On maximal partial spreads in $PG(n, q)$

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Abstract

In this paper we construct maximal partial spreads in $PG(3, q)$ which are a $\log q$ factor larger than the best known lower bound. For $n \geq 5$ we also construct maximal partial spreads in $PG(n, q)$ of each size between $cnq^{n-2} \log q$ and $c'q^{n-1}$.

1 Introduction

A set of lines partitioning the points of $PG(n, q)$ is called a *spread*. It has to contain $\frac{q^{n+1}-1}{q^2-1}$ lines which shows that for the existence of such a structure n has to be odd. This condition is also sufficient, see [13]. A *partial spread* in $PG(n, q)$ is a set of pairwise disjoint lines. It is called *maximal* if it is not contained properly in a larger partial spread. In Hirschfeld's books [12], [13] the terminology *k-span* is used for a partial spread of size k. In the next theorem we summarize some results about the size of a maximal partial spread.

Result 1.1 *Let $\mathcal{S}$ be a maximal partial spread in $PG(n, q)$, which is not a spread. Then*

(i) For $n = 3$, $|\mathcal{S}| \geq 2q$ (Glynn [6]) and $|\mathcal{S}| \leq q^2 + 1 - b$, where $q + b$ is the size of the smallest non-trivial blocking set in $PG(2, q)$ (Bruen [3]);

(ii) For $n \geq 5$ odd, $|\mathcal{S}| \leq \frac{q^{n+1}-1}{q^2-1} - b$, where $q + b$ is the size of the smallest non-trivial blocking set in $PG(2, q)$ (Govaerts and Storme [7]);

(iii) For $n \geq 4$ even, $|\mathcal{S}| \leq q\frac{q^n-1}{q^2-1} - q + 1$ (Beutelspacher [1]).

There are some improvements on the upper bound in (i), see [2]. From the construction side, the following results are known.

Result 1.2 *(i) In $PG(3, q)$, $q \geq 7$ odd, there are maximal partial spreads of any size between $\frac{q^2+1}{2} + 6$ and $q^2 - q + 2$ (Heden [9], [10], [11]);*

(ii) In $PG(3, q)$, $q > q_0$ even, there are maximal partial spreads of any size between $\frac{5q^2+q+16}{8}$ and $q^2 - q + 2$ (Govaerts, Heden, Storme [8]; Jungnickel, Storme [14]).

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For dimensions greater than 3 we only mention a result we will use and which shows that Result 1.1 (iii) is sharp:

Result 1.3 *(Beutelspacher [1]) In $PG(n,q)$, n even there is a maximal partial spread of size $q\frac{q^n-1}{q^2-1}-q+1$. The q^2 points not covered by the lines of the spread form a plane minus a line.*

In this paper we construct partial spreads in various dimensions and of various sizes. Note that the smallest ones in dimensions at least 4 were already found by Beutelspacher [1], Theorems 5 and 6. Here it is formulated in a slightly more general form.

Result 1.4 *(Beutelspacher [1]) Let b denote the size of the smallest non-trivial (that is not containing a hyperplane) blocking set with respect to lines in $PG(n,q)$. Then all maximal partial spreads in $PG(n,q)$ of size below $\frac{b}{q+1}$ arise from the following construction: take a hyperplane H in $PG(n,q)$ and let $\mathcal{P}$ be a partial spread in H covering all but r of its points. Adding r lines which are pairwise disjoint and meet H in the points not covered by $\mathcal{P}$, we get a maximal partial spread of size $(\frac{q^n-1}{q-1}-r)/(q+1)+r$.*

In $PG(3,q)$ our smallest example is a $\log q$ factor worse than the theoretical lower bound in Result 1.1 (i). For dimension $n \geq 5$ we construct examples for each size between $cnq^{n-2}\log q$ and a size almost identical with the largest possible size of a maximal partial spread.

In some cases we will need some basic properties of inversive planes, i.e. $3-(q^2+1, q+1, 1)$ designs for which a good reference is [4].

2 Covers and fractional covers of hypergaphs

In this section we summarize some definitions and results about hypergraphs. At the end we formulate the geometric lemmas needed in later sections which use these hypergraph results. For the definition of a hypergraph and for more details see [5].

