ON GEOMETRIC CONSTRUCTIONS OF (k, g)-GRAPHS

ANDRÁS GÁCS AND TAMÁS HÉGER

ABSTRACT. We give new constructions for k-regular graphs of girth $6, 8$ and 12 with a small number of vertices. The key idea is to start with a generalized n-gon and delete some lines and points to decrease the valency of the incidence graph.

1. INTRODUCTION

In 1960 Ferenc Kárteszi [10] proved the following theorem.

THEOREM 1.1. *A regular graph with valency k and girth 6 has at least $2((k-1)^2 + (k-1) + 1)$ vertices with equality if and only if it is the incidence graph of a projective plane of order $k - 1$.*

Kárteszi also started the systematic study of the more general problem of determining the least number $c(k, g)$ of vertices a regular graph of valency k and girth g can have. Graphs with $c(k, g)$ points are called (k, g)-*cages*. This problem is still open for most of the cases, for a survey, we refer to Wong [15] or to the website of Royle [13].

By counting the number of vertices of distance $1, 2, ...$ from a vertex or an edge, the following lower bound for $c(k, g)$ is easily proved.

PROPOSITION 1.2. *(Moore bound)*

$$c(k,g) \geq \begin{cases} 1 + k + k(k-1) + \cdots + k(k-1)^{\frac{g-1}{2}-1} & \text{for } g \text{ odd;} \\ 2\left(1 + (k-1) + (k-1)^2 + \cdots + (k-1)^{\frac{g}{2}-1}\right) & \text{for } g \text{ even.} \end{cases}$$

We shall refer to this as the Moore bound, though originally this name came from an upper bound for the number of vertices a regular graph can have with given valency and diameter.

A graph is called a *(k,g)-graph*, if it is regular of valency k and girth g. (k, g)-graphs satisfying equality in the Moore bound will be called *Moore graphs* (some authors only use this term for graphs with g odd). There is a large literature about Moore graphs, for instance the Hoffmann-Singleton theorem deals with the case $g = 5$.

Note that for $g = 4$ a (unique) Moore graph exists for every k: the complete bipartite graph on $2k$ vertices has valency k and girth 4. When $g = 2n \geq 6$, one can characterize Moore graphs as the incidence graph of certain generalized n-gons. The *incidence graph* of a set system in general is a bipartite graph, where the two vertex classes correspond to points and sets respectively, and edges correspond to incident point-set pairs.

The first author was supported by Bolyai grant and OTKA grant F 043772. Both authors were supported by OTKA grants T 043758 and T 049662, and TÉT grant E16/04.

Definition 1.3. *Let $\mathcal{P}$ be a finite set and $\mathcal{L}$ a set of subsets of $\mathcal{P}$ called points and lines, respectively. The pair $(\mathcal{P}, \mathcal{L})$ is called a generalized n-gon of order (s, t), if it satisfies the following axioms.*

- *there are $s+1$ lines through every point;*
- *every line contains $t+1$ points;*
- *the diameter and the girth of the incidence graph is n and $2n$, respectively.*

It is straightforward to check that a regular graph G with girth $2n$ and valency k is a Moore graph if and only if it is the incidence graph of a generalized n-gon of order $(k-1, k-1)$. Feit and Higman [8] proved that generalized n-gons with $s, t \geq 2$ can exist only if $n \in \{3, 4, 6, 8\}$ and that for $n = 8$, $s = t$ cannot occur. Hence Moore graphs with $k \geq 3$ and g even can exist only if $g \in \{6, 8, 12\}$. There are examples whenever $k-1$ is a prime power and it is wide open if there exist examples for other values of k.

When $g = 6, 8$ or 12, but $k-1$ is not a prime power, so there is no known generalized $g/2$-gon of order $(k-1, k-1)$, then one can do the following. Start from a Moore graph with valency $q+1$, where q is the smallest prime power bigger than or equal to k, and delete vertices from the graph to make it k-regular. The first one to use this idea (for $g = 6$) seems to be Brown [7]. In [1] Abreu, Funk, Labbate and Napolitano use a method which is in fact equivalent to Brown's method applied for the projective plane $\mathrm{PG}(2, q)$. In a recent paper by Araujo, Gonzalez, Montellano and Oriol [2], the same idea was used for the $g = 8$ and 12 cases, too.

In this paper we apply the same method, but use more from the geometrical structure of generalized n-gons to improve the previous constructions, hence the upper bounds for $c(k, g)$, $g \in \{6, 8, 12\}$. (In fact, our main improvements are for $g = 6$ and $g = 8$.)

In section 2 we explain the construction method, and give two constructions that work in every generalized n-gon.

In section 3 we consider the $g = 6$ case, that is, projective planes. The best construction we will find is when k is close to the square of a prime power. In this case we will also prove that one cannot hope for a better construction by deleting vertices from a Moore graph.

Section 4 is devoted to the $g = 8$ case. We only achieve an improvement when k is a prime-power (that is, one has to start with a Moore graph of valency $k+1$) and cannot prove that this construction is best possible.

In Section 5 we summarize what follows for $c(k, g)$ from the constructions of the previous sections.

2. The construction method

In all the constructions of this paper we will look for regular subgraphs of the incidence graph of a generalized n-gon by deleting a set of points and a set of lines. The girth of a graph of this kind is at least $2n$, since the original girth is exactly $2n$. In all interesting cases (when k is not too small) it is the direct consequence of the Moore bound that the girth of the resulting graph cannot be larger than $2n$.

It is easy to see that we need the following.

DEFINITION 2.1. *The pair $(\mathcal{P}_0, \mathcal{L}_0)$ in the generalized n-gon $(\mathcal{P}, \mathcal{L})$ is called a t-good structure, if there are t lines of $\mathcal{L}_0$ through any point not in $\mathcal{P}_0$, and there are t points of $\mathcal{P}_0$ on any line not in $\mathcal{L}_0$.*

General construction method. Suppose $(\mathcal{P}_0, \mathcal{L}_0)$ is a t-good structure in the generalized n-gon $(\mathcal{P}, \mathcal{L})$ of order q. Deleting points and lines of $\mathcal{P}_0$ and $\mathcal{L}_0$, respectively, the incidence graph of the resulting structure is $(q+1-t)$-regular with girth at least $2n$.

