Complete arcs on the parabolic quadric $\mathrm{Q}(4, q)$

Jan De Beule [a,1]

[a] *Department of Pure Mathematics and Computer Algebra, Ghent University, Krijgslaan 281, S22, B-9000 Gent, Belgium*

jdebeule@cage.ugent.be
[url]http://www.jandebeule.be;http://cage.ugent.be/∼jdebeule

András Gács [b,2]

[b] *Department of Computer Science, Eötvös Loránd University, H-1117 Budapest, Pázmány Péter sétány 1/C, Hungary*

gacs@cs.elte.hu [url]http://cs.elte.hu/∼gacs

Abstract

Using the representation $T_2(\mathcal{O})$ of $\mathrm{Q}(4, q)$ and algebraic methods, we prove that complete $(q^2 - 1)$-arcs of $\mathrm{Q}(4, q)$ do not exist when $q = p^h$, p odd prime and $h > 1$. As a by-product we prove an embeddability theorem for the direction problem in $\mathrm{AG}(3, q)$.

Key words: maximal arc, generalized quadrangle, parabolic quadric
1991 MSC: 05B25, 51E12, 51E20, 51E21

1 Introduction

A (finite) *generalized quadrangle* (GQ) is an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint non-empty sets of objects called points and lines (respectively), and for which $\mathrm{I} \subseteq (\mathcal{P} \times \mathcal{B}) \cup (\mathcal{B} \times \mathcal{P})$ is a symmetric point-line incidence relation satisfying the following axioms:

[1] The first author thanks the Fund for Scientific Research - Flanders (Belgium) for a research grant.

[2] The second author was supported by OTKA grants T 043758, T 049662 and F 043772; and TÉT grant E-16/04. The preparation of the final version was supported by a Bolyai grant

(i) each point is incident with $1+t$ lines ($t \geqslant 1$) and two distinct points are incident with at most one line;

(ii) each line is incident with $1+s$ points ($s \geqslant 1$) and two distinct lines are incident with at most one point;

(iii) if x is a point and L is a line not incident with x, then there is a unique pair $(y, M) \in \mathcal{P} \times \mathcal{B}$ for which $x \mathrm{I} M \mathrm{I} y \mathrm{I} L$.

The integers s and t are the parameters of the GQ and $\mathcal{S}$ is said to have order (s,t). If $s = t$, then $\mathcal{S}$ is said to have order s. If $\mathcal{S}$ has order (s,t), then $|\mathcal{P}| = (s+1)(st+1)$ and $|\mathcal{B}| = (t+1)(st+1)$ (see e.g. [11]). The *dual* $\mathcal{S}^D$ of a GQ $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is the incidence structure $(\mathcal{B}, \mathcal{P}, \mathrm{I})$. It is again a GQ.

A GQ $\mathcal{S}' = (\mathcal{P}', B', \mathrm{I}')$ of order (s', t') is called a *subquadrangle* of the GQ $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ of order (s,t) if $\mathcal{P}' \subset \mathcal{P}$, $\mathcal{B}' \subset \mathcal{B}$ and I' is the restriction of I to $(\mathcal{P}' \times \mathcal{B}') \cup (\mathcal{B}' \times \mathcal{P}')$.

An *ovoid* of a GQ $\mathcal{S}$ is a set $\mathcal{O}$ of points of $\mathcal{S}$ such that every line is incident with exactly one point of the ovoid. An ovoid of a GQ of order (s,t) has necessarily size $1+st$. An *arc* or a *partial ovoid* of a GQ is a set $\mathcal{K}$ of points such that every line contains *at most* one point of $\mathcal{K}$. An arc $\mathcal{K}$ is called *complete* if and only if $\mathcal{K} \cup \{p\}$ is not an arc for any point $p \in \mathcal{P} \setminus \mathcal{K}$, in other words, if $\mathcal{K}$ cannot be extended. It is clear that any arc of a GQ of order (s,t) contains $1 + st - \rho$ points, $\rho \geq 0$, with $\rho = 0$ if and only if $\mathcal{K}$ is an ovoid.

