - 1$ lines that are incident with at least t points of the graph of f, we prove that the graph of f is contained in these $t - 1$ lines. We conjecture that the graph of f is contained in an algebraic curve of degree $t - 1$ and prove the conjecture for $t = 2$ and $t = 3$. These results apply to functions that determine less than $p - 2\sqrt{p-1} + \frac{11}{4}$ directions. In particular, the proof of the conjecture for $t = 2$ and $t = 3$ gives new proofs of the result of Lovász and Schrijver [7] and the result in [5] respectively, which classify all functions which determine at most $2(p-1)/3$ directions.

1. Introduction

Let p be a prime and let f be a function from $\mathbb{F}_p$, the finite field with p elements, to $\mathbb{F}_p$. Any such function has a unique representation as a polynomial of degree at most $p - 1$ and, conversely, each polynomial $\phi(X)$ of degree at most $p-1$ defines a distinct function $x \mapsto \phi(x)$. The function $x \mapsto f(x)^i$ is understood to be the i-th power of the image of $f(x)$, will sometimes be abbreviated as f^i, and should not be confused with the i-fold composition of f.

This article is concerned with functions $f(x)$ for which there is an $n > 2\sqrt{p-1} - \frac{11}{4}$ with the property that, for all $i = 1, \ldots, n$, the function $f(x)^i$ has degree at most $p - 2 - n + i$. By degree we mean the degree of the polynomial of degree at most $p-1$ which represents the function $x \mapsto f(x)^i$, that is the degree of the residue of $f(x)^i$ in the quotient ring $\mathbb{F}_p[x]/(x^p - x)$. Nearly all polynomials that appear in this article will come from this ring.

We define $I(f)$ to be the maximum such n plus one. An alternative definition is given by

$$I(f) = \min\{i + j \mid \sum_{x \in \mathbb{F}_p} x^j f(x)^i \neq 0\}.$$

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To see this note that the sum $-\sum_{x\in\mathbb{F}_p} g(x)$ is equal to the coefficient of x^{p-1} in g, for any polynomial $g(X)$ of degree at most $p-1$. Thus, for all $n \leq I(f)-1$, the sum $\sum_{x\in\mathbb{F}_p} x^{n-i}f(x)^i = 0$ implies that $f(x)^i$ has degree at most $p-2-n+i$.

Note that if the degree of f is at least two then $I(f(x)) = I(f(x)-mx-c)$ for all $m, c \in \mathbb{F}_p$.

Let $M(f)$ be the number of elements c of $\mathbb{F}_p$ for which $x \mapsto f(x)+cx$ is a permutation of $\mathbb{F}_p$, in other words $f(X)+cX$ is a permutation polynomial. Alternatively, $-c$ does not occur as a direction determined by the function f, i.e. $-c \neq (f(y)-f(x))/(y-x)$ for all $x, y \in \mathbb{F}_p$, $x \neq y$. Indeed, the motivation to look at the properties of functions f for which $I(f)$ is large, stems from the desire to classify those functions that determine few directions.

Although the results in the first section relate to functions over a field with a prime number of elements they more or less extend to all finite fields, some care having to be taken with the parity of the characteristic in a few places. However, the motivation to study functions with the above property is the fact that if the field is a prime field then $I(f)$ is greater than $M(f)$. Let us check this first.

Let

$$\pi_k(Y) = \sum_{x\in\mathbb{F}_p} (f(x)+xY)^k = \sum_{i+j=k} \sum_{x\in\mathbb{F}_p} \binom{i+j}{i} x^j f(x)^i Y^j.$$

By [6, Lemma 7.3], if $x \mapsto f(x)+cx$ is a permutation, then $\pi_k(c) = 0$ for all $0 < k < p-1$. For $k < p-1$ the polynomial $\pi_k(Y)$ has degree at most $k-1$, since the coefficient of Y^k is $\sum_{x\in\mathbb{F}_p} x^k = 0$. Therefore it is identically zero for all $0 \leq k-1 < M(f)$, unless $M(f) = p-1$ in which case f is linear. The binomial coefficient occurring in the coefficent of Y^j, $\binom{i+j}{i}$ is non-zero since $i+j < p$. Hence, if f is not linear then $I(f)-1 \geq M(f)$.

The purpose of this note is to say something about the graph of the function f given that $I(f) > 2\sqrt{p-1} - \frac{7}{4}$. We shall then apply these results to functions for which $M(f)$ is large. Previously, in [7], [5] and [2], although the proofs centered on functions for which $I(f)$ is large, all assumed that $M(f)$ was reasonably large too. Here we eliminate this necessity. Moreover, in previous articles $I(f)$ was required to be much larger, at least $(p+4)/3$, to be able to draw conclusions.