Complements of $\mathcal{P}_0$ and $\mathcal{L}_0$ will be denoted by $\mathcal{P}_1$ and $\mathcal{L}_1$, respectively. We will also call points of $\mathcal{P}_0$ *deleted points* and lines of $\mathcal{L}_0$ *deleted lines*. Note that since we start with and get a regular bipartite graph, $|\mathcal{P}_0| = |\mathcal{L}_0|$ and $|\mathcal{P}_1| = |\mathcal{L}_1|$ holds.

We end this section with two constructions that work in any generalized n-gon $(\mathcal{P}, \mathcal{L})$. For a point p or line l of the generalized n-gon, also p and l will denote the corresponding vertices of the incidence graph. We define the distance $d(x, y)$ for a pair $x, y \in \mathcal{P} \cup \mathcal{L}$ to be the distance in the incidence graph, that is, the length of the shortest path connecting x and y.

CONSTRUCTION 2.2. *([2]) Take a generalized n-gon $(\mathcal{P}, \mathcal{L})$ of order (q, q). Let $p_1, \dots, p_t \in \mathcal{P}$ all incident with a line l_1 and let $l_2, \dots, l_t$ be lines through p_1. Delete every line and point x for which $d(x, p_i) \le n-2$ or $d(x, l_i) \le n-2$ for some $i \in \{1, \dots, t\}$. This gives a t-good structure of size $tq^{n-2} + q^{n-3} + \dots + q + 1$.*

The upper bound for $c(k, 6)$ and $c(k, 8)$ coming from the above construction was already proved (with another method) by Lazebnik, Ustimenko and Woldar [11].

CONSTRUCTION 2.3. *Take a generalized $n = 3, 4, 6$-gon $(\mathcal{P}, \mathcal{L})$ of order (q, q). Let $p \in \mathcal{P}$ and $l \in \mathcal{L}$, where p is not on l. Deleting every line and point that are at distance at most $n-2$ from p or l, we get a 1-good structure of size $q^{n-2} + 2q^{n-3} + q^{n-4} + \dots + q + 1$.*

Proof. We only give the proof for the $n = 4$ and $n = 6$ cases, the proof of the $n = 3$ case will be given in the next section.

First we show that $(\mathcal{P}_0, \mathcal{L}_0)$ is a 1-good structure when n is even (that is $n = 4, 6$). Let p_1 be a point not deleted. Then $d(p, p_1) = n$ and $d(l, p_1) = n-1$, hence the neighbours of p_1 are at distance $n-1$ from p and at distance n or $n-2$ from l. Since the shortest circle in the graph is of length $2n$, there is a unique path of length $n-1$ connecting l and p_1, and hence there is exactly one neighbour of p_1 which was deleted. Dually, we deleted exactly one neighbour of a non-deleted line, thus $(\mathcal{P}_0, \mathcal{L}_0)$ is 1-good.

We only calculate the size in the $n = 6$ case, the proof of the $n = 4$ case is similar to the one to be presented. Denote by l_1 the vertex incident to p in the unique path between p and l (in the incidence graph). Let A_i denote the vertices of the graph, which are of distance i from p and $i+1$ from l_1 ($i = 1, \dots, 5$), and similarly, denote by B_i the vertices of the graph, which are of distance i from l_1 and $i+1$ from p ($i = 1, \dots, 5$) (see Figure 1). Then l is either in B_2 or in B_4.

Let $A_0 = \{p\}$ and $B_0 = \{l_1\}$. Each vertex of A_i or B_i ($0 \le i \le 4$) is incident to q vertices of A_{i+1} or B_{i+1}, respectively; and each vertex of A_i or B_i ($1 \le i \le 5$) is incident to a unique vertex of A_{i-1} or B_{i-1}, respectively. The only remaining edges (besides the one

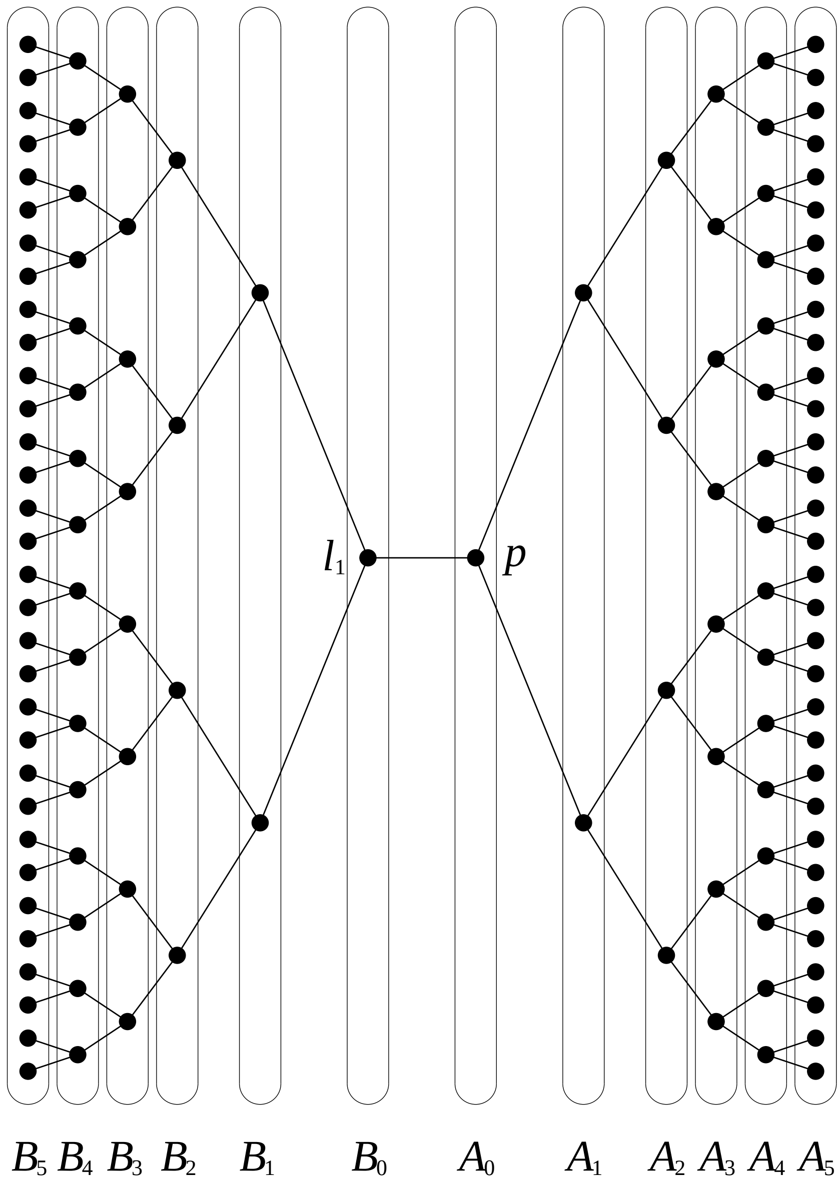

FIGURE 1

between p and l_1) are those between A_5 and B_5, here we see a regular bipartite graph of valency q.