In this paper we consider a classical finite generalized quadrangle of order q, which consists of the points and lines, together with the natural incidence, of the non-singular parabolic quadric $\mathrm{Q}(4,q)$ in $\mathrm{PG}(4,q)$. It is well-known, (see e.g. [11]) that this GQ has ovoids. A particular example of an ovoid is any elliptic quadric $\mathrm{Q}^-(3,q)$ contained in it. When q is prime, these are the only ovoids [2], when q is a prime power, other examples are known, see e.g. [15] for a list of references.

Concerning complete arcs of a GQ of order (s,t), we state the following results from [11].

Theorem 1 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (s,t). Any $(st - \rho)$-arc of $\mathcal{S}$ with $0 \leq \rho < t/s$ is contained in a uniquely defined ovoid of $\mathcal{S}$.*

Theorem 2 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (s,t). Let $\mathcal{K}$ be a maximal partial ovoid of size $st - t/s$ of $\mathcal{S}$. Let $\mathcal{B}'$ be the set of lines incident with no point of $\mathcal{K}$, and let $\mathcal{P}'$ be the set of points on at least one line of $\mathcal{B}'$ and let I' be the restriction of I to points of $\mathcal{P}'$ and lines of $\mathcal{B}'$. Then $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', \mathrm{I}')$ is a subquadrangle of order $(s, \rho = t/s)$.*

Applying these theorems to the *GQ* $\mathrm{Q}(4,q)$ implies only that an arc of size q^2 cannot be complete. When $\mathcal{K}$ is a complete (q^2-1)-arc of $\mathrm{Q}(4,q)$, then the second theorem implies that the subquadrangle $\mathcal{S}'$ is a hyperbolic quadric $\mathrm{Q}^+(3,q)$ contained in $\mathrm{Q}(4,q)$.

When q is even, it is known that no complete arcs of size $q^2-\rho$, $0<\rho<q$ exist [5]. No analogous theorem is known for q odd. It is even not known whether or not $Q(4,q)$, q odd, has complete (q^2-1)-arcs. It is probably a bit unexpected but this problem seems to be quite hard. Using a combination of geometrical and algebraic methods, the problem can be solved when $q=p^h$, p odd prime, $h>1$. The main result is as follows.

Theorem 3 *Suppose that $q=p^h$, p an odd prime, $h>1$. Then $\mathrm{Q}(4,q)$ has no complete (q^2-1)-arcs.*

The restriction $h>1$ seems to be unexpected, but complete (q^2-1)-arcs of $\mathrm{Q}(4,q)$ are known when $q=3,5,7,11$. These examples were first found by T. Penttila. References and more information on particular examples will be given in the last section.

In the second section an alternative representation of $\mathrm{Q}(4,q)$ is given and a complete (q^2-1)-arc of $\mathrm{Q}(4,q)$ will be described in this representation. In the third section algebraic techniques are applied to prove the main result.

2 Tits' representation of $\mathrm{Q}(4,q)$

An *oval* of $\mathrm{PG}(2,q)$ is a set of $q+1$ points $\mathcal{C}$, such that no three points of $\mathcal{C}$ are collinear. When q is odd, it is known that all ovals of $\mathrm{PG}(2,q)$ are conics. When q is even, several other examples and infinite families are known, see e.g. [9]. The GQ $T_2(\mathcal{C})$ is defined as follows. Let $\mathcal{C}$ be an oval of $\mathrm{PG}(2,q)$, embed $\mathrm{PG}(2,q)$ as a plane in $\mathrm{PG}(3,q)$ and denote this plane by π_∞. Points are defined as follows:

(i) the points of $\mathrm{PG}(3,q)\setminus\mathrm{PG}(2,q)$;
(ii) the planes π of $\mathrm{PG}(3,q)$ for which $|\pi\cap\mathcal{C}|=1$;
(iii) one new symbol (∞).