Other articles that are relevant here are [4] and [1] which deal with functions f over a finite field $\mathbb{F}_q$, where q is a prime power, for which $M(f) \geq (q-1)/2$ and [10] which bounds $M(f)$ in terms of the degree of f.

2. Properties of functions for which $I(f)$ is more than $2\sqrt{p-1} - \frac{7}{4}$

Let f be a function from $\mathbb{F}_p$ to $\mathbb{F}_p$ which, as an element of $\mathbb{F}_p/(X^p - X)$, has degree at least three. Note that if the degree of f is either one or two then $I(f)$ and $M(f)$ are easily determined.

Write $I(f) = 2s+1+\epsilon$ where s is some integer satisfying $s > \sqrt{p-1} - \frac{11}{8}$ and $\epsilon = 0$ if $I(f)$ is odd and $\epsilon = 1$ if $I(f)$ is even. This implies $I(f) > 2\sqrt{p-1} - \frac{7}{4}$.

By definition

$$\sum_{x \in \mathbb{F}_p} f(x)^i x^j = 0,$$

for all $0 < i+j \leq 2s+\epsilon$, and the degree of f, which we write as f°, satisfies $f^\circ \leq p-2s-1-\epsilon$ and more generally $(f^i)^\circ \leq p-2s-2+i-\epsilon$, for all $i = 1, \ldots, 2s+\epsilon$.

Let

$$V = \{(F_1, F_2, \ldots, F_s) \mid F_i \in \mathbb{F}_p[X],\ F_i^\circ \leq s-i\}.$$

The set V consists of s-tuples of polynomials and is a vector space over $\mathbb{F}_p$ of dimension $s(s+1)/2$.

Consider the linear map ψ_1 from V to $\mathbb{F}_p[X]$ defined by

$$\psi_1((F_1, F_2, \ldots, F_s)) = F_1 f + F_2 f^2 + \ldots + F_s f^s.$$

We want to bound the dimension of the subspace $\mathrm{Im}(\psi_1)$, the image of ψ_1. Note that the dimension of a subspace U of a vector space of polynomials is equal to the number of distinct degrees of polynomials that appear in U.

LEMMA 2.1. *If $f^\circ > (p-3)/2s$ then at most $(p-3)/2-s-\epsilon$ of the numbers in the interval $[s+1, \ldots, p-1]$ occur as degrees of polynomials in $\mathrm{Im}(\psi_1)$.*

Proof. The maximum degree of a polynomial in $\mathrm{Im}(\psi_1)$ is $p-s-2-\epsilon$ so we are only concerned with the interval $[s+1, \ldots, p-s-2-\epsilon]$. Given any two polynomials g and h in $\mathrm{Im}(\psi_1)$, the product gh can be written as a $\sum_{i=1}^{2s} G_i f^i$, where $G_i^\circ \leq 2s-i$ for some G_i. Thus, since $I(f) \geq 2s+1$, it follows that $(gh)^\circ \neq p-1$ and if $\epsilon = 1$ then $(gh)^\circ \neq p-2$, since Xgh cannot have degree $p-1$ in this case.

If $\epsilon = 0$ then only half of the numbers in the interval $[s+1, \ldots, p-s-2]$ can occur as degrees of polynomials in $\mathrm{Im}(\psi_1)$, that is at most $(p-3)/2-s$.

If $\epsilon = 1$ and m a number in the interval $[(p+1)/2, \ldots, p-s-3]$ occurs as a degree of a polynomial in $\mathrm{Im}(\psi_1)$, then neither $p-1-m$, nor $p-2-m$ occur as degrees of a polynomial in $\mathrm{Im}(\psi_1)$. Thus, if a positive number d of the numbers in the interval $[(p+1)/2, \ldots, p-s-3]$ occur as a degree of a polynomial in $\mathrm{Im}(\psi_1)$, then at most $(p-3)/2-s-d-1$ of the numbers in the interval $[s+1, \ldots, (p-3)/2]$ occur as a degree of a polynomial in $\mathrm{Im}(\psi_1)$. If $g \in \mathrm{Im}(\psi_1)$ then $g^\circ \neq (p-1)/2$, since $(g^2)^\circ \neq p-1$. Thus, overall at most $(p-5)/2-s$ of the numbers in the interval $[s+1, \ldots, p-s-3]$ can occur as degrees of polynomials in $\mathrm{Im}(\psi_1)$. The case $d = 0$ does not occur since $f, f^2, \ldots, f^s \in \mathrm{Im}(\psi_1)$ and it is not possible that all these polynomials have degree less than $(p-1)/2$. □

Let t be a positive integer with the property that $I(f)-1-\epsilon = 2s > (p-1-2\epsilon)/t+t-3$ and $2 \leq t \leq \sqrt{p-1}$. Note that $t < s+2$, so the following lemma is not trivial.