Note that the vertex sets corresponding to points of the generalized hexagon are B_5, B_3, B_1, A_0, A_2, A_4. These are all of distance at most 4 from p, except for B_5. Hence all non-deleted points are in B_5. Since all points from B_3, B_1, A_0, A_2, A_4 are deleted, $|\mathcal{P}_0| = q^3 + q + 1 + q^2 + q^4 +$ the number of vertices deleted from B_5. Hence to finish the proof, we have to count the vertices of B_5 of distance 1 or 3 from l. We distinguish two cases according to whether l is in B_2 or B_4.

For $l \in B_2$, all we have is $l - B_3 - B_4 - B_5$ paths, so the number in question is q^3.

For $l \in B_4$, there are three different types of paths: $l - B_5$, $l - B_3 - B_4 - B_5$ and $l - B_5 - A_5 - B_5$. The number of vertices (of B_5) reached is q, $1 \cdot (q-1) \cdot q$ and $q \cdot q \cdot (q-1)$, respectively. This gives again q^3 vertices. (Note that the girth of the graph assures that we did not count any vertex more than once.) □

Note that the second construction is better than the first one for all three cases, but improvement is achieved only for $t = 1$. As we shall see in the next section, for $n = 3$ one can generalize Construction 2.3 to $t > 1$, and this was already done in [1].

3. The $g = 6$ case: constructions from a projective plane

This section is devoted to the $g = 6$ case, that is, generalized triangles. These are usually called projective planes. It is easy to see that the following definition is equivalent to that of a generalized 3-gon of order (q, q).

Definition 3.1. *Let $\mathcal{P}$ be a finite set and $\mathcal{L}$ a set of subsets of $\mathcal{P}$ called points and lines, respectively. The pair $(\mathcal{P}, \mathcal{L})$ is called a projective plane of order q, if it satisfies the following axioms.*

- *there are $q + 1$ lines through every point and $q + 1$ points on every line;*
- *there is a unique line through any two distinct points and a unique intersection point of any two distinct lines.*

Note that the role of lines and points is symmetric in the definition, hence for every definition and result we also have a dual definition and result by changing the words point and line to each other. It is easy to see (either from the above definiton, or from the Moore bound) that the number of points and lines is $q^2 + q + 1$.

First we give constructions which only use the definition of projective planes. They are not new, see the remarks after the constructions.

Construction 3.2. *Choose lines $l_1, \ldots, l_t$ through a point p_1 and let $p_2, \ldots, p_t$ be $t-1$ other points on l_1. Let $\mathcal{P}_0$ be the union of points on the l_is $(i = 1, \ldots, t)$ and $\mathcal{L}_0$ be the set of lines through any p_i $(i = 1, \ldots, t)$. Then $(\mathcal{P}_0, \mathcal{L}_0)$ is t-good of size $tq + 1$.*

Proof. Take a line $e \in \mathcal{L}_1$. Then e intersects every line l_i in one point, therefore e contains t points of $\mathcal{P}_0$. Take a point $p \in \mathcal{P}_1$. We deleted the t lines through p going through some p_i $(1 \le i \le t)$. The calculation of the size is easy. □

This construction is the $n = 3$ case of Construction 2.2 (from [2]) and seems to be originally due to Brown [7]. In a recent paper, though with a different terminology, Abreu, Funk, Labbate and Napolitano give the same construction ([1] Construction (i) on page 126).

Construction 3.3. *Let l_1 be a line, $p_1 \notin l_1$, $p_2, \ldots, p_t \in l$, finally, let $l_2, \ldots, l_t$ be the lines joining p_1 to the p_is $(2 \le i \le t)$. Let $\mathcal{P}_0$ consist of all points on the l_is and let $\mathcal{L}_0$ consist of all the lines through the p_is $(1 \le i \le t)$. Then $(\mathcal{P}_0, \mathcal{L}_0)$ is t-good of size $tq+3-t$.*

Proof. Lines in $\mathcal{L}_1$ do not contain p_i $(1 \le i \le t)$, hence they meet the l_is in t different points, while from points in $\mathcal{P}_1$ (which are not on l_i $(1 \le i \le t)$) we deleted the t lines which connect the point with some p_i. The calculation of the size is easy.

□

This construction, though with a different terminology, can be found in [1] (Construction (ii) on page 126).

REMARK. *Note that for $t = 1$, the second construction is slightly better than the first one (recall that we need t-good sets as large as possible). If $t = 2$, then the two constructions above are the same. This proves a conjecture of the just mentioned paper* [1] *(page 127, Remark 3.7). In the second construction $(\{p_1, \ldots, p_t\}, \{l_1, \ldots, l_t\})$ is a so called degenerate subplane. This can be generalized by taking a subplane S of order k and deleting all the lines through the points of S and all the points on the lines which meet S in $k+1$ points. We do not give any details, since this gives rise to smaller t-good sets than the previous ones.*

We continue with a construction that is better than the previous ones, but only works when q is a square prime power. First some definitions and basic facts. A subset B of the points of a projective plane is called a *Baer subplane*, if it has size $q + \sqrt{q} + 1$ and meets every line in 1 or $\sqrt{q} + 1$ points. Easy calculation shows that through a point out of the set there is a unique $(\sqrt{q} + 1)$-secant, while through points in the set the number of $(\sqrt{q} + 1)$-secants is $\sqrt{q} + 1$. Hence the number of $(\sqrt{q} + 1)$-secants is $q + \sqrt{q} + 1$. After this, one can easily deduce that B, together with its intersections with $(\sqrt{q} + 1)$-secants, forms a projective plane of order $\sqrt{q}$. The $(\sqrt{q}+1)$-secants are sometimes called *the lines of B*.