Lines are defined as follows:

(a) the lines of $\mathrm{PG}(3,q)$ which are not contained in $\mathrm{PG}(2,q)$ and meet $\mathcal{C}$ (necessarily in a unique point);
(b) the points of $\mathcal{O}$.

Incidence between points of type (i) and (ii) and lines of type (a) and (b) is the inherited incidence of $\mathrm{PG}(3,q)$. In addition, the point (∞) is incident with no line of type (a) and with all lines of type (b). It is straightforward to show that this incidence structure is a GQ of order q. The following theorem (see e.g. [11]) allows us to use this representation.

Theorem 4 *The GQs $T_2(\mathcal{C})$ and $\mathrm{Q}(4,q)$ are isomorphic if and only if $\mathcal{C}$ is a conic of the plane $\mathrm{PG}(2,q)$.*

Since all ovals of $\mathrm{PG}(2,q)$, q odd, are conics, $T_2(\mathcal{C}) \cong \mathrm{Q}(4,q)$ when q is odd. From now on we suppose that q is odd.

Let $\mathcal{K}$ be a complete k-arc of $T_2(\mathcal{C})$. Since $\mathrm{Q}(4,q) \cong T_2(\mathcal{C})$ has a collineation group acting transitively on the points (see e.g. [10]), we can suppose $(\infty) \in \mathcal{K}$. This implies that $\mathcal{K}$ contains no points of type (ii). It is clear that no two points of type (i) of $\mathcal{K}$ determine a line meeting $\mathcal{C}$ in a point. Hence the existence of $\mathcal{K}$ implies the existence of a set U of $k-1$ points of type (i) such that no two points determine a line meeting π_∞ in $\mathcal{C}$. It is easy to see that the converse is also true: from a set U of $k-1$ points in $AG(3,q)$ with the property that all lines joining at least two points of U are disjoint from $\mathcal{C}$, we can find an arc $\mathcal{K}$ of $T_2(\mathcal{C})$ of size k by adding ∞ to U. The completeness of $\mathcal{K}$ is equivalent to the maximality of U.

In the next section it is proved that such a set U cannot be maximal when $|U| = q^2-2$, $q = p^h$, p prime, $h > 1$.

3 Directions in $\mathrm{AG}(3,q)$

This chapter is devoted to the direction problem in $\mathrm{AG}(3,q)$, where $q = p^h$ is a prime power. Embed $\mathrm{AG}(3,q)$ in $\mathrm{PG}(3,q)$ in such a way that the infinite plane is $X_3 = 0$. Consider a set $U = \{(a_i, b_i, c_i, 1) : i = 1, \ldots, k\} \subset \mathrm{AG}(3,q)$. Let $D = \{(a_i - a_j, b_i - b_j, c_i - c_j, 0) : i \neq j\}$ denote the *set of directions determined by U*. The set D is a subset of the infinite plane $X_3 = 0$ and consists of those points through which there is an affine line with at least two points of U. It is easy to see that if $|U| \geq q^2+1$, then it determines all directions: for any direction there are q^2 lines in the parallel class, so by the pigeon-hole principle, there is a line with at least two points. In the previous section we saw that if $|U| = q^2$ and D is disjoint from a conic $\mathcal{C}$, then by adding to U the point ∞ we find an ovoid of $T_2(\mathcal{C})$ and vice versa.

What we are looking for is a result assuring that, if $|U| = q^2-2$ and D is disjoint from a conic, then U can be extended to a set of q^2 points still not determining points of the conic. We will prove a bit more, since we will only use the fact that D misses at least $p+2$ points (this is true when $q = p^h$, $h > 1$).