LEMMA 2.2. *Either the function f has less than t zeros or it has more than $s+2$ zeros.*

Proof. If $f^\circ \leq (p-3)/2s$ then the statement is trivial since $f^\circ \leq (p-3)/2s < t-t(t-3)/s < t$, unless $t = 2$. If $t = 2$ then $2s < (p-5)/2$ and $f^\circ \leq 2$ which is not the case since we have assumed $f^\circ \geq 3$.

Let r be the number of distinct zeros of f and suppose that $t \leq r \leq s$. We will deal with the cases $r = s+1$ and $r = s+2$ at the end of the proof.

A zero of f is a zero of any polynomial in $\mathrm{Im}(\psi_1)$, so all non-zero polynomials in $\mathrm{Im}(\psi_1)$ have degree at least r. Thus, applying Lemma 2.1,

$$\dim \mathrm{Im}(\psi_1) \leq (p-3)/2 - s - \epsilon + s - r + 1 = (p-1)/2 - r - \epsilon,$$

and so $\mathrm{Ker}(\psi_1)$, the kernel of ψ_1 satisfies

$$\dim \mathrm{Ker}(\psi_1) \geq s(s+1)/2 - (p-1)/2 + r + \epsilon.$$

Let $(F_1, \ldots, F_s) \in \mathrm{Ker}(\psi_1)$. Then $F_1 f + F_2 f^2 + \ldots + F_s f^s = 0$ and for all x such that $f(x) \neq 0$

$$-F_1 = F_2 f + \ldots + F_s f^{s-1}.$$

The degree of this equation is at most $p - s - 3$ and it holds for all elements that are not zeros of f, of which there are at least $p - s - 2$ by assumption, so it holds for all elements of $\mathbb{F}_p$. Therefore a zero of f is a zero of the polynomial F_1, which implies, if F_1 is not zero then it has degree at least r. By definition it has degree at most $s - 1$.

Define a linear map ψ_2 from $\mathrm{Ker}(\psi_1)$ to $\mathbb{F}_p[X]/(X^p - X)$ by

$$\psi_2((F_1, F_2, \ldots, F_s)) = F_2 f + F_3 f^2 + \ldots + F_s f^{s-1} = -F_1.$$

A non-zero polynomial in the $\mathrm{Im}(\psi_2)$ has degree at least r and at most $s - 1$ and so

$$\dim \mathrm{Ker}(\psi_2) \geq s(s+1)/2 - (p-1)/2 + r + \epsilon - (s - r).$$

Let $(F_1, \ldots, F_s) \in \mathrm{Ker}(\psi_2)$. Then $F_2 f + F_3 f^2 + \ldots + F_s f^{s-1} = 0$ and for all x such that $f(x) \neq 0$

$$-F_2 = F_3 f + \ldots + F_s f^{s-2}.$$

The degree of this equation is at most $p - s - 4$ and since it holds for at least $p - s - 2$ elements of $\mathbb{F}_p$, it holds for all elements of $\mathbb{F}_p$. Therefore a zero of f is a zero of the polynomial F_2, which implies, if F_2 is not zero then it has degree at least r, and by definition it has degree at most $s - 2$.

Now we define recursively maps ψ_j, for $j = 3, 4, \ldots, s - t + 1$, from the $\mathrm{Ker}(\psi_{j-1})$ to $\mathbb{F}_p[X]/(X^p - X)$ by

$$\psi_j((F_1, F_2, \ldots, F_s)) = F_j f + F_{j+1} f^2 + \ldots + F_s f^{s-j+1}.$$

Arguing as before, non-zero polynomials in the $\mathrm{Im}(\psi_j)$ have degree at least r and at most $s - j + 1$ and so the dimension of $\mathrm{Im}(\psi_j)$ is at most $s - j - r + 2$. Therefore

$$\dim \mathrm{Ker}(\psi_j) \geq s(s+1)/2 - (p-1)/2 + r + \epsilon - (s-r) - (s-r-1) - \ldots - (s-j-r+2).$$

In particular

$$\dim \mathrm{Ker}(\psi_{s-r+1}) \geq (2rs - p + 1 - r(r-3) + 2\epsilon)/2,$$

which is greater than zero since $2rs - r(r-3)$ is minimised while r ranges between t and $s + 2$ when $r = t$, and $2ts - t(t-3) > p - 1 - 2\epsilon$.

