ON $(k,6)$-GRAPHS ARISING FROM PROJECTIVE PLANES

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Abstract. We study a construction method (first used in a paper by Brown [7] and having been rediscovered by several authors recently) producing small $(k,6)$-graphs. We prove that under some conditions the known constructions are best possible in the sense that one cannot hope for smaller examples from this method. Both algebraic and combinatorial tools are used.

1. Introduction

A (k,g)-*graph* is a simple k-regular graph of girth g. The problem of determining the smallest number $c(k,g)$ of vertices a (k,g)-graph may have is very difficult for most values of k and g and has been studied by several authors. A (k,g)-graph with $c(k,g)$ vertices is called a (k,g)-*cage*. An easy combinatorial lower bound for $c(k,g)$ is given by the Moore bound, and graphs attaining equality in this bound are called Moore graphs. Since the present paper is about the $g=6$ case, we only formulate the bound for this: $c(k,6) \geq 2((k-1)^2+(k-1)+1)$. It was shown by Kárteszi [9] that a Moore graph for the $g=6$ case is the incidence graph of a porjective plane of order $k-1$. Hence a Moore graph exists for $g=6$ if and only if there exists a projetive plane of order $k-1$. This is true in a more general setting: when $g \geq 6$ is even, then a Moore graph is the incidence graph of a so-called generalized $g/2$-gon of order $k-1$. These exist for $g=6,8,12$ whenever $k-1$ is a prime power. We refer to the survey of Wong [1], to the dynamic survey of Exoo [2] and the web page of Royle [3] for further introduction and results on (k,g)-graphs, cages and Moore graphs.

Considering the cases $g=6,8,12$, many papers have focused on constructing small (k,g)-graphs as induced regular subgraphs of the incidence graphs of generalized polygons. Some authors use 0/1 matrices to construct the adjacency matrix of the (k,g)-graphs, but these also turn out to give rise to subgraphs of generalized polygons ([4], [5], [6], [7], [8]). When $g=6$, the generalized polygon is a projective plane. For these, essentially 2 constructions are known, which we will describe in the next section. Our main result (to be stated in Section 2) is that under certain conditions, one cannot hope for better constructions from this technique. However, it should be mentioned already at this point, that we only consider induced subgraphs of the incidence graph. We will make some comments on this in the last section.

The paper is organised as follows. In Section 2 we describe the construction method, list the known constructions by this method and state the result. The proof will be given in Section 3. Finally, in Section 4 we make some comments and discuss some open problems.

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We end this introduction with some facts about projective planes. For proofs and more infromation, we refer to [11].

A *projective plane* is an incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ consisting of a point set $\mathcal{P}$, a line set $\mathcal{L}$ and an incidence relation $\mathcal{I}$ between points and lines satisfying that any two lines have a unique point in common and any two points are incident with a unique line. One can prove that besides two degenerate examples, any porjective plane has an order q with the property that any line is incident with $q+1$ points, any point is incident with $q+1$ lines and both the number of points and the number of lines are q^2+q+1. The two degenerate projective planes are the following.

π_1: there is a line l and a point p such that all points are incident with l and all lines are incident with p;

π_2: there is a non-incident point-line pair (p, l) such that points besides p are incident with l and lines besides l are incident with p.

Note that the axioms of a projective plane are symmetric, so there is a duality between points and lines.

A *subplane* of a projective plane $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is a subset $\mathcal{P}'$ of points and $\mathcal{L}'$ of lines of the original plane such that $(\mathcal{P}', \mathcal{L}', \mathcal{I}')$ is a porjective plane on its own right, where $\mathcal{I}'$ is the restriction of $\mathcal{I}$ on the pair $(\mathcal{P}', \mathcal{L}')$. The subplane is *degenerate*, if it is π_1 or π_2 (that is, it is degenerate as a projective plane). The order of a non-degenerate subplane is always at most the square root of the order of the big plane.

It is easy to see that any projective plane has degenerate subplanes of both types. A degenerate subplane of type π_1 is an incident point-line pair (p, l) together with some other points on l and some other lines through p. A degenerate subplane of type π_2 is a non-incident point-line pair (p, l) together with some more points on l and the lines joining p to these points.

The projective plane $\mathrm{PG}(2, p)$ defined over the finite field $\mathrm{GF}(p)$ where p is prime, does not have non-degenerate subplanes. On the other hand, for any square prime power q, $\mathrm{PG}(2, q)$ contains non-degenerate subplanes of order $\sqrt{q}$, these are called *Baer subplanes*. Moreover, one can partition the point set of the plane into the point sets of $q - \sqrt{q} + 1$ disjoint Baer subplanes. For any prime power q, all subplanes of $\mathrm{PG}(2, q)$ have order a power of the same prime as q.