CONSTRUCTION 3.4. *Suppose that in our projective plane there are t disjoint Baer subplanes $B_1, \ldots, B_t$ with the property that no two of them has a common $(\sqrt{q} + 1)$-secant. Let $\mathcal{P}_0$ consist of the union of the B_is, and $\mathcal{L}_0$ of all lines intersecting one of the B_is in $\sqrt{q} + 1$ points. Then $(\mathcal{P}_0, \mathcal{L}_0)$ is t-good of size $t(q + \sqrt{q} + 1)$.*

Proof. First of all note that by the above listed properties, all lines meet $\mathcal{P}_0$ in either t or $\sqrt{q} + t$ points. Lines in $\mathcal{L}_1$ meet any of the t deleted subplanes in one point, hence we deleted t points from them. Let $p \in \mathcal{P}_1$ be an arbitrary point not deleted. For every $1 \leq i \leq t$ there is a unique line through p meeting B_i in $\sqrt{q} + 1$ points, and these lines are different for different i-s, so there are t lines deleted from p. The calculation of the size is easy. □

In general, it is not true (or at least not known) that any projective plane of square order has a Baer subplane, but it is true for the ones coordinatized by the finite field GF(q). These are denoted by PG$(2, q)$ and can be defined as follows. Let V denote a 3-dimensional vector space over GF(q). Let $\mathcal{P}$ and $\mathcal{L}$ consist of the 1- and 2-dimensional subspaces of V, respectively, and define incidence as inclusion. To make lines become subsets of points, one can identify lines with the set of 1-dimensional subspaces it contains. The pair $(\mathcal{P}, \mathcal{L})$ is a projective plane of order q. When q is square, PG$(2, q)$ does contain Baer subplanes, all of them are isomorphic to PG$(2, \sqrt{q})$. Moreover, any two disjoint Baer subplanes have distinct $(\sqrt{q} + 1)$-secants. This is a particular case of a theorem due to Sved [14]. Even more is true: PG$(2, q)$ can be partitioned into $q - \sqrt{q} + 1$ disjoint Baer subplanes. For more about projective planes, Baer subplanes and for the proofs of the listed properties, we refer to [9].

THEOREM 3.5. *For any square prime power q and $t \leq q - \sqrt{q} + 1$, Construction 3.4 works in the plane $PG(2, q)$.*

Proof. By the listed facts about $\mathrm{PG}(2, q)$, one can find $q - \sqrt{q} + 1$ disjoint Baer subplanes. Choosing only t of these will be appropriate, since all we need is that the $(\sqrt{q}+1)$-secants are distinct, and this follows from the above mentioned result by Sved. ☐

In [1] Section 4, there is a construction for $q = 4, 9$ and 16 giving a graph of the same size as the one in Construction 3.4 here. The authors conjecture that this construction should work for general q. Theorem 3.5 solves this open problem. (The method is quite different there, but it seems that essentially its equivalent to deleting Baer subplanes from $\mathrm{PG}(2, q)$.)

After this, it is natural to ask if one could improve this construction by finding larger t-good structures. We will prove that, at least for $t \leq 2\sqrt{q}$, Construction 3.4 is best possible. We also want to study, if there are more constructions. We will prove the following theorems.

THEOREM 3.6. *In an arbitary projective plane of order q, every t-good structure with $t \leq 2\sqrt{q}$ has size at most $t(q + \sqrt{q} + 1)$.*

THEOREM 3.7. *In any projective plane a 1-good pair $(\mathcal{P}_0, \mathcal{L}_0)$ is one of the constructions 3.2, 3.3, 3.4.*

THEOREM 3.8. *In $PG(2, q)$, $q > 256$, every 2-good structure is one of the construcions 3.2, 3.3, 3.4.*

In the proof of Theorem 3.4 we will use the so called *standard equations.* For any point set S in a projective plane of ordrer q, denote by n_i the number of i-secants to S. Recall that both the number of points and lines is $q^2 + q + 1$. By counting the total number of lines, incident pairs (P, l) with $P \in S$, and triples (P, Q, l) with $P \neq Q \in S$, we obtain the following three equation:

$$\sum_{i=0}^{q+1} n_i = q^2 + q + 1,$$

$$\sum_{i=0}^{q+1} i n_i = |S| \, (q+1) \, ,$$

$$\sum_{i=0}^{q+1} i \, (i-1) \, n_i = |S| \, (|S| - 1) \, .$$

For a t-good structure $(\mathcal{P}_0, \mathcal{L}_0)$, let n_i^0 denote the number of i-secants to $\mathcal{P}_0$ in $\mathcal{L}_0$ and n_i^1 the number of i-secants to $\mathcal{P}_0$ in $\mathcal{L}_1$. Then the total number of i-secants to $\mathcal{P}_0$ is $n_i = n_i^0 + n_i^1$. Since $(\mathcal{P}_0, \mathcal{L}_0)$ is t-good, by definition

$$n_i^1 = \begin{cases} q^2 + q + 1 - |\mathcal{L}_0| & \text{for } i = t \\ 0 & \text{otherwise} \end{cases}$$

Using this, the standard equations and $|\mathcal{P}_0| = |\mathcal{L}_0|$, we obtain what we shall refer to as the *standard equations for t-good structures*:

$$\sum_{i=0}^{q+1} n_i^0 = |\mathcal{L}_0|,$$

$$\sum_{i=0}^{q+1} i n_i^0 = |\mathcal{L}_0|(q+1+t) - t(q^2+q+1),$$

$$\sum_{i=0}^{q+1} i(i-1)n_i^0 = |\mathcal{L}_0|^2 + |\mathcal{L}_0|(t^2-t-1) - t(t-1)(q^2+q+1).$$

Proof of Theorem 3.6. Using the standard equations for t-good structures, we get

$$0 \le \sum_{i=0}^{q+1}(i-(\sqrt{q}+t))^2 n_i^0 = \sum_{i=0}^{q+1} i(i-1)n_i^0 - \sum_{i=0}^{q+1}(2(\sqrt{q}+t)-1)in_i^0 + \sum_{i=0}^{q+1}(\sqrt{q}+t)^2 n_i^0 =$$
$$|\mathcal{L}_0|^2 + |\mathcal{L}_0|\left[t^2-t-1-(2(\sqrt{q}+t)-1)(q+1+t)+(\sqrt{q}+t)^2\right] +$$
$$(q^2+q+1)\left[(2t(\sqrt{q}+t)-t)-t(t-1)\right] =$$
$$|\mathcal{L}_0|^2 - 2\left[(q+1)t+\sqrt{q}(q-\sqrt{q}+1)\right]|\mathcal{L}_0| + (q^2+q+1)(2t\sqrt{q}+t^2) =$$
$$\left(|\mathcal{L}_0| - t(q+\sqrt{q}+1)\right)\left(|\mathcal{L}_0| - (t+2\sqrt{q})(q-\sqrt{q}+1)\right),$$