Let $(F_1, F_2, \ldots, F_s)$ be a non-zero element of $\mathrm{Ker}(\psi_{s-r+1})$. The fact that $F_{s-r+1} f + \ldots + F_s f^r = 0$ implies that for all x that are not zeros of f

$$-F_{s-r+1} = F_{s-r+2} f + \ldots + F_s f^{r-1}.$$

However, the degree of this equation is at most $p-2s+r-3 \leq p-s-3$ and, since it holds for at least $p-s-2$ elements, it holds for all elements of $\mathbb{F}_p$. Therefore a zero of f is a zero of the polynomial F_{s-r+1}, which implies that F_{s-r+1} is zero since it has degree at most $r-1$. Similarly $F_{s-r+2}, F_{s-r+3}, \ldots, F_s$ are zero. Now $(F_1, F_2, \ldots, F_s) = (F_1, F_2, \ldots, F_{s-r}, 0, \ldots, 0) \in \text{Ker}(\psi_{s-r+1}) \subseteq \text{Ker}(\psi_{s-r}) \subseteq \ldots \subseteq \text{Ker}(\psi_1)$. Recursively $(F_1, F_2, \ldots, F_{s-r-j}, 0, \ldots, 0) \in \text{Ker}(\psi_{s-r-j})$ implies $F_{s-r-j} = 0$ for $j = 0, 1, \ldots, s-r-1$, and hence $(F_1, F_2, \ldots, F_s) = 0$. We have shown that if $(F_1, \ldots, F_s) \in \text{Ker}(\psi_{s-r+1})$ then $(F_1, F_2, \ldots, F_s) = 0$. Thus the dimension of $\text{Ker}(\psi_{s-r+1})$ is zero, which is not the case.

Let us finally deal with the cases $r = s+1$ and $r = s+2$. In these cases, since the zeros of f are zeros of any polynomial in $\text{Im}(\psi_1)$, every polynomial in $\text{Im}(\psi_1)$ has degree at least $s+1$. Lemma 2.1 implies

$$\dim \text{Im}(\psi_1) \leq (p-3)/2 - s - \epsilon,$$

and so

$$\dim \text{Ker}(\psi_1) \geq s(s+1)/2 - (p-3)/2 + s + \epsilon,$$

which is greater than zero since $s > \sqrt{p-1} - \frac{11}{8}$ and $p \geq 5$.

Let $(F_1, \ldots, F_s) \in \text{Ker}(\psi_1)$. Then $F_1 f + F_2 f^2 + \ldots + F_s f^s = 0$ and for all x such that $f(x) \neq 0$,

$$-F_1 = F_2 f + \ldots + F_s f^{s-1}.$$

The degree of this equation is at most $p-s-3$ and it holds for all elements that are not zeros of f, of which there are at least $p-s-2$ by assumption, so it holds for all elements of $\mathbb{F}_p$. The degree of F_1 is at most $s-1$ and has at least $s+1$ zeros, since it is zero whenever f is zero. Therefore $F_1 = 0$ and arguing as before $F_2 = \ldots = F_s = 0$, and we have shown that the dimension of $\text{Ker}(\psi_1)$ is zero, which is not the case. □

LEMMA 2.3. *If the function f has more than $s+2$ zeros then it has at least $I(f)+3-t$ zeros.*

Proof. Since f has more than $s+2$ zeros, the image of ψ_1 contains no polynomials of degree less than $s+3$. Thus, by Lemma 2.1, the dimension of $\text{Im}(\psi_1) \leq (p-3)/2 - s - \epsilon$.

Therefore the dimension of $\text{Ker}(\psi_1)$ is at least $s(s+1)/2 - (p-3)/2 + s + \epsilon > 0$.

Again, let r be the number of distinct zeros of f, so $r \geq s+3$, and let

$$g(X) = (X^p - X)/((X^p - X), f(X)),$$

so the degree of g is $p - r$.

Define a linear map ϕ_2 from $\text{Ker}(\psi_1)$ to $\mathbb{F}_p[X]/(X^p - X)$ by

$$\phi_2((F_1, F_2, \ldots, F_s)) = F_1 + F_2 f + \ldots + F_s f^{s-1} = -F_1.$$

Let $(F_1, F_2, \ldots, F_s) \in \text{Ker}(\psi_1)$. Since $F_1 f + F_2 f^2 + \ldots + F_s f^s = 0$ it follows that for all x such that $f(x) \neq 0$ we have $F_1 + F_2 f + \ldots + F_s f^{s-1} = 0$ and so there is a polynomial $k(X)$ with the property that

$$F_1 + F_2 f + \ldots + F_s f^{s-1} = g(X) k(X).$$

The degree of the left-hand side of this equality is at most $p-s-3-\epsilon$ so the degree of k is at most $r-s-3-\epsilon$. Thus, $\dim \text{Im}(\phi_2) \leq r-s-2-\epsilon$ and therefore

$$\dim \text{Ker}(\phi_2) \geq s(s+1)/2 - (p-3)/2 + s + \epsilon - (r-s-2-\epsilon).$$