2. t-good structures and the main theorem

For a non-degenerate projective plane of order q, the incidence graph (between lines and points) is easily seen to be a $(q+1, 6)$ graph. As was mentioned in the introduction, this is the best possible construction for a $(q+1, 6)$ graph. When there is no projective plane of order $k-1$, one possibility to construct a (k, g) graph is to consider the first $q \geq k$ for which a projective plane of order q exists, consider the incidence graph and try to delete vertices of this graph to make it k-regular. It is easy to see that for this we need the following structure within the plane.

Definition 2.1. *Let $(\mathcal{P}_0, \mathcal{L}_0)$ be a pointset and a lineset such that*

- *$\forall P \notin \mathcal{P}_0$ there are exactly t lines in $\mathcal{L}_0$ through P,*

- *$\forall l \notin \mathcal{L}_0$ there are exactly t points in $\mathcal{P}_0$ on l.*

*Then the pair $(\mathcal{P}_0, \mathcal{L}_0)$ is called a t-*good structure*.*

It is straightforward to check that to get a $(q + 1 - t)$-regular (induced) subgraph of the incidence graph, we have to delete vertices corresponding to points and lines of a t-good structure. The larger the point- and lineset of a t-good structure is, the smaller $(q + 1 - t, 6)$-graph we get. Essentially two types of t-good structures are known when $t < \sqrt{q}$ (see also [8]):

- **Construction 1: completely deleted subplanes:** A point P (or a line l) is *completely deleted* if P and all the lines through P (l and all the points on l) are deleted. Take a (possibly degenerate) subplane with t points and t lines, and delete all its points and lines completely. This is obviously t-good. It is easy to prove that t-good structures consisting of t completely deleted points and lines are subplanes.
- **Construction 2: disjoint Baer-subplanes:** The points and the lines of t disjoint Baer-subplanes form a t-good structure. (Note that the points of a Baer-subplane meet every line in 1 or $\sqrt{q} + 1$ points.)

A little calculation shows that the size $|\mathcal{P}_0| = |\mathcal{L}_0|$ of the t-good structure we get from the above constructions is $tq + 1$, $tq - t + 3$, $tq - (t_1 - 1)t$ or $t(q + \sqrt{q} + 1)$ according to wether we use the first one with a degenerate subplane of type π_1 or the first one with a degenerate subplane of type π_2 or the first one with a non-degenerate subplane of order t_1 or the second one.

Hence the second construction is much better than any type of the first one, but we need the existence of Baer subplanes in the projective plane of order q. By the remarks at the end of the introduction, whenever q is square, we can use $\mathrm{PG}(2, q)$, where the second construction is possible.

When q is not a square, we have to use the first construction, which is always better with a degenerate subplane than with a non-degenerate one and for $t = 1$ the better one is to start from a subplane of type π_1; for $t \geq 3$ the better one is to start from a subplane of type π_2; finally, for $t = 2$ the two types give the same size for the arising graph.

In [8] the following were proved.

Theorem 2.2. *Let $(\mathcal{P}_0, \mathcal{L}_0)$ be a t-good structure in a projective plane of order q.*

(i) *If $t \leq 2\sqrt{q}$, then $|\mathcal{P}_0| = |\mathcal{L}_0| \leq t(q+\sqrt{q}+1)$ and in case of equality $\mathcal{P}_0$ meets every line in either t or $\sqrt{q} + t$ points. Moreover, if the plane is $PG(2, q)$ and $t < \frac{\sqrt[4]{q}}{2}$, then equality holds only for Construction 2;*
(ii) *If $t = 1$, then the 1-good structure is one of the above constructions;*
(iii) *If $t = 2$, the plane is $PG(2, q)$ and $q > 256$, then the 2-good structure is one of the above constructions.*

By the above result, one cannot hope better t-good structures than Construction 2 for not too large t when q is square. The purpose of this paper is to pove a similar result when q is a prime. We will prove the following.

Theorem 2.3. *Let p be a prime and let $(\mathcal{P}_0, \mathcal{L}_0)$ be a t-good structure in $PG(2,p)$, $t < \frac{\sqrt[4]{p}}{2}$. Then $|\mathcal{P}_0| = |\mathcal{L}_0| \leq \max(tq+1, tq+t-3)$.*

The proof will be given in the next section. We end this section by recalling a result of Blokhuis, Storme and Szőnyi which we will use in the proof. Recall that a t-fold blocking set is a set of points that meets every line in at least t points.