The theory of directions in the affine plane has proved to be very useful in many aspects. Using and developing techniques originally due to Rédei [12], in the last years the classification of all sets of size q (which is the analogue of the size q^2 case in 3-dimensions) and determining at most half of the directions was achieved by Ball, Blokhuis, Brouwer, Storme and Szőnyi, see [1], [4].

It was perhaps surprising, when Ball and Lavrauw [3] proved theorems for the 3-dimensional case with the much weaker condition that D misses at least q from the q^2+q+1 directions. They considered the $|U| = q^2$ case, while in our result we have $|U| = q^2 - 2$. Also Ball and Lavrauw were the first to use the $T_2(\mathcal{C})$ representation of $\mathrm{Q}(4,q)$ and use results about directions. Our approach is similar to theirs, but we will use more geometrical arguments (mainly due to Sziklai [13]) to help the use of the Rédei polynomial. Also some ideas of a paper by Szőnyi [14] will be used, more details will be discussed at the end of the section.

Denote by O the complement of D in the infinite plane. From now on by "plane" we will always mean a plane different from $X_3 = 0$.

Lemma 1 *Suppose the plane π has at least one point in common with O. Then $|\pi \cap U| \leq q$.*

If $|U| = q^2$, then the points of O can be characterized by the property that $|U \cap \pi| = q$ for all planes π containing the point in question.

Proof. Let $p \in O \cap \pi$. The q affine lines in π through p cover the affine points of π and on each there is at most one point of U, hence $|\pi \cap U| \leq q$.

For the second statement, suppose that π is a plane through the infinite point $p \in O$. Note that there are q affine planes through the infinite line of π partitioning the points of the affine part, hence they all meet U in exactly q points. Finally suppose $p \notin O$, that is, p is a determined direction, but all planes through p have q points in common with U. Let l be an affine line with direction corresponding to p with $r \geq 2$ points of U. Counting the number of points of U on planes through l, we have $q^2 = |U| = r + (q+1)(q-r)$, a contradiction. $\square$

Lemma 2 *For any $x, y, z, w \in GF(q)$, $(y, z, w) \neq (0, 0, 0)$, the multiplicity of $-x$ in the multi-set $\{ya_i + zb_i + wc_i : i = 1, \ldots, k\}$ is the same as the number of common points of U and the plane $yX_0 + zX_1 + wX_2 + xX_3 = 0$.*

Proof. The point $(a_i, b_i, c_i, 1)$ is on the plane if and only if $ya_i + zb_i + wc_i + x = 0$. $\square$

Define the Rédei polynomial of U as follows

$$R(X,Y,Z,W) = \prod_{i=1}^{k}(X + a_iY + b_iZ + c_iW) = X^k + \sum_{i=1}^{k} \sigma_i(Y,Z,W)X^{k-i}$$

Here $\sigma_i(Y,Z,W)$ is the i-th elementary symmetric polynomial of the multi-set $\{a_iY + b_iZ + c_iZ : i\}$ and is either zero or has degree i. The use of R is that it translates intersection properties of U with planes to algebraic conditions.

Lemma 3 *Consider three fixed field elements y, z and w. Substitute $Y = y$, $Z = z$ and $W = w$ in $R(X,Y,Z,W)$ and consider $R(X,y,z,w)$ as a polynomial in X. Then the multiplicity of the root x equals the number of common points of U and the plane with equation $yX_0 + zX_1 + wX_2 + xX_3 = 0$.*

Proof. The proof follows immediately by Lemma 2. □

To demonstrate the use of R we prove the following theorem.