Let $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ be a hypergraph. The *covering number* $\tau(\mathcal{H})$ of $\mathcal{H}$ is the minimum size for a subset of $V(\mathcal{H})$ meeting every edge. A function $\phi : V(\mathcal{H}) \to \mathbf{R}_0^+$ is called a *fractional covering of* $\mathcal{H}$, if $\sum_{P\in E}\phi(P) \geq 1$ holds for every $E \in E(\mathcal{H})$. The *fractional covering number of* $\mathcal{H}$ is defined to be $\tau^*(\mathcal{H}) = \min_\phi(\sum_{P\in V(\mathcal{H})}\phi(P))$. An easy calculation shows that for a d-regular, r-uniform hypergraph $\tau^*(\mathcal{H}) = |E(\mathcal{H})|/d = |V(\mathcal{H})|/r$. It is also easy to see that $\tau \geq \tau^*$ always holds.

With a standard random method it is not very difficult to prove that $\tau(\mathcal{H}) \leq (1 + \log d)\tau^*(\mathcal{H})$, where d is the maximum degree of the hypergraph. Lovász proved that a greedy algorithm can also be used to find a covering of such a small cardinality, see [5].

We will use the following corollary for bipartite graphs.

Lemma 2.1 *Let G be a bipartite graph with bipartition $A \cup B$. Suppose that the degree of points in B is at least d. Then there is a set $A' \subseteq A$, $|A'| \leq |A|(1+\log(|B|))/d$, such that any $b \in B$ is adjacent to a point of A'.*

If one wants to improve the constants in the constructions that follow, one possibility is to use random choice and elementary counting with the corresponding binomial coefficients instead of the previous lemma. We will only give the details for the 3-dimensional case. Now we summarize the particular geometric situations where we will use Lemma 2.1.

Lemma 2.2 *Suppose that $C_1, ..., C_q$ are circles of an inversive plane through a point P and partitioning the rest of the points. For any k such that $6.1 \log q \leq k \leq q$ we can find indices $i_1, ..., i_k$ such that $C_{i_1} \cup C_{i_2} \cup ... \cup C_{i_k}$ meets all circles of the plane.*

(Note that if the interval $[6.1 \log q, q]$ is not empty (that is $6.1 \log q \leq q$), then $q > 32$.)

Proof: We construct a bipartite graph on some circles of the plane. Let the vertex class A contain the C_i-s and the class B contain all circles not through P. We draw an edge between two circles of different classes if they intersect. The degree of any point of B is at least $\frac{q+1}{2}$, so by Lemma 2.1, we can choose at most $\frac{q \cdot (1+\log(q^3-q))}{(q+1)/2} \leq 6.1 \log q$ from the C_i-s meeting any circle. Adding more C_i-s to this set we can find a desired set of any size between $6.1 \log q$ and q. □

Remark 2.3 It is not very difficult to see that for q odd and if our inversive plane is Miquelian, we really need $c \log q$ from the C_i-s to block all the circles. The points of the Miquelian inversive plane of order q, q odd, can be identified with points of $AG(2, q)$ together with an extra point, ∞. Circles through ∞ are lines of the plane, while the rest of the circles are the ones in $AG(2, q)$ with equation $(x - a)^2 + (y - b)^2 = r$, where $a, b \in GF(q)$, $0 \neq r \in GF(q)$. If we let P in Lemma 2.2 be the point ∞, then the C_i-s are parallel lines partitioning the points. Without loss of generality suppose that these are the vertical lines, that is lines with equation $x = c$. What we need is a set $c_1, ..., c_k$ with the property that for any a and $r \neq 0$ there exists a c_i such that $r - (c_i - a)^2$ is a square in $GF(q)$. In particular for $a = 0$, there is no r such that $r - c_i^2$ is a non-square for all i-s. Among these quadratic equations at least $k/2$ are different. This set of relations satisfies the conditions of [16] Lemma 1. Hence $k \geq c \log q$.

This also shows, that with an elementary random argument $6 \log q$ can be decreased to $2 \log_2 q$. We have $\binom{q}{k}$ different choices for the set $\{c_1, ..., c_k\}$. For a fixed a and $r \neq 0$ there are at most $(q - 1)/2$ c-s with $r - (c - a)^2$ a non-square, so at most $\binom{(q-1)/2}{k}$ choices are wrong. This gives altogether at most $(q^2 - q)\binom{(q-1)/2}{k}$ wrong choices, which is smaller than $\binom{q}{k}$ for $k \geq 2 \log q$.