Theorem 2.4 (Blokhuis-Storme-Szőnyi [10]). *A t-fold blocking set in $PG(2,q)$, $t < \frac{\sqrt[4]{q}}{2}$, has at least $t(q+\sqrt{q}+1)$ points and in case of equality it is the union of t disjoint Baer-subplanes.*

3. Proof of Theorem 2.3

Throughout this section $(\mathcal{P}_0, \mathcal{L}_0)$ will denote a t-good structure in $\mathrm{PG}(2,q)$. For a while q will be an arbitrary power of a prime p, only at the end we will have to suppose q is prime. We will suppose $t \geq 2$ (Theorem 2.2 (ii) proves the $t = 1$ case).

Definition 3.1. *We call a line* bad *if it does not meet $\mathcal{P}_0$ in $t \mod p$ points. Dually, we call a point bad if it does not have $t \mod p$ deleted lines through it.*

Note that if we supposed that $q = p$ prime and not a power of p, then in the above definiton $t \mod p$ and exactly t would be the same.

Since most lines meet a t-good structure in exactly t points, the property "bad" is a relaxation of being not typical. Clearly, the bad lines and points have to be deleted lines and points.

Definition 3.2. *Let the* index *of a point P be the number of bad lines going through it. Dually the index of a line is the number of bad points on it.*

We use an algebraic lemma due to Szőnyi and Weiner concerning polynomials which let us conclude that there are no average indices.

Lemma 3.3 (Szőnyi-Weiner). *Let $f(X,Y) = a_0X^n + a_1(Y)X^{n-1} + \ldots + a_n(Y)$, $g(X,Y) = b_0X^{n-1} + b_1(Y)X^{n-2} + \ldots + b_{n-1}(Y)$ be polynomials of two variables over $GF(q)$ with $a_0 \neq 0$, $\deg(a_i) \leq i$, $\deg(b_i) \leq i$. For a $y \in \mathrm{GF}(q)$ let the degree of the greatest common divisor of $f(X,y)$ and $g(X,y)$ be $n - k_y$. Then for any $y \in GF(q)$ fixed, there exists a polynomial $D(Y)$ such that*

- *$\deg(D) \leq k_y(k_y-1)$;*
- *$D(Y) \not\equiv 0$;*
- *if for some $y' \in \mathrm{GF}(q)$ $n - k_{y'} \geq n - k_y$, then $(Y-y)^{k_y - k_{y'}} \mid D(Y)$.*

Proof. See [12]. □

Note that with the notations of the above result, for any $y \in GF(q)$, we have

$\sum_{y' \in GF(q): k_y > k_{y'}} (k_y - k_{y'}) \leq k_y(k_y-1)$. (This is simply saying that $D(Y)$ cannot have more roots counted with multiplicity than its degree.)

PROPOSITION 3.4. *Let k be the index of a point and denote by N the number of bad lines. Then*

$$k^2 - k(q+1) + N \geq 0.$$

Proof. Let us consider $\mathcal{P}_0$ as a collection of affine points and points on the line at infinity: $\mathcal{P}_0 = \{(x_i, y_i) \in \mathrm{AG}(2,q), i = 1, \ldots, s\} \cup \{(m_i), i = 1, \ldots, l\}$, where (m_i) is the common point of lines having slope $m_i \in$GF(q). The common point of vertical lines (∞) may be in $\mathcal{P}_0$ as well but it does not matter. We may suppose that the point in question $P = (m)$ is on the line at infinity and that the line at infinity is good. For a $z \in$GF(q), denote the index of the infinite point (z) by k_z. Consider the following polynomial:

$$H(M,B) = \sum_{i=1}^{s} \left(1 - (B + x_i M - y_i)^{q-1}\right) + \sum_{i=1}^{l} \left(1 - (M - m_i)^{q-1}\right).$$

Note that since $\forall x \in \mathrm{GF}(q)\colon x^{q-1} \neq 1 \iff x = 0$, the line $y = m_0 x + b_0$ meets $\mathcal{P}_0$ in $H(m_0, x_0) \mod p$ points. Replacing the variable M in $H(M,B)$ with a fixed field element m, $H(m,B)$ is a polynomial of one variable satisfying that the degree of $\gcd(B^q - B, H(m,B) - t)$ is exactly $q - k_m$. Let $R = \sum_{m' \in \mathrm{GF}(q)} k_{m'}$, this is the number of non-vertical bad lines (hence $R \leq N$). By Lemma 3.3,

$$k_m q - R = \sum_{m' \in \mathrm{GF}(q)} (k_m - k_{m'}) \leq \sum_{m' \colon k_m - k_{m'} \geq 0} (k_m - k_{m'}) \leq k_m(k_m - 1),$$

thus

$$0 \leq k_m^2 - (q+1)k_m + R \leq k_m^2 - (q+1)k_m + N.$$

□

Note that by duality the inequality 3.4 also holds for indices of lines.