Define recursively linear maps ϕ_j for $j = 3, 4, \ldots, s$, from the kernel of ϕ_{j-1} to $\mathbb{F}_p[X]/(X^p - X)$ by

$$\phi_j((F_1, F_2, \ldots, F_s)) = F_{j-1} + F_j f + \ldots + F_s f^{s-j+1}.$$

Let $(F_1, F_2, \ldots, F_s) \in \mathrm{Ker}(\phi_{j-1})$. Then

$$F_{j-2} + F_{j-1} f + \ldots + F_s f^{s-j+2} = 0.$$

Every one of the r zeros of f is a zero of F_{j-2}, which has degree at most $s - j + 2 < r - 1$. Thus $F_{j-2} = 0$. Since $F_{j-1} f + F_j f^2 + \ldots + F_s f^{s-j+2} = 0$ it follows that for all x such that $f(x) \neq 0$ we have $F_{j-1} + F_j f + \ldots + F_s f^{s-j+1} = 0$ and so there is a polynomial $k_j(X)$ with the property that

$$F_{j-1} + F_j f + \ldots + F_s f^{s-j+1} = g(X) k_j(X).$$

The degree of the left-hand side of this equality is at most $p - s - j - 1 - \epsilon$, so the degree of k_j is at most $r - s - j - 1 - \epsilon$. Thus, the dimension of $\mathrm{Im}(\phi_j) \leq r - s - j - \epsilon$. Hence, for $j \leq r - s - 1$, the dimension of the kernel of ϕ_j is at least

$$s(s+1)/2 - (p-3)/2 + s + \epsilon - [(r - s - 2 - \epsilon) + (r - s - 3 - \epsilon) + \ldots + (r - s - j - \epsilon)].$$

Let us suppose that $r \leq 2s + 1$ and consider the above in the case $j = r - s - 1$.

The dimension of the kernel of ϕ_{r-s-1} is at least

$$s(s+1)/2 - (p-3)/2 + s + \epsilon - (r - s - 2 - \epsilon)(r - s - 1 - \epsilon)/2.$$

Now if $(F_1, F_2, \ldots, F_s) \in \mathrm{Ker}(\phi_{r-s-1})$ then $F_1 = \ldots = F_{r-s-2} = 0$ and

$$F_{r-s-1} + F_{r-s-2} f + \ldots + F_s f^{2s-r+1} = g(X) k_{r-s}(X).$$

The degree of the left-hand side of this equality is at most $p - r - 1$, so $k_{r-s} = 0$. Each of the r zeros of f is therefore a zero of F_{r-s-1}, which has degree at most $2s - r + 1 \leq r - 5$. Thus $F_{r-s-1} = 0$. Similarly $F_{r-s-2} = \ldots = F_s = 0$ and so the kernel of ϕ_{r-s-1} is zero. Therefore

$$0 \geq s(s+1)/2 - (p-3)/2 + s + \epsilon - (r - s - 2 - \epsilon)(r - s - 1 - \epsilon)/2.$$

If $r \leq 2s + 3 - t + \epsilon$ then this implies that

$$(p - 1 - 2\epsilon)/t + (t - 3) \geq 2s,$$

which it is not. □

The previous two lemmas have the following consequence. Recall that $\epsilon = 0$ if $I(f)$ is odd and $\epsilon = 1$ if $I(f)$ is even.

THEOREM 2.4. *If $I(f) > (p - 1 - 2\epsilon)/t + t - 2 + \epsilon$ for some integer t then every line meets the graph of f in at least $I(f) + 3 - t > (p-1)/t + 1$ points or at most $t - 1$ points.*

Proof. The line $y = mx + c$ meets the graph of f, $\{(x, f(x)) \mid x \in \mathbb{F}_p\}$, in the point (x, y), whenever $mx + c = f(x)$. Define $f_1(x) = f(x) - mx - c$. Since, for all $0 < i + j < I(f)$ we have $\sum x^i f(x)^j = 0$ it follows that $\sum x^i f_1(x)^j = 0$. Thus $I(f_1) \geq I(f)$. Lemma 2.2 and Lemma 2.3 imply f_1 has at most $t - 1$ zeros or at least $I(f) + 3 - t > (p-1)/t + 1$ zeros. □

Note that if $f(x) = x^t$ and t divides $p+1$ then $I(f) = (p+1)/t + t - 3$ so the bound is the more or less best possible for the short lines, assuming that for some p and t there will be a and b such that $x^t = ax + b$ has t solutions.