hence either $|\mathcal{L}_0| \le t(q+\sqrt{q}+1)$ or $|\mathcal{L}_0| \ge (t+2\sqrt{q})(q-\sqrt{q}+1)$ (it is easy to check that the first root is smaller than the second one). Assuming $0 \le t \le 2\sqrt{q}$ and $|\mathcal{L}_0| \ge (t+2\sqrt{q})(q-\sqrt{q}+1)$, the number of vertices in the $(q+1-t)$-regular graph induced by $\mathcal{L}_1$ and $\mathcal{P}_1$ would be

$$|\mathcal{L}_1| + |\mathcal{P}_1| \le 2\left(q^2+q+1-(t+2\sqrt{q})(q-\sqrt{q}+1)\right) <$$
$$2\left(q^2+q+1-t(2q-t+1)\right) = 2\left((q-t)^2+(q-t)+1\right),$$

contradicting the Moore-bound. Therefore $|\mathcal{L}_0| \le t(q+\sqrt{q}+1)$ must hold. □

Based on a result of Blokhuis, Storme and Szőnyi, one can characterize equality in the previous bound for PG$(2,q)$. A subset of the points of a projective plane is called a *t-fold blocking set*, if it meets every line in at least t points. For $t = 1$, it is simply called a *blocking set*.

THEOREM 3.9. *(Blokhuis, Storme, Szőnyi* [5]*) In PG$(2,q)$ a t-fold blocking set has at least $t\left(q+\sqrt{q}+1\right)$ points for $t < \frac{\sqrt[4]{q}}{2}$, and equality holds if and only if the set is the union of t disjoint Baer-subplanes.*

In the proof of Theorem 3.6 equality holds exactly when $n_i^0 \neq 0 \iff i = \sqrt{q}+t$, that is, every line in $\mathcal{L}_0$ intersects $\mathcal{P}_0$ in $\sqrt{q}+t$ points. The lines in $\mathcal{L}_1$ meet $\mathcal{P}_0$ in t points, hence $\mathcal{P}_0$ is a t-fold blocking set.

COROLLARY 3.10. *If $t < \frac{\sqrt[4]{q}}{2}$ and $(\mathcal{P}_0, \mathcal{L}_0)$ is a t-good structure in PG$(2,q)$ with $|\mathcal{P}_0| = t\left(q+\sqrt{q}+1\right)$, then $\mathcal{P}_0$ is the union of t disjoint Baer-subplanes and the lines in $\mathcal{L}_0$ are those that intersect one of the Baer-subplanes in $\sqrt{q}+1$ points, hence we have Construction 3.4.*

For the proof of 3.7 and 3.8, we need some more definitions and results about projective planes.

It is easy to check, that any blocking set contains at least $q+1$ points with equality if and only if it is a line.

THEOREM 3.11 (Bruen [6]). *In any projective plane of order q a blocking set not containing a line has size at least $q+\sqrt{q}+1$ with equality if and only if it is a Baer subplane.*

LEMMA 3.12. *Let $(\mathcal{P}_0, \mathcal{L}_0)$ be a t-good structure, $t<\sqrt{q}$. Then $\mathcal{P}_0$ is a blocking set.*

Proof. Assume that there exists a line l not meeting $\mathcal{P}_0$. Then l must be in $\mathcal{L}_0$. Since any point p on l is in $\mathcal{P}_1$, there has to be exactly $t-1$ lines from $\mathcal{L}_0$ different from l through p, therefore $|\mathcal{L}_0| = 1+(q+1)(t-1) = tq-q+t$. On the other hand, taking a line $e \in \mathcal{L}_1$, we can see at least $(q+1-t)t$ deleted lines intersecting e, thus $tq+t-t^2 \leq tq-q+t$, which cannot occur for $t<\sqrt{q}$. □

Proof of Theorem 3.7. First note that by Lemma 3.12, $\mathcal{P}_0$ is a blocking set. Since a line not deleted meets $\mathcal{P}_0$ in exactly $t=1$ points, every line joining two deleted points has to be deleted, and dually, the intersection of two deleted lines is in $\mathcal{P}_0$. We distinguish three cases according to the maximum number ν of points in $\mathcal{P}_0$ such that no three of them is collinear:

CASE 1: $\nu=2$. Then $\mathcal{P}_0$ is contained in a line, but since it is a blocking set, it has to be the full line. It's easy to see that this is Construction 3.2.

CASE 2: $\nu=3$. In this case $|\mathcal{P}_0| \leq q+2$, because it cannot contain two pairs of points on two different lines, for that would lead to $\nu \geq 4$. By Theorem 3.11, $\mathcal{P}_0$ has to contain a line, thus $|\mathcal{P}_0| = q+2$. It's easy to see that this is Construction 3.3.

CASE 3: $\nu \geq 4$. Assume that $\mathcal{P}_0$ contains a full line l. Then by $\nu \geq 4$, there must be at least two points of $\mathcal{P}_0$ not on l, but then the lines joining these two points to the points of l are all deleted, thus $|\mathcal{L}_0| \geq 2q+2$, contradicting the upper bound in Theorem 3.6. Therefore $\mathcal{P}_0$ is a blocking set that does not contain a full line, hence by Theorem 3.11 and Theorem 3.6 it is a Baer-subplane, which is Construction 3.4. □

For the proof of Theorem 3.8, we need one more lemma.

LEMMA 3.13. *If $t=2$ and $q \geq 5$, then $|\mathcal{P}_0| = |\mathcal{L}_0| \geq 2q+1$ with equality if and only if we have Construction 3.2.*

Proof. Let $p \in \mathcal{P}_1$. There are $q-1$ lines from $\mathcal{L}_1$ through p all containing exactly 2 points of $\mathcal{P}_0$, hence the size of $|\mathcal{P}_0| = |\mathcal{L}_0| = 2q-2+c$, where c denotes the number of deleted points on the two deleted lines through p. By Lemma 3.12, $\mathcal{P}_0$ is a blocking set, so we can deduce that $c \geq 2$. Hence $|\mathcal{P}_0| \geq 2q$ with equality if and only if the two deleted lines through p meet $\mathcal{P}_0$ in 1 point. One can repeat this counting from any $p \in \mathcal{P}_1$ to deduce, that if $|\mathcal{P}_0| = 2q$, then all lines from $\mathcal{L}_0$ meet $\mathcal{P}_0$ in 1 or $q+1$ points. It is easy to see that this cannot be true for a set of $2q$ points.