Theorem 5 *If $|U| = q^2 - 1$, then it can be extended to a set of q^2 points determining the same set of directions.*

Proof. We have $|U| = q^2 - 1$, hence for any infinite line l with equation $yX_0 + zX_1 + wX_2 = X_3 = 0$, if l meets O, then $q - 1$ affine planes through l meet U in q points and one in $q - 1$ points (see Lemma 1). For the Rédei polynomial this means that $R(X,y,z,w)$ has $q-1$ q-fold roots and one $(q-1)$-fold root. Hence there is a constant S depending on y, z, w for which

$$(X - S)R(X,y,z,w) = (X^q - X)^q$$

holds. It is easy to see that

$$S = \sigma_1(y,z,w) = (\sum_i a_i)y + (\sum_i b_i)z + (\sum_i c_i)w.$$

This implies, that if we add to U the point $(-(\sum_i a_i), -(\sum_i b_i), -(\sum_i c_i), 1)$, then for the Rédei polynomial of the new set, $R^*(X,y,z,w) = (X^q - X)^q$ holds for any line $yX_0 + zX_1 + wX_2 = X_3 = 0$ meeting O. By Lemma 1 and 2, this means that the new set determines the same directions. □

Note that by the end of the previous section, this result implies that a partial ovoid of size q^2 in $T_2(\mathcal{C})$ can always be extended to an ovoid.

From now on suppose that q is odd and $|U| = q^2 - 2$. After translation (not affecting O or D) we can suppose $\sum_i a_i = \sum_i b_i = \sum_i c_i = 0$. Note that this implies $\sigma_1 \equiv 0$.

Lemma 4 *All planes meeting O have q, $q-1$ or $q-2$ points in common with U. If l is a line of the infinite plane with at least one point in common with O, then either we have $q-1$ planes thorugh l with q points of U and one with $q-2$ points, or $q-2$ planes with q points of U and two with $q-1$ points.*

Proof. This is an easy consequence of Lemma 1. $\square$

Lemma 5 *If the infinite line with equation $yX_0 + zX_1 + wX_2 = X_3 = 0$ has at least one common point with O, then*

$$R(X, y, z, w)(X^2 - \sigma_2(y, z, w)) = (X^q - X)^q. \tag{1}$$

Proof. From the previous lemma and Lemma 3 we know that

$$R(X, y, z, w)(X - S)(X - S') = (X^q - X)^q,$$

where S and S' are not necessarily different and depend on y, z, w. Considering the first three terms on both sides and taking into account that $\sigma_1 = 0$, we have $(X - S)(X - S') = X^2 - \sigma_2(y, z, w)$. $\square$

Note that this implies that for these y, z, w, we can explicitly determine R in terms of σ_2:

$$R(X, y, z, w) = X^q(X^{q^2-q-2} + \sigma_2 X^{q^2-q-4} + \sigma_2^2 X^{q^2-q-6} + \cdots),$$

hence $\sigma_{2l+1}(y, z, w) = 0$, $\sigma_{2l}(y, z, w) = \sigma_2(y, z, w)^l$ for $l = 0, 1, ..., \frac{q^2-q-2}{2}$.

Let $S_k(Y, Z, W) = \sum_i (a_i Y + b_i Z + c_i W)^k$, the k-th power sum of the multi-set $\{a_i Y + b_i Z + c_i W : i\}$.

Lemma 6 *If the line with equation $yX_0 + zX_1 + wX_2 = X_3 = 0$ has at least one common point with O, then $S_k(y, z, w) = 0$ for odd k and $S_k(y, z, w) = -2\sigma_2^{k/2}(y, z, w)$ for even k.*

Proof. We prove the statement by induction on k. For $k = 1$ we have $S_1 = \sigma_1 = 0$. For the induction step suppose we have proved the statement for $1, ..., k-1$ and recall that by the Newton formulas, we have

$$k\sigma_k = S_1\sigma_{k-1} - S_2\sigma_{k-2} + - \ldots - (-1)^k S_k \sigma_0,$$

where σ_0 is defined to be 1. If k is odd, then except for S_k, we know that all summands are zero, so $S_k = 0$. If k is even, then replacing the terms $S_i\sigma_{k-i}$, i odd with zero and replacing the terms $S_i\sigma_{k-i}$, i even with $-2\sigma_2^{k/2}$, finally, using that the left hand side is $k\sigma_2^{k/2}$, we are done. $\square$