The property that the graph of f is incident with at most $t-1$ points or more than $(p-1)/t + 1$ points of a line indicates that the following conjecture may hold.

Conjecture 2.5. *If $I(f) > (p - 1 - 2\epsilon)/t + t - 2 + \epsilon$ for some integer t then the graph of f is contained in an algebraic curve of degree $t - 1$.*

To prove the conjecture it is sufficient to prove that the $\mathrm{Ker}(\psi_{s-t+1})$, where ψ_{s-t+1} is as defined in the proof of Lemma 2.2, is not $\{0\}$. We shall prove the conjecture by other means for $t = 2$ and $t = 3$ in the following section.

We finish this section by proving Conjecture 2.5 under additional hypothesis.

Theorem 2.6. *If $I(f) > (p - 1 - 2\epsilon)/t + t - 2 + \epsilon$ and there are $t - 1$ lines incident with at least t points of the graph of f then the graph of f is contained in the union of these $t - 1$ lines.*

Proof. After a suitable affine transformation we can assume that one of the $t - 1$ lines, incident with at least t points of the graph of f, is the line $Y = 0$ and that the lines $Y = m_i X + c_i$, $i = 1, 2, \ldots, t - 2$, are the other $t - 2$ lines incident with at least t points of the graph of f.

Recall that $I(f) = 2s + 1 + \epsilon$.

Let $V = \{(F_1, F_2, \ldots, F_{t-1}) \mid F_i^\circ \leq s - i\}$. The dimension of V is $(t-1)s-(t-1)(t-2)/2$ which is greater than $(p - 3)/2 - \epsilon - s$, since by assumption $2st > p - 2\epsilon - 1 + t^2 - 3t$.

Define a linear map ψ from V to $\mathbb{F}_p[X]$ by

$$\psi((F_1, F_2, \ldots, F_{t-1})) = F_1 f + F_2 f^2 + \ldots F_{t-1} f^{t-1}.$$

Since $I(f) > 2s$ the product of any two polynomials in the image of ψ cannot have degree $p-1$. The maximum degree of any polynomial in the image of ψ is $p - s - 2 - \epsilon$, so only half of the numbers in the interval $[s + 1 + \epsilon, \ldots, p - s - 2 - \epsilon]$ can occur amongst the degrees of polynomials in the image of ψ. Thus at most $(p - 3)/2 - s - \epsilon$, which is less than the dimension of V. Hence in the image of ψ there is a polynomial of degree at most $s + \epsilon$ or ψ has a non-trivial kernel.

The line $Y = 0$ is incident with at least t points of the graph of f and so by Theorem 2.4 it is incident with at least $I(f) + 3 - t$ points of the graph of f. Therefore f has at least $I(f)+3-t$ distinct zeros and any polynomial in the image of ψ has the zeros of f amongst its zeros and so must have degree at least $I(f) + 3 - t$. Since this number is larger than $s + \epsilon$ we conclude that ψ has a non-trivial kernel.

Let $(F_1, F_2, \ldots, F_{t-1}) \in V$ be such that

$$\sum_{j=1}^{t-1} F_j f^j = 0. \tag{2.1}$$

For all $i = 1, \ldots, t - 2$ we have that the line $Y = m_i X + c_i$ is incident with t, and hence by Theorem 2.4, at least $I(f) + 3 - t$ points of the graph of f. Therefore, there are at least $I(f)+3-t$ solutions to the equation

$$\sum_{j=1}^{t-1} F_j (m_i X + c_i)^j = 0. \tag{2.2}$$

However, this equation has degree at most s and so is an identity. These $t-2$ equations are linear and homogeneous in the F_j and will have a unique solution up to a scalar factor whenever the determinant

$$\begin{vmatrix} m_1X+c_1 & \dots & (m_1X+c_1)^{t-2} \\ . & \dots & . \\ . & \dots & . \\ . & \dots & . \\ m_{t-2}X+c_{t-2} & \dots & (m_{t-2}X+c_{t-2})^{t-2} \end{vmatrix}$$

is non-zero. This is a Vandermonde matrix whose determinant is non-zero since the lines are distinct.