Finally, suppose that $|\mathcal{P}_0| = 2q+1$. The above counting shows that through a point of $\mathcal{P}_1$, the two deleted lines meet $\mathcal{P}_0$ in 1 and 2 points, respectively. Hence lines of $\mathcal{L}_0$ are 1-,2-, or $(q+1)$-secants to $\mathcal{P}_0$. Let $p \in \mathcal{P}_0$. There are $q+1$ lines through p, so even if they all belong to $\mathcal{L}_0$, one of them has to have at least 2 more points of $\mathcal{P}_0$, so we can deduce that there is a line $l \in \mathcal{L}_0$ with all of its point in $\mathcal{P}_0$. Take any two points from $\mathcal{P}_0$ not on l. The line through them contains at least 3 points from $\mathcal{P}_0$, hence all of its points are in $\mathcal{P}_0$. Hence the deleted points are exactly the points of two lines. The dual of this argument implies that the deleted lines are the lines going through two points. It is easy to see that we have Construction 3.2. □

Recall that for $t = 2$, Constructions 3.2 and 3.3 are the same. Now we are ready to prove Theroem 3.8.

Proof of Theorem 3.8. By the previous lemma, we see that a possible counterexample would have $|\mathcal{P}_0| \geq 2q + 2$. But this implies that $\mathcal{P}_0$ is a double blocking set: if a line l had at most 1 point from $\mathcal{P}_0$, then, since through the non-deleted points of l there were exactly one more deleted line, we would have $|\mathcal{L}_0| \leq q + 1 + q$, a contradiction.

Using the result of Blokhuis, Szőnyi and Storme (Theorem 3.9) for $t = 2$, we deduce that $|\mathcal{P}_0| \geq 2(q + \sqrt{q} + 1)$ with equality if and only if $\mathcal{P}_0$ is the union of two Baer subplanes, that is, we have Construction 3.4. Theorem 3.6 completes the proof. □

Note that almost everything goes through for an arbitrary projective plane of order q. The only moment when we had to use that we are in $\mathrm{PG}(2, q)$ is (after deducing that $\mathcal{P}_0$ is a double blocking set) when we used the result of Blokhuis, Storme and Szőnyi.

We end this section by listing some results without proofs, which are only interesting from the finite geometry point of view.

The lower bound $|\mathcal{L}_0| \geq (q + 1 - t)t$ is sharp if and only if $t = \sqrt{q}$ and $\mathcal{P}_0$ consists of the points of a maximal $(k, \sqrt{q})$-arc. In this case $\mathcal{P}_0$ is not a blocking set. If $\mathcal{P}_0$ is a blocking set, then one can add t to the lower bound, hence $|\mathcal{P}_0| \geq (q+2-t)t$ for $t < \sqrt{q}$. One can prove that assuming $t < \sqrt{q}$, this is sharp only if $t = 1$. However, for $t = \sqrt{q}+1$, a unital and its tangents form a t-good pair with $|\mathcal{P}_0| = (q + 2 - t)t$ and in this example $\mathcal{P}_0$ is a blocking set.

Small t-good structures can be constructed with the help of subplanes by deleting the lines through the points of a subplane of order s and the points that are on the lines intersecting the subplane in $s + 1$ points. This is an $(s^2 + s + 1)$-good structure of size $(s^2 + s + 1)q - (s - 1)(s^2 + s + 1)$.

It is easy to prove that for $t < \frac{q+1}{2}$, if $\mathcal{P}_0$ and $\mathcal{L}_0$ consist of all points on t given lines and all lines on t given points, respectively, then $(\mathcal{P}_0, \mathcal{L}_0)$ is t-good if and only if the points and lines in question form a (possibly degenerate) subplane.

There are t-good structures of size larger than $tq + 1$ when $t = \frac{q+1}{2}$, for example take the external points and the secants of an oval. Note that the graph constructed this way is quite far from the Moore bound, since t is large. However, considering this in $\mathrm{PG}(2, q)$, where the oval is a conic arising from a polarity, one can identify the secants and the external points using the polarity. This "half graph" is regular of girth five exactly when $q \equiv 3 \pmod 4$. Replacing the external points with internal points and secants with skew lines, we get a similar example which works for $q \equiv 1 \pmod 4$. This construction is due to Jason Williford (see [13]).

4. The $g = 8$ case: constructions from a generalized quadrangle

In this section we first give the necessary definitions and recall some results about generalized quadrangles. It is straightforward to check that the following definition is equivalent to the one given in the introduction (for $g = 8$).

Definition 4.1. *Let $\mathcal{P}$ be a finite set and $\mathcal{L}$ a set of subsets of $\mathcal{P}$ called points and lines, respectively. The pair $(\mathcal{P}, \mathcal{L})$ is called a generalized quadrangle of order (s, t), if it satisfies the following axioms.*

- *there are $s + 1$ lines through every point;*
- *every line has $t + 1$ points;*
- *for any point p and line l not through p, there is a unique line through p intersecting l.*

Note that the role of points and lines in the definition of a generalized quadrangle is symmetric, hence changing the role of points and lines, one finds another generalized quadrangle (of order (t, s)). This generalized quadrangle is not necessarily isomorphic to the original one even if $s = t$ holds. Taking any definition or result, one can change the words point and line with each other to find a dual definition or result.

The point-line incidence graph of such a structure is a Moore graph (with $g = 8$ and $k = s + 1$) if and only if $s = t$. So from now on we suppose that $s = t$, in this case one usually sais that *the generalized quadrangle has order s* and denotes the structure by $GQ(s)$.

For any subset of the points U, $U^\perp$ denotes the set of points collinear with all points of U, and $U^{\perp\perp}$ the set of points collinear with all points of $U^\perp$. One can similarly define $W^\perp$ and $W^{\perp\perp}$ for a set W of lines. Next we summarize some easy consequences of the definition.

Lemma 4.2. *For any $GQ(s)$*

- *(i) there are $(s + 1)(s^2 + 1)$ points and the same number of lines;*
- *(ii) for any two non-collinear points u and v, $\left|\{u, v\}^\perp\right| = s + 1$,*
- *(iii) for any two non-collinear points u and v, $\left|\{u, v\}^{\perp\perp}\right| \leq s + 1$;*
- *(iv) for any two skew lines l and m, $\left|\{l, m\}^\perp\right| = s + 1$,*
- *(v) for any two skew lines l and m, $\left|\{l, m\}^{\perp\perp}\right| \leq s + 1$.*

Proof. (i) Fix a point p. There are $s + 1$ lines through p, hence the number of collinear points to p is $1 + (s + 1)s$. By the third axiom of GQ-s, all lines not through p have a unique point collinear to p, hence the number of lines is $s + 1 + (s + 1)s^2 = (s + 1)(s^2 + 1)$. The number of points is the same by duality.