Lemma 2.4 *Suppose that $U_1, ..., U_{(q^{n-1}-1)/(q^2-1)}$ are 3-dimensional subspaces in $PG(n, q)$ through a fixed line l and partitioning the rest of the points (hence $n \geq 5$ odd). Then for any k such that $4nq^{n-4} \log q \leq k \leq (q^{n-1} - 1)/(q^2 - 1)$ one can choose k from the U_i-s in such a way that they meet any line except for those contained in one of the rest of the U_i-s.*

Proof: Similarly to the previous proof, we construct a bipartite graph. The vertex class A will consist of the U_i-s, while the class B will consist of the lines not contained in any of the U_i-s and not meeting l. We put an edge between two vertices if the corresponding 3-dimensional subspace and line meet each other. It is easy to see that the degree of any vertex in B is $q + 1$. The size of A is $\frac{q^{n-1}-1}{q^2-1} \leq 2q^{n-3}$ and the size of B is less than the number of lines in the space, which is smaller than $4q^{2n-2}$, so by Lemma 2.1 we find at most $2q^{n-3}\log(4q^{2n-2})/(q+1) \leq 4nq^{n-4}\log q$ U_i-s with the desired property. Adding more U_i-s, the set can be larger. □

Lemma 2.5 *Suppose that $U_1, ..., U_{(q^{n-1}-q)/(q^2-1)-q+1}$ are 3-dimensional subspaces in $PG(n,q)$ through a fixed line l forming a partial spread in the derived geometry $PG(n,q)/l$ coming from Lemma 1.3. Then for any k such that $4nq^{n-4}\log q \leq k \leq \frac{q^{n-1}-q}{q^2-1} - q + 1$ one can choose k from the U_i-s in such a way that they meet any line except for those contained in one of the rest of the U_i-s.*

Proof: The proof is almost identical with the previous one. Before constructing the very same graph note that, according to 1.3, the points not covered by the U_i-s form a 4-dimensional subspace minus one of the U_i-s, U_1 say. If we let U_1 be one of the k chosen subspaces, then we do not have to worry about lines contained in the 4-dimensional subspace in question. The rest of the lines meet this subspace in at most one point, so the only difference from the previous proof is that here the vertices in B have degree at least q instead of $q+1$, and the size of A is a bit smaller (which makes the estimate better), the rest is the same. □

3 Maximal partial spreads in $PG(3,q)$

In this section we construct maximal partial spreads of size $nq+1$, where $c\log q \leq n \leq q$. Before the formulation, we need some definitions and known results. For more about these and for proofs see [12].

Suppose that l_1, l_2 and l_3 are pairwise disjoint lines in $PG(3,q)$. Then there are exactly $q+1$ lines meeting all of them, $l'_1, ..., l'_{q+1}$, say. Such a system is called a *regulus*. The number of lines meeting all the l'_i-s is again $q+1$, this set is called the *opposite regulus* of the regulus $\{l'_1, ..., l'_{q+1}\}$. This means that through any three pairwise skew lines there is a unique regulus in $PG(3,q)$. The lines of a regulus and the lines of the opposite regulus cover the same $(q+1)^2$ points of $PG(3,q)$. If R is a regulus then R^{opp} will denote the opposite regulus.

A spread of $PG(3,q)$ is called *regular*, if it contains the whole regulus determined by any three of its lines. This implies that the set of lines and reguli of a regular spread form a Miquelian inversive plane. A spread is regular if and only if for any line l not contained in it, the lines meeting l form a regulus. Regular spreads exist for any prime power q.

Construction 3.1 *Let $\mathcal{S}$ denote a regular spread in $PG(3,q)$ and l an arbitrary line in $\mathcal{S}$. Since the lines and reguli of $\mathcal{S}$ form an inversive plane, for any k such that $6.1\log q \leq k \leq q$ we can use Lemma 2.2 to find reguli $\mathcal{R}_1, ..., \mathcal{R}_k, \mathcal{R}_{k+1}, ..., \mathcal{R}_q$ with the following properties:*

(i) For any $i \neq j$: $\mathcal{R}_i \cap \mathcal{R}_j = \{l\}$;

(ii) any line from $\mathcal{S} \setminus \{l\}$ is contained in exactly one of the $\mathcal{R}_i$-s;

(iii) taking the union of lines contained in $\mathcal{R}_1, ..., \mathcal{R}_k$ and removing l we get a point set meeting every line of $PG(3,q)$ except for those in $\mathcal{R}_{k+1}, ..., \mathcal{R}_q$.