Now we only have to find a solution of these equations and this is easily done. Define polynomials σ_j in X of degree at most j by

$$\prod_{i=1}^{t-2}(Y - m_i X - c_i) = \sum_{j=0}^{t-2} \sigma_{t-2-j} Y^j.$$

For all $i = 1, \dots, t-2$ we have

$$\sum_{j=0}^{t-2} \sigma_{t-2-j}(m_i X + c_i)^j = 0.$$

Putting $F_j = \sigma_{t-1-j} F_{t-1}$ in Equation 2.2 we get

$$F_{t-1} \sum_{j=1}^{t-1} \sigma_{t-1-j}(m_i X + c_i)^j = (m_i X + c_i) F_{t-1} \sum_{j=0}^{t-2} \sigma_{t-2-j}(m_i X + c_i)^j = 0.$$

Substituting the solution into Equation 2.1 we have

$$F_{t-1} \sum_{j=1}^{t-1} \sigma_{t-1-j} f^j = 0.$$

For every $x \in \mathbb{F}_p$ that is not a zero of F_{t-1}

$$\sum_{j=1}^{t-1} \sigma_{t-1-j} f^j = 0.$$

This equation has degree at most $p - 2s + t$ and has at least $p - (s - t + 1)$ solutions and so is an identity. Thus for all $x \in \mathbb{F}_p$

$$0 = f \sum_{j=0}^{t-2} \sigma_{t-2-j} f^j = f \prod_{i=1}^{t-2} (f - m_i x - c_i).$$

□

3. Classification of functions for which $I(f)$ is more than $(p+5)/3$ and consequences for functions determining few directions

Firstly we note that we have proved what Lovász and Schrijver proved in [7], with no restriction on $M(f)$.

Theorem 3.1. *If $I(f) \geq (p+1)/2$ then f is linear.*

Proof. This is an immediate corollary of Theorem 2.4 with $t = 2$. □

Since $I(f) \geq M(f) + 1$ it follows that if $M(f) \geq (p-1)/2$ then f is linear.

The theorem itself holds for all finite fields $\mathbb{F}_q$, that is if $I(f) \geq (q+1)/2$ then f is linear, although there do not seem to be any geometric applications in the case q is not a prime.

Now we shall prove a generalised version of the main theorem in [5], where the hypothesis on f was $M(f) \geq (p+2)/3$. This was weakened slightly in [2], where the hypothesis on f was $I(f) \geq (p+5)/3$ and $M(f) \geq (p-1)/6$. In both cases the conclusion was that the graph of f is contained in the union of two lines. Here we have no hypothesis on $M(f)$ which allows the possibility that $f(X)$ is of degree 2, so our conclusion is slightly weaker.

Theorem 3.2. *If $I(f) \geq (p+5)/3$ then the graph of f is contained in an algebraic curve of degree 2.*

Proof. Since $(p+5)/3 > (p-1-2\epsilon)/3 + 1 + \epsilon = (p+2+\epsilon)/3$, Theorem 2.4 implies that there is a line incident with at least $(p+5)/3$ points of the graph of f or every line is incident with at most 2 points of the graph of f. In the latter case, Segre's theorem [8] implies that the graph of f is contained in an algebraic curve of degree 2.

Thus we can assume that there is a line meeting the graph of f in at least $(p+5)/3$ points and after making a suitable affine transformation we can assume that this is the line $Y = 0$. In other words f has at least $(p+5)/3$ distinct zeros.

Recall that $I(f) = 2s + 1 + \epsilon$, so $2s \geq (p-1)/3 - \epsilon$. By Theorem 3.1 we can assume that $s < p/4$.

Let V be a vector space of pairs of polynomials of dimension $2s-1$ defined by $V = \{(A, B) \mid A^\circ \leq s-1, B^\circ \leq s-2\}$. Define a linear map ϕ from V to $\mathbb{F}_p[X]$ by

$$\phi((A, B)) = Af + Bf^2.$$

The maximum degree of any polynomial in the image of ϕ is $p-s-2$. Arguing as in the previous lemmas, only half of the degrees in the range $[s+1+\epsilon, \dots, p-s-2-\epsilon]$ can occur amongst the polynomials in the image of ϕ. Since $(p-2s-3)/2 - \epsilon \leq (4s+\epsilon-5)/2 \leq 2s-2 \leq dim(V)$, the image of ϕ contains a polynomial of degree at most s or ϕ has a non-trivial kernel.

Any polynomial g in the image of ϕ has at least $(p+5)/3$ zeros, since any zero of f is a zero of g. However, $(p+5)/3 > s$, so we can conclude that ϕ has a non-trivial kernel.

Let A and B be such that

$$Af + Bf^2 = 0.$$

Note that the degree restriction on A and B implies that $Af \bmod X^p - X$ is equal to A times ($f \bmod X^p - X$) and $Bf^2 \bmod X^p - X$ is equal to B times ($f^2 \bmod X^p - X$).

By removing any common factors, if necessary, we can assume $(A, B) = 1$. This equation has degree at most $p-s-2$ and it holds for all $x \in \mathbb{F}_p$, so it is an identity. Thus A divides f^2 and B divides f. Moreover A and B have no common factors so f/B has the same zeros as f^2, and since f^2 has the same zeros as f, f/B has the same zeros as f. Since B divides f, the zeros of B are zeros of f and so the zeros of B are zeros of f/B.