(ii) There are $s + 1$ lines through v, all of them have a unique point collinear to u.

(iii) Choose two different points $a, b \in \{u, v\}^\perp$. Then $\{u, v\}^{\perp\perp} \subseteq \{a, b\}^\perp$, hence (ii) implies (iii).

(iv) and (v) are the dual of (ii) and (iii). □

A non-collinear point-pair u, v is called *regular* if $\left|\{u,v\}^{\perp\perp}\right| = s+1$ holds. One can similarly define a regular line-pair. Next we list some properties of regular pairs that will be needed for our constructions.

LEMMA 4.3. *Suppose the point-pair (u_0, u_1) is regular and let $\{u_0, u_1\}^{\perp} = \{v_0, \dots, v_s\}$, $\{u_0, u_1\}^{\perp\perp} = \{u_0, \dots, u_s\}$. Denote by L' the set of lines joining a point u_i to a point v_j.*

- *(i) Any u_i is collinear to any v_j, but no different u_i and u_j or v_i and v_j can be collinear.*
- *(ii) L' contains $(s+1)^2$ lines;*
- *(iii) for any u_i, u_j $(i \neq j)$, $\{u_i, u_j\}^{\perp} = \{v_0, \dots, v_s\}$, and for any v_i, v_j $(i \neq j)$, $\{v_i, v_j\}^{\perp} = \{u_0, \dots, u_s\}$;*
- *(iv) all lines through an u_i or v_i are in L';*
- *(v) through any point not in $\{u_0, \dots, u_s\} \cup \{v_0, \dots, v_s\}$, there is a unique line in L'.*

Proof. Any u_i is collinear to any v_j by definition of the perp. If an u_i and an u_j were collinear, then the line joining them and any v_k would contradict the third axiom of GQ-s. This proves (i) and (ii).

(iii) follows from (i) and Lemma 4.2 (ii).

For (iv) note that there are $s+1$ lines through a point, and we see $s+1$ lines through any u_i or v_i in L'.

For (v) first suppose that there are at least two lines in L' through a point $p \notin \{u_0, \dots u_s\} \cup \{v_0, \dots v_s\}$. Without loss of generality suppose that p is collinear to u_i and u_j. Then $\{u_i, u_j\}^{\perp}$ contains at least $s+2$ points contradicting Lemma 4.2 (ii). Hence the number of points on the lines of L' is $2(s+1) + (s+1)^2(s-1) = (s+1)(s^2+1)$, this is the number of points of the GQ, hence every point is on a line of L'. □

CONSTRUCTION 4.4. *Suppose the $GQ(s)$ has a regular point-pair (u, v). Pick a point $p \notin \{u,v\}^{\perp} \cup \{u,v\}^{\perp\perp}$. Let $\mathcal{P}_0 = \{u,v\}^{\perp} \cup \{u,v\}^{\perp\perp} \cup p^{\perp}$. Let $\mathcal{L}_0$ consist of lines joining a point of $\{u,v\}^{\perp}$ to a point of $\{u,v\}^{\perp\perp}$ together with lines through p. Then $(\mathcal{P}_0, \mathcal{L}_0)$ is 1-good with $|\mathcal{P}_0| = |\mathcal{L}_0| = s^2 + 3s + 1$.*

Proof. By Lemma 4.3, through a point $q \in \mathcal{P}_1$ there is exactly one line joining a point of $\{u,v\}^{\perp}$ to a point of $\{u,v\}^{\perp\perp}$ (and no lines through p, since points collinear to p were deleted). For a line $l \in \mathcal{L}_1$, there is a unique line through p meeting l by the definition of generalized quadrangles. For the size note, that by Lemma 4.3, there is a unique line through p joining a point of $\{u,v\}^{\perp}$ to a point of $\{u,v\}^{\perp\perp}$. Hence $|\mathcal{L}_0| = (s+1)^2 + s + 1 - 1 = s^2 + 3s + 1$. □

CONSTRUCTION 4.5. *Suppose the $GQ(s)$ has a regular point-pair (u, v) and a regular line pair (l, m). Suppose also that there are no points from $\{u,v\}^{\perp} \cup \{u,v\}^{\perp\perp}$ on the lines of either $\{l,m\}^{\perp}$ or $\{l,m\}^{\perp\perp}$. Let $\mathcal{P}_0$ consist of the points from $\{u,v\}^{\perp} \cup \{u,v\}^{\perp\perp}$ together with points from lines in $\{l,m\}^{\perp} \cup \{l,m\}^{\perp\perp}$. Dually, let $\mathcal{L}_0$ consist of the lines from $\{l,m\}^{\perp} \cup \{l,m\}^{\perp\perp}$ together with lines through points of $\{u,v\}^{\perp} \cup \{u,v\}^{\perp\perp}$. Then $(\mathcal{P}_0, \mathcal{L}_0)$ is 1-good with $|\mathcal{P}_0| = |\mathcal{L}_0| = s^2 + 4s + 3$.*

Proof. Let $p \in \mathcal{P}_1$. By Lemma 4.3, there is a unique line joining a point of $\{u, v\}^\perp$ to a point of $\{u, v\}^{\perp\perp}$, and since p is not in $\mathcal{P}_0$, there is no line in $\{l, m\}^\perp \cup \{l, m\}^{\perp\perp}$ through p. Hence there are s lines in $\mathcal{L}_1$ through p. The dual of this argument (using the dual of Lemma 4.3) implies that on any line in $\mathcal{L}_1$ there are exactly s points. The calculation of the size is easy. □

Looking through the literature of generalized quadrangles, it turns out that GQ-s with both regular point- and line-pairs only exist for q even. Here we show an example where our constructions work.

DEFINITION 4.6. *The symplectic generalized quadrangle of order q denoted by $W(q)$ is the following: as point-set, we take all points of the 3-dimensional projective geometry $PG(3, q)$. The lines are the totally isotropic lines with respect to a symplectic polarity of $PG(3, q)$.*

W(q) is a generlized quadrangle of order q. For the proof of this and further properties of W(q), we refer to [4].