(Property (iii) comes from the fact that for any line m of $PG(3,q)$, $m \notin \mathcal{S}$, the lines of $\mathcal{S}$ meeting m form a regulus, that is a circle of the inversive plane in Lemma 2.2.) Choose lines $l_1, ..., l_{q+1}$ from $\cup_{i=k+1}^{q+1} \mathcal{R}_i^{opp}$ in such a way that they cover each point of l and for every $k+1 \leq i \leq q+1$ there is at least one line chosen from $\mathcal{R}_i^{opp}$.

Then the set

$$\cup_{i=1}^{k} \mathcal{R}_i \cup \{l_1, ..., l_{q+1}\} \setminus \{l\}$$

is a maximal partial spread of size $(k+1)q+1$.

Proof: First we prove that the lines in question are pairwise disjoint. Since the lines of a regulus and those of its opposite regulus cover the same points, all we have to check is that $l_i \cap l_j = \emptyset$ for any $i \neq j$. Suppose to the contrary that $l_i \in \mathcal{R}_u^{opp}$ and $l_j \in \mathcal{R}_v^{opp}$ meet in a point P. This means that both $\mathcal{R}_u$ and $\mathcal{R}_v$ has a line through P contradicting the fact that these lines are coming from the spread $\mathcal{S}$.

Finally our partial spread is maximal, since by (ii) and (iii), a possible extra line would have to be contained in one of the $\mathcal{R}_i$-s with $k+1 \leq i \leq q$, but we have a line from every opposite regulus. □

This construction, together with the proof, is almost identical with the one of Beutelspacher [1]. The only extra trick is that $\mathcal{R}_1, ..., \mathcal{R}_k$ is not chosen arbitrarily. This enables one to construct maximal partial spreads as small as $7q \log q$, while the smallest one in [1] has size roughly $q^2/2$. For q odd it can be shown that Construction 3.1 cannot produce much smaller examples, see the remark after Lemma 2.2. This remark also shows that using a random selection we can construct partial spreads of size $(2 \log q + 1)q + 1$.

4 Partial spreads in $PG(n,q)$, $n \geq 5$ odd

In this section we construct maximal partial spreads in $PG(n,q)$, $n \geq 5$ odd, of size s for any $cnq^{n-2} \log q \leq s \leq \frac{q^{n+1}-1}{q^2-1} - q + 1$.

Construction 4.1 *Fix a line l in $PG(n,q)$, $n \geq 5$ odd. Let $U_1, \ldots, U_{(q^{n-1}-1)/(q^2-1)}$ be 3-dimensional subspaces through l partitioning the rest of the points of the geometry (this is equivalent to a spread in the derived geometry $PG(n,q)/l$). Let $4nq^{n-4} \log q \leq k \leq \frac{q^{n-1}-1}{q^2-1}$. According to Lemma 2.4, we can suppose that there are k U_i-s, $U_1, ..., U_k$ say, meeting every line not contained in one of the chosen $\frac{q^{n-1}-1}{q^2-1}$ subspaces. Let $\mathcal{S}_i$ be a spread in U_i through l $(i = 1, \ldots, k)$ and $\mathcal{P}_i$ be a maximal partial spread in U_i $(i = k+1, \ldots, \frac{q^{n-1}-1}{q^2-1})$. The union of lines in the $\mathcal{S}_i$-s and the $\mathcal{P}_i$-s is a maximal partial spread in $PG(n,q)$.*

It is straightforward that our construction really gives a maximal partial spread. Since there are lots of different size maximal partial spreads as candidates for the $\mathcal{P}_i$-s, this implies the following:

Theorem 4.2 *For any odd q and for $q > q_0$ even, in $PG(n,q)$, $n \geq 5$ odd, there are maximal partial spreads of any size between $9nq^{n-2} \log q$ and $\frac{q^{n+1}-1}{q-1} - q + 1$.*

Proof: Consider Construction 4.1 with $k = 4nq^{n-4} \log q$ and the $\mathcal{P}_i$-s the smallest partial spreads constructed in 3.1 (that is of size $7q \log q$). A little calculation shows that this gives a maximal partial spread of size at most $8nq^{n-2}\log q$ in $PG(n,q)$. Now changing one of the $\mathcal{P}_i$-s to a maximal partial spread of size around $5q^2/16$ (see Result 1.2), we find a maximal partial spread of size at most $9nq^{n-2}\log q$. From this size we can go up one by one by changing one of the $\mathcal{P}_i$-s to another example coming from 1.2, or increasing k if necessary. The largest size we can have (apart from $k=\frac{q^{n-1}-1}{q^2-1}$ giving a spread) is the case $k=\frac{q^{n-1}-1}{q^2-1}-1$ and the only partial spread used being of size q^2-q+2 (see Result 1.2). This gives a maximal partial spread of size $k=\frac{q^{n+1}-1}{q^2-1}-q+1$. □

5 Maximal partial spreads in $PG(n,q)$, $n \geq 6$ even

The previous construction cannot be repeated literally, since if n is even, then it is not possible to cover all points of $PG(n,q)$ with 3-dimensional subspaces through a line l. But almost the same trick works, since we can cover almost all of the points this way using Theorem 1.3.