PROPOSITION 3.5. *Let k be the index of a point and let $t \leq \frac{\sqrt{q}}{2}$. Then either $k \leq t$ or $k \geq q+1-t$.*

Proof. Using Proposition 3.4 and that $N \leq |\mathcal{L}_0| \leq t(q + \sqrt{q} + 1)$ by Theorem 2.2 (i), we see that

$$k^2 - k(q+1) + t(q + \sqrt{q} + 1) \geq 0.$$

After a little counting this quadratic inequality under the condition $t \leq \frac{\sqrt{q}}{2}$ implies the desired result.

□

Now we see that the points (and the lines) can be split into two groups: ones with small and others with large index. We shall consider the latter group now.

PROPOSITION 3.6. *If $t \leq \frac{\sqrt{q}}{2}$, then points with large index are completely deleted.*

Proof. Suppose to the contrary that there is a point P with large index and a non-deleted line l through P. Thus $|l \cap \mathcal{P}_0| = t$. We count the number of deleted lines according to the points of l. On each of the $q+1-t$ not deleted points of l we see exactly t deleted lines and at least $q+1-t$ more through P (since the bad lines are deleted). Thus

$|\mathcal{L}_0| \geq (q+1-t)t+q+1-t$, but this contradicts the upper bound in Theorem 2.2 (i) if $t \leq \frac{\sqrt{q}}{2}$. □

PROPOSITION 3.7. *Suppose $t \leq \frac{\sqrt{q}}{2}$. The points with large indices block the bad lines.*

Proof. Suppose to the contrary that there exists a bad line l on which every point has index at most c, where $c \leq t$, and suppose that there exists a point on l with index precisely c. Then the total number of bad lines is at most $(c-1)(q+1)+1$. Using the inequality 3.4 we obtain that

$$0 \leq c^2 - c(q+1) + (c-1)(q+1) + 1 = c^2 - q,$$

a contradiction since $t < \sqrt{q}$. □

Note that by the previous two propositions the existence of a point or a line with large index is equivalent. For instance a point with large index is completely deleted, that is, all lines through it are deleted, hence it is bad (the number of deleted lines through it is not $t \mod p$). On this bad point there should exist a line with large index.

PROPOSITION 3.8. *Suppose $t \leq \frac{\sqrt{q}}{2}$ and also $t < p$. The points and the lines with large indices form a (possibly degenerate) subplane.*

Proof. Suppose P_1 and P_2 are points with large indices. We will show that the line l connecting P_1 and P_2 has large index. By duality this will imply that the substructure of points and lines with large indices satisfy that there is a unique line through any two points and a unique point on any two lines, hence it is a subplane.

Suppose to the contrary that the index of l is at most t. Then there are at least $q+1-t$ good points on l, each having $t \mod p$, thus at least t deleted lines through them (this is where we use $t < p$). By Proposition 3.6, all lines through P_1 and P_2 are deleted. Since l is a common deleted line on each of these points we can deduce that $|\mathcal{L}_0| \geq (q+1-t)(t-1)+2q+1$, but this contradicts the upper bound in Theorem 2.2 (i) if $t \leq \frac{\sqrt{q}}{2}$. □

PROPOSITION 3.9. *Suppose $t < \frac{\sqrt[4]{q}}{2}$ and also $t < p$. If there are no points with large index, then we have Construction 2 (that is, $\mathcal{P}_0$ and $\mathcal{L}_0$ are the points and lines of t disjoint Baer subplanes).*

Proof. By Proposition 3.7 there are no bad lines, thus every line is a $t \mod p$ secant to $\mathcal{P}_0$. Since $t < p$, $\mathcal{P}_0$ must be a t-fold blocking set. Thus by Theorem 2.4 and Theorem 2.2 (i) we see that $(\mathcal{P}_0, \mathcal{L}_0)$ is the union of t disjoint Baer-subplanes. □

It might seem at first glance that Proposition 3.8 proves that (if we do have points and lines with large indices), we have Constrution 1 from the previous section. However, since the property of being deleted and having large index is not equivalent, it is not so easy. This is the moment when we have to suppose q is prime.