THEOREM 4.7. *In $W(q)$ Construction 4.4 always works. Construction 4.5 works if and only if q is even.*

Proof. By [12], all point-pairs are regular of W(q), and the sets $\{u, v\}^\perp$ and $\{u, v\}^{\perp\perp}$ consist of points of a non-symplectic line l and points of $l^\perp$, respectively. There is at least one regular line-pair if and only if all line pairs are regular if and only if q is even.

If q is even, then for two skew lines l and m of W(q), the sets $\{l, m\}^\perp$ and $\{l, m\}^{\perp\perp}$ are the two opposite reguli on a hyperbolic quadric. Hence after choosing l and m for Construction 4.5, all we have to do is choose u and v to be two points determining a non-isotropic line disjoint from the hyperbolic quadric in question. □

5. ORDER OF (k, g)-CAGES

In this section we summarize the consequences of our constructions. All improvements depend on how close a prime power is to k.

THEOREM 5.1. *Denote by q the smallest prime power greater or equal to $k - 1$. If q is a square, then*

$$c(k, 6) \leq 2\left(kq - (q - k)(\sqrt{q} + 1) - \sqrt{q}\right).$$

Proof. We need to delete t Baer subplanes from PG$(2, q)$ using Construction 3.4 (see also Theorem 3.5) with $t = q + 1 - k$. Hence the number of points of the incidence graph of the resulting structure is

$$2\left((q^2 + q + 1) - (q + 1 - k)(q + \sqrt{q} + 1)\right).$$

A little calculation shows that this equals the formula stated. □

If the smallest prime power $q \geq k-1$ is not a square, then one can use (the previously known) Construction 3.3 to find an upper bound on $c(k,6)$.

Note that it is very rare that the smallest prime power $q \geq k-1$ is a square. If q is not a square, then even if $q+1$ is a square prime power, Constructions 3.3 and 3.2 starting from a plane of order q are better than Construction 3.4 starting from a plane of order $q+1$.

By Theroem 3.6, one cannot hope for a better bound on $c(k,6)$ using the same construction method. However, there is one example known when $c(k,6)$ is smaller than the one coming from Theorem 5.1: there is a construction due to Baker [3] (see also [13]) for a $(7,6)$ graph (that is, regular of valency 7 and girth 6) with 90 vertices. Our method would start with a plane of order 7, and even if there was a Baer subplane of order $\sqrt{7}$, Construction 3.4 would give a graph on $2\left((7^2+7+1)-(7+\sqrt{7}+1)\right) \approx 92.7$ vertices.

THEOREM 5.2. *Suppose that k is a prime power. If k is even, then $c(k,8) \leq 2(k^3-3k-2)$. If k is odd, then $c(k,8) \leq 2(k^3-2k)$.*

Proof. One should start with $W(k)$ and use Construction 4.4 or 4.5 according to whether k is odd or even (see also Theorem 4.7). Hence the number of points of the incidence graph of the resulting structure is $2(k^3+k^2+k+1)-2\left|\mathcal{P}_0\right|$. □

Finally, our slight improvement for the $g=12$ case is the following.

THEOREM 5.3. *Suppose k is a prime power. Then $c(k,12) \leq 2(k^5-k^3)$.*

Proof. One should start with a generalized hexagon of oder k and use Construction 2.3. □

Acknowledgement. We thank Tamás Szőnyi for his many suggestions and help.

REFERENCES

[1] M. ABREU, M. FUNK, D. LABBATE, V. NAPOLITANO, On (minimal) regular graphs of girth 6. *Australas. J. Combin.* **35** (2006), 119-132.

[2] G. ARAUJO, D. GONZÁLEZ, J.J. MONTELLANO-BALLESTEROS, O. SERRA, On upper bounds and connectivity of cages. *Australas. J. Combin.* **38** (2007), 221-228.

[3] R. D. BAKER, Elliptic semi-planes I. Existence and classification. *Proceedings of the Eighth Southeastern Conference on Combinatorics, Graph Theory and Computing* (Louisiana State Univ., Baton Rouge, La., 1977), pp. 61–73. Congressus Numerantium, No. XIX, Utilitas Math., Winnipeg, Man., 1977.

[4] S. BALL, ZS. WEINER, An Introduction to Finite Geometry, http://www-ma4.upc.es/ simeon/IFG.pdf.

[5] A. BLOKHUIS, L. STORME, T. SZŐNYI, Lacunary polynomials, multiple blocking sets and Baer subplanes. *J. London Math. Soc. (2)* **60** (1999), no. 2, 321-332.

[6] A. A. BRUEN, Blocking sets in finite projective planes. *SIAM J. Appl. Math.* **21** (1971), 380-392.

[7] W.G. BROWN, On Hamiltonian regular graphs of girth six. *J. London Math. Soc.* **42** (1967), 514-520.

[8] W FEIT, G. HIGMAN, The nonexistence of certain generalized polygons, *J. Algebra* **1** (1964), 114-131.

[9] J.W.P. HIRSCHFELD, Projective geometries over finite fields. *Clarendon Press, Oxford,* 1979, 2nd edition, 1998.

[10] F. KÁRTESZI Piani finiti ciclici come risoluzioni di un certo problema di minimo (Italian). *Boll. Un. Mat. Ital. (3)* **15** (1960) 522-528.

[11] F. Lazebnik, V.A. Ustimenko, A.J. Woldar, New upper bounds on the order of cages. The Wilf Festschrift (Philadelphia, PA, 1996). *Electron. J. Combin.* **4** (1997), no. 2, Research Paper 13, approx. 11 pp. (electronic).

[12] S.E. Payne, J.A. Thas, Finite generalized quadrangles, Research Notes in Mathematics, **110** *Pitman Advanced Publishing Program, Boston, MA,* (1984) vi+312 pp.

[13] G. Royle, Cages of higher valency, http://people.csse.uwa.edu.au/gordon/cages/allcages.html.

[14] M. Sved, Baer subspaces in the n-dimensional projective space, *Combinatorial mathematics, X* (Adelaide, 1982), 375–391, *Lecture Notes in Math.*, **1036**, Springer, Berlin, 1983.

[15] P.K. Wong, Cages — a survey, *J. Graph Theory* **6** (1982), no. 1, 1–22.

Author's addresses:

András Gács, Tamás Héger
Department of Computer Science, Eötvös Loránd University,
H-1117 Budapest, Pázmány Péter sétány 1/C, HUNGARY

e-mail: gacs@cs.elte.hu, hetamas@cs.elte.hu