Theorem 6 *If $|U| = q^2 - 2$, $q = p^h$ and $|O| \geq p + 2$, then U can be extended* *by two points to a set of q^2 points determining the same directions.*

Proof. Consider the previously found equation with $k = p + 1$. We have

$$S_{p+1}(y, z, w) + 2\sigma_2(y, z, w)^{\frac{p+1}{2}} = 0$$

for all lines $yX_o + zX_1 + wX_2 = X_3 = 0$ meeting O. This is a dual curve of degree at most $p+1$ vanishing on at least $(p+2)$ pencils, hence it is identically zero. This means that the equation $S_{p+1} = -2\sigma_2^{\frac{p+1}{2}}$ is an identity of polynomials. On the other hand, it is easy to see that S_{p+1} has only terms of the form $Y^{p+1}, Z^{p+1}, W^{p+1}$ and $Y^pW, YW^p, Z^pW, ZW^p, Y^pZ, YZ^p$. By Lemma 7 (after this proof), this implies that σ_2 is reducible. By Equation (1) (see Lemma 5) it has to be of the form $(AY + BZ + CW)^2$.

Add to U the points $(A, B, C, 1)$ and $(-A, -B, -C, 1)$. It is easy to see that for the Rédei polynomial of the new set, we have $R^*(y, z, u) = (X^q - X)^q$ whenever the (infinite) line with equation $yX_0 + zX_1 + uX_2 = X_3 = 0$ has at least one point in common with O. By Lemma 1, we are done. □

Lemma 7 *Let I consist of all linear combinations of the following monomials: Y^{p+1}, Z^{p+1}, W^{p+1}, Y^pW, YW^p, Z^pW,ZW^p,Y^pZ, YZ^p. Suppose σ is a homogeneous polynomial of degree 2 for which $\sigma^{\frac{p+1}{2}}$ is in I. Then σ is not irreducible.*

Proof. Suppose σ is irreducible. Then by replacing Y, Z and W by Y', Z' and W' which are linear combinations of the original variables, we can achieve $\sigma = Z'^2 - W'Y'$. Note that such a transformation permutes elements of I. But $(Z'^2 - W'Y')^{\frac{p+1}{2}}$ is obviously not in I, a contradiction. □

We end this section with some remarks about the connection of our technique to those used in the above mentionned papers [14] and [3].

In [14] Szőnyi proves that if a set of $q - c\sqrt{q}$ points in $\mathrm{AG}(2, q)$ determines at most $\frac{q-1}{2}$ directions, then it can be extended to a set of q points determining the same directions. The proof is analogous to ours, but there it can be shown that the curve defined by the missing factors of R cannot have too many points, unless it has a linear component implying that there is a point that can be added to the set without changing the set of determined directions. In our situation instead of a curve we have the surface $X^2 - \sigma_2(Y, Z, W) = 0$. This might be the equation of a hyperbolic quadric having as many points as we can deduce from the conditions, so we cannot obtain a contradiction. This is why we needed some extra ideas and this is why it seems difficult to use Szőnyi's method to extend the result for sets of size $q^2 - c\sqrt{q}$. Applying our method to the planar case we can prove the following.

Theorem 7 *Suppose that U is a set of $q-2$ points in* $\mathrm{AG}(2,q)$*, q odd, missing at least $p+2$ directions. Then U can be extended to a set of q points determining the same directions.*

Finally, some words about the connection to the paper [3]. There the authors use the very same Rédei polynomial, but recognize lines with a clever evaluation in R, while we only recognize planes. The advantage in this is that the Rédei polynomial (after plugging in y, z, w) becomes a bit nicer. Our method also applies to every problem attacked in [3], but of course only together with the extra ideas the authors develop there.