Multiplying by $Bf - A$ and rearranging we see that

$$B^2 f^3 = A^2 f$$

for all $x \in \mathbb{F}_p$, and so

$$Bf^3 = A^2(f/B),$$

for all $x \in \mathbb{F}_p$, such that $B(x) \neq 0$. If x is a zero of B then the left-hand side of this equation is zero and the right hand side is also zero since any zero of B is a zero of f/B. This equation holds for all $x \in \mathbb{F}_p$, it has degree less than p, and so is an identity, in the sense that B times ($f^3 \bmod X^p - X$) is equal to $A^2(f/B)$.

Thus A^2 divides f^3 and B^2 divides f. Again, since A and B have no common factors f/B^2 has the same zeros as f^3, and since f^3 has the same zeros as f, f/B^2 has the same zeros as f. Therefore the zeros of B are zeros of f/B^2.

Repeating the above argument we conclude that

$$Bf^{i+1} = A^i(f/B^{i-1}),$$

for all $i = 1, 2, \ldots$, so long as the degree of this equation is less than p, in other words whenever $B^\circ + (f^{i+1})^\circ \leq p-1$, which is certainly whenever $i \leq s+2$. Thus B^{s+2} divides f, so the degree of B is at most 3. Now we can conclude that $B^\circ + (f^{i+1})^\circ \leq p-1$ whenever $i \leq 2s-3$. Thus B^{2s-3} divides f. The polynomial f/B^{2s-3} has at least $(p+5)/3$ zeros, so $B^\circ \leq 1$ and the equation is an identity for $i = 2s-1$. Now we can conclude that B^{2s-1} divides f and the polynomial f/B^{2s-1} has at least $(p+5)/3$ zeros, which implies $f^\circ - (2s-1)B^\circ \geq (p+5)/3$ which in turn implies $(2s-1)B^\circ \leq 2s-2$ and so B is constant. Now A^{2s-1} divides f^{2s} and the quotient has at least $(p+5)/3$ zeros. Hence $p - 2 - (2s-1)A^\circ \geq (p+5)/3$, which gives $A^\circ \leq 1$.

The graph of f is contained in the algebraic curve

$$A(X)Y + B(X)Y^2 = 0,$$

which is of degree two. □

Corollary 3.3. *If $M(f) \geq (p+2)/3$ then the graph of f is contained in the union of two lines.*

Proof. Since $I(f) \geq M(f)+1$, by Theorem 3.2 the graph of f is contained in an algebraic curve of degree two. If this curve is irreducible then f determines every direction, since

$$((y^2 + ay + b) - (x^2 + ax + b))/(y - x) = x + y + a.$$

If not then after a suitable affine transformation there exsits a linear polynomial $A(X) = aX + b$ and a constant polynomial $B(X) = c$ such that

$$f(x)(ax + b + cf(x)) = 0,$$

for all $x \in \mathbb{F}_p$. □

We can also prove Corollary 3.3 as a corollary to Theorem 2.4.

Proof. Since $I(f) \geq M(f)+1$ Theorem 2.4 implies that every line meets the graph of f in at least $(p+5)/3$ points or at most 2 points. If a point of the graph of f is incident only with lines incident with at most 2 points of the graph of f, then $M(f) \leq 1$. Therefore, every point of the graph of f is incident with a line which is incident with at least $(p+5)/3$ points of the graph of f. The graph of f is a set of p points and so is contained in the union of two such lines. □

In the article [9] T. Szőnyi proves that if $M(f) \geq 2$ and the graph of f is contained in the union of two lines then f is affinely equivalent to a generalized example of Megyesi, which is constructed using cosets of the multiplicative group. For more details of this construction see [9] or [5].

In the article [3] A. Biró proves that if the graph of f is contained in the union of two lines then $I(f) = p-1$ or $I(f) = (p-1)/3$ or $I(f) \leq (p-1)/4$ and classifies all examples when $I(f) = p-1$, $I(f) = (p-1)/3$ or $I(f) = (p-1)/4$.

If the graph of f is contained in an irreducible curve of degree 2 then f is of degree two and $I(f) = (p-1)/2$.

We are unaware of any results concerning $I(f)$ (or $M(f)$) obtained under the assumption that the graph of f is contained in an algebraic curve of degree three.

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Simeon Ball
Departament de Matemàtica Aplicada IV,

Universitat Politècnica de Catalunya, Jordi Girona 1-3, Mòdul C3, Campus Nord,
08034 Barcelona, Spain
simeon@mat.upc.es

András Gács
Eötvös University Budapest,
Pázmány P. sétány 1/c,
Budapest, Hungary H-1117
gacs@cs.elte.hu