PROPOSITION 3.10. *Suppose $q = p$ is prime, $t \leq \frac{\sqrt{p}}{2}$ and that there are points and lines having large indices forming a degenerate subplane of type π_1 (see the end of the introduction). Then $|\mathcal{P}_0| = tp + 1$.*

Proof. Let l be a line (with large index) containing all points with large index. Take a point $P \in l$ with small index (such a point should exist, since otherwise we would have at least $1+(q+1)(q-t)$ deleted lines). Then every line through P except l has to be good, since bad lines are blocked by points with large indices (Proposition 3.7). Good lines meet $\mathcal{P}_0$ in t points (as $q=p$ now). The points of l are all deleted (by the dual of Proposition 3.6), thus there are exactly $q+1+(t-1)q=tq+1$ deleted points. □

PROPOSITION 3.11. *Suppose $q=p$ is prime, $t \le \frac{\sqrt{p}}{2}$ and that there are points and lines having large indices forming a degenerate subplane of type π_2 (see the end of the introduction). Then $(\mathcal{P}_0,\mathcal{L}_0)$ is a completely deleted degenerate subplane, that is, we have a particular case of Construction 1 of the previous section.*

Proof. Denote the points with large indices by $P_1,\dots,P_k$ in such a way, that $P_2,\dots,P_k$ are collinear, but P_1 is not. Here $k \ge 3$, since otherwise the degenerate subplane in question is of type π_1. Denote by l the line through $P_2,\dots,P_k$. Note that by Proposition 3.6 and its dual, all lines through any of the P_is and all points on any line joining two of the P_is are deleted.

First we prove that $k \le t$. By the above remark, counting only the number of lines through the P_is, we see at least $1+(k-1)q+(q+1)-(k-1)$ deleted lines. This would contradict Theorem 2.2 (i) for $k \ge t+1$.

Pick a point P on l with small index and denote by c the number of deleted points on the line PP_1 besides P and P_1. Counting the elements of $\mathcal{P}_0$ from P we get that $|\mathcal{P}_0| = q+1+(t-1)(q-1)+c+1$, as there are $q+1$ deleted points on l, $t-1$ further deleted points on the $q-1$ good lines through P not incident with P_1 and $c+1$ more points on the line PP_1 (note that by Proposition 3.7 the only deleted lines through P are l and PP_1). This implies that c must be independent from the coice of P. Counting the deleted points via the lines passing through P_1 we get that $|\mathcal{P}_0| = 1+(k-1)(q-1)+(q+1)+c(q+1-(k-1))$. Hence we have the following equation: $q+1+(t-1)(q-1)+c+1 = 1+(k-1)(q-1)+(q+1)+c(q+1-(k-1))$. After a little manipulation this implies $c(k-2)=(c+k-t)(q-1)$.

Since $c \le q-1$, we need to have $k-2 \ge c+k-t$, hence $c \le t-2$. Using this and the already proved bound on k, we get that $(c+k-t)(q-1) = c(k-2) \le (t-2)^2 < q-1$ by the assumptions. Hence the only possibility is that $c=0$ and $k=t$. It is easy to see that this implies that the pair $(\mathcal{P}_0,\mathcal{L}_0)$ is exactly the degenerate subplane of points and lines with large indices. □

Proof of Theorem 2.3

Suppose there are no points or lines with large indices. Then by Proposition 3.9 we need to have Baer subplanes, hence p is a square, contradiction.

If there are points and lines with large indices, then by Proposition 3.8 these form a subplane. Since $\mathrm{PG}(2,p)$ does not have non-degenerate subplanes, we can use either Proposition 3.10 or Proposition 3.11 to finish the proof.

4. Concluding remarks

Theorem 2.3 implies that under some assumptions, one should not hope better constructions from the method this paper is about. It should be stressed however, that our method could only handle the case when one is looking for induced subgraphs of the incidence graph of the projective plane. Recently Balbuena et. al. managed to find slightly better constructions than Construction 1 by considering not induced subgraphs as well (that is, one is allowed to delete incidences from the plane, not only points and lines). However, when q is square, still Construction 2 seems to be the best one.

The other weekness of the result is that it is only about planes of prime order. We conjecture however that the result can be generalized to $\mathrm{PG}(2, q)$ for an arbitrary prime power, provided that $t < \frac{\sqrt[4]{q}}{2}$ and also $t < p$. Note that in most of Section 3 we did not have to suppose that q is prime, only the proofs of the last two propositions before the proof of Theorem 2.3 do not seem to work without the assumption.

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