4 Related results

As mentioned in the introduction, the only examples of complete (q^2-1)-arcs of $\mathrm{Q}(4,q)$ are known for $q = 3, 5, 7$ and 11. It is not clear if examples exist for all odd primes. The examples for $q = 5, 7$ and 11 were found using a computer and were recently described in detail in [7]. The example for $q = 3$ can easily be constructed from an elliptic quadric contained in $\mathrm{Q}(4,q)$.

We mention that the non-existence of complete (q^2-1)-arcs of $\mathrm{Q}(4,q)$, $q = 9$ was shown using an exhaustive computer search [6].

For general q and complete arcs of different size, the following theorem is part of a slightly more general theorem in [8].

Theorem 8 *Suppose that $\mathcal{K}$ is a complete $(q^2+1-\delta)$-arc of* $\mathrm{Q}(4,q)$*, $\delta \leq \sqrt{q}$. Then δ is even.*

Putting this together with Theorem 3, a maximal partial ovoid which is not an ovoid of $\mathrm{Q}(4,q)$, $q = p^h$, p odd, $h > 1$, has size at most $q^2 - 3$.

Acknowledgement

The first author thanks the department of Computer Science at Eötvös Loránd University in Budapest, and especially András Gács, Péter Sziklai, Tamás Szőnyi and Zsuzsa Weiner for their hospitality.

References

[1] S. Ball. The number of directions determined by a function over a finite field. *J. Combin. Theory Ser. A*, 104(2):341–350, 2003.

[2] S. Ball, P. Govaerts, and L. Storme. On ovoids of parabolic quadrics. *Des. Codes Cryptogr.*, 38(1):131–145, 2006.

[3] S. Ball and M. Lavrauw. On the graph of a function in two variables over a finite field. *J. Algebraic Combin.*, 23(3):243–253, 2006.

[4] A. Blokhuis, S. Ball, A. E. Brouwer, L. Storme, and T. Szőnyi. On the number of slopes of the graph of a function defined on a finite field. *J. Combin. Theory Ser. A*, 86(1):187–196, 1999.

[5] M. R. Brown, J. De Beule, and L. Storme. Maximal partial spreads of $T_2(\mathcal{O})$ and $T_3(\mathcal{O})$. *European J. Combin.*, 24(1):73–84, 2003.

[6] M. Cimráková and V. Fack. Searching for maximal partial ovoids and spread in generalized quadrangles. *Bull. Belg. Math. Soc. Simon Stevin*, 12(5):697–706, 2005.

[7] S. De Winter and K. Thas. Partial ovoids and spreads in generalized quadrangles, and related combinatorial structures. *Discrete Math.*. submitted.

[8] P. Govaerts, L. Storme, and H. Van Maldeghem. On a particular class of minihypers and its applications. III. Applications. *European J. Combin.*, 23(6):659–672, 2002.

[9] J. W. P. Hirschfeld. *Projective geometries over finite fields*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1998.

[10] J. W. P. Hirschfeld and J. A. Thas. *General Galois geometries*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1991. Oxford Science Publications.

[11] S. E. Payne and J. A. Thas. *Finite generalized quadrangles*, volume 110 of *Research Notes in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1984.

[12] L. Rédei. *Lacunary polynomials over finite fields*. North-Holland Publishing Co., Amsterdam, 1973. Translated from the German by I. Földes.

[13] P. Sziklai. Directions in $\mathrm{AG}(3, q)$ and applications. *Note di Matematica, Lecce.* to appear.

[14] T. Szőnyi. On the number of directions determined by a set of points in an affine Galois plane. *J. Combin. Theory Ser. A*, 74(1):141–146, 1996.

[15] J. A. Thas. Ovoids, spreads and m-systems of finite classical polar spaces. In *Surveys in combinatorics, 2001 (Sussex)*, volume 288 of *London Math. Soc. Lecture Note Ser.*, pages 241–267. Cambridge Univ. Press, Cambridge, 2001.