Construction 5.1 *Fix a line l in $PG(n,q)$, $n \geq 6$ even. Let $U_1, \dots U_{(q^{n-1}-q)/(q^2-1)-q+1}$ be 3-dimensional subspaces through l forming a maximal partial spread in the derived geometry $PG(n,q)/l$ coming from Theorem 1.3. Let $4nq^{n-3}\log q \leq k \leq \frac{q^{n-1}-q}{q^2-1}-q+1$. According to 2.5, we can suppose that there are k U_i-s, $U_1, ..., U_k$ say, meeting every line not contained in one of the chosen $\frac{q^{n-1}-q}{q^2-1}-q+1$ subspaces. Let $\mathcal{S}_i$ be a spread in U_i through l $(i=1,\dots,k)$ and $\mathcal{P}_i$ be a maximal partial spread in U_i $(i=k+1,\dots,\frac{q^{n-1}-q}{q^2-1}-q+1)$. The union of lines in the $\mathcal{S}_i$-s and the $\mathcal{P}_i$-s is a maximal partial spread in $PG(n,q)$.*

Again it is straightforward that the construction works. Similarly to the previous section, we have the following:

Theorem 5.2 *For arbitrary odd q and for $q > q_0$ even, in $PG(n,q)$, $n \geq 6$ even, there are maximal partial spreads of any size between $9nq^{n-2}\log q$ and $\frac{q^{n+1}-q}{q^2-1}-q^3+q^2-2q+2$.*

Proof: The proof is the same as the proof of Theorem 4.2. Here the largest value of the interval comes from $k=\frac{q^{n-1}-q}{q^2-1}-q$ and the only partial spread used being of size q^2-q+2. □

References

[1] A. Beutelspacher, Blocking sets and partial spreads in finite projective spaces, *Geometriae Dedicata* **9** (1980) 425-449.

[2] A. Blokhuis and K. Metsch, On the size of a maximal partial spread, *Des. Codes Cryptogr.* **3** (1993), 187-191.

[3] A. Bruen, Partial spreads and replaceable nets, *Canad J. Math.* **23** (1971), 381-392.

[4] P. Dembowski, Finite Geometries, Springer, Berlin, 1968 and 1997.

[5] Z. Füredi, Matchings and covers in hypergraphs, *Graphs and Combin.* **4** (1988), 115-206.

[6] D.G. Glynn, A lower bound for maximal partial spreads in $PG(3,q)$, *Ars Combin.* **13** (1982), 39-40.

[7] P. Govaerts and L. Storme, On a particular class of minihypers and its applications, I: The result for general q. *Des. Codes Cryptogr.*, to appear.

[8] P. Govaerts, L. Storme and O. Heden, On the spectrum of sizes of maximal partial spreads in $PG(3,q)$, q even, manuscript.

[9] O. Heden, Maximal partial spreads and the modular n-queen problem, *Discrete Math.* **120** (1993), 75-91.

[10] O. Heden, Maximal partial spreads and the modular n-queen problem II, *Discrete Math.* **142** (1995), 97-106.

[11] O. Heden, Maximal partial spreads and the modular n-queen problem III, *Discrete Math.* **243** (2002), 135–150.

[12] J.W.P. Hirschfeld, Finite projective spaces of three dimensions, *Oxford Univ. Press* (1985).

[13] J.W.P. Hirschfeld and J.A. Thas, General Galois geometries, *Oxford Univ. Press* (1991).

[14] D. Jungnickel and L. Storme, Maximal partial spreads in $PG(3,4)$ and maximal sets of mutually othogonal Latin squares of order 16, *Discr. Math.*, to appear.

[15] T. Szőnyi, Blocking sets in finite planes and spaces, *Ratio Mathematica* **5** (1992), 93-106.

[16] T. Szőnyi, Note on the existence of large minimal blocking sets in Galois planes, *Combinatorica* **12** (1992), 227-235.

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