On divisibility properties of integers of the form ab+1

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Abstract

We give estimates for the cardinality of the sets $\mathcal{A}$, $\mathcal{B} \subseteq \{1, 2, \dots, N\}$ with the property that $ab+1$ is squarefree for all $a \in \mathcal{A}$, $b \in \mathcal{B}$.

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1 Introduction

P. Erdős and A. Sárközy [1] studied the following question: how large can $|\mathcal{A}|$ be if $A \subseteq \{1, 2, \dots, N\}$ and $a + a'$ is squarefree for all $a \in \mathcal{A}$, $a' \in \mathcal{A}$? G. N. Sárközy [7] extended the problem to the case of two different sequences and k^{th}-power-free sums. He proved the following result: let k and $N > N_0(k)$ be positive integers, $\mathcal{A}, \mathcal{B} \subseteq \{1, 2, \dots, N\}$. If $|\mathcal{A}|\,|\mathcal{B}| > c(k)N^{2-\frac{1}{k}}$, then there exists a prime p such that $p^k \le \sqrt{N}$ and $p^k \mid a + b$ for some $a \in \mathcal{A}$, $b \in \mathcal{B}$.

He also gave a lower bound: there exist sequences $\mathcal{A}, \mathcal{B} \subseteq \{1, 2, \ldots, N\}$ such that $p^k \nmid a + b$ for every prime p, $a \in \mathcal{A}$, $b \in \mathcal{B}$, and

$$|\mathcal{A}|\,|\mathcal{B}| > \frac{(k-1)N(\log N)^{k-1}}{65e^4(3k)^{k-1}}.$$

(See [2], [3] and [4] for other somewhat related results.)

A. Sárközy [6] asked the multiplicative analog of the problem: how large set $\mathcal{A}$ can be selected from $\{1, 2, \ldots N\}$ so that $aa' + 1$ is squarefree for all $a \in \mathcal{A}, a' \in \mathcal{A}$? In this paper we will prove the following:

Theorem 1 *If $\mathcal{A}, \mathcal{B} \subseteq \{1, 2, \ldots, N\}$ and $ab + 1$ is squarefree for all $a \in \mathcal{A}$, $b \in \mathcal{B}$ then we have:*

$$|\mathcal{A}|\,|\mathcal{B}| \ll N^{\frac{3}{2}}(\log N)^2.$$

The proof will use the large sieve. Next we will give a lower bound in the case $|\mathcal{A}| = |\mathcal{B}|$.

Theorem 2 *a) There exist sequences $\mathcal{A}, \mathcal{B} \subseteq \{1, 2, \ldots, N\}$ such that*

$$|\mathcal{A}| = |\mathcal{B}| \gg (\log N)^2$$

and $a+b$ is squarefree for all $a \in \mathcal{A}$, $b \in \mathcal{B}$.

b) There exist sequences $\mathcal{A}, \mathcal{B} \subseteq \{1, 2, \ldots, N\}$ such that

$$|\mathcal{A}| = |\mathcal{B}| \gg (\log N)^2$$

and $ab+1$ is squarefree for all $a \in \mathcal{A}$, $b \in \mathcal{B}$.

In the special case $\mathcal{A} = \mathcal{B}$ we will prove

Theorem 3 *a) There exists a sequence* $\mathcal{A} \subseteq \{1, 2, \ldots, N\}$ *such that*

$$|\mathcal{A}| \gg \log N$$

and $a + a'$ *is squarefree for all* $a \in \mathcal{A}$, $a' \in \mathcal{A}$.

b) There exists a sequence $\mathcal{A} \subseteq \{1, 2, \ldots, N\}$ *such that*

$$|\mathcal{A}| \gg \log N$$

and $aa' + 1$ *is squarefree for all* $a \in \mathcal{A}$, $a' \in \mathcal{A}$.

Theorem 3a) is not new and, indeed, it is due to P. Erdős and A. Sárközy [1]. We will give another proof for this fact. The interesting feature of these theorems is that the proofs are based on graph theory.

Throughout the paper the following notations will be used: $\Lambda(n)$ is the Mangoldt function, $\pi(x)$ is the number of prime numbers not exceeding x, $\tau(x)$ is the number of positive divisors of x, and $K_{r,r}$ is the complete bipartite graph with vertex sets of cardinality r.

2 Proofs

Proof of Theorem 1

Let us assume that for a positive integer m there are $f(m)$ residue classes mod m which contain no element of $\mathcal{A}$ and there are $g(m)$ residue classes mod m which contain no element of $\mathcal{B}$. Denote by $\rho(m)$ the number of residue classes mod m which contain an element of $\mathcal{A}$. We shall need the following lemma:

Lemma 1 *There exists at most one prime p such that $\sqrt[4]{\frac{N}{2}} \le p \le \sqrt[4]{N}$ and $\rho(p^2) < p^c$ where $c = 2\frac{\log|\mathcal{A}| - \log 2}{\log N}$.*

Proof of Lemma 1

Assume that there exist two primes: p_1, p_2, which satisfy the condition of Lemma 1. It follows from the Chinese Remainder Theorem that there are at most $\rho(p_1^2)\rho(p_2^2)$ residue classes mod $p_1^2p_2^2$ which contain an element of $\mathcal{A}$. Using this and $\sqrt[4]{\frac{N}{2}} \le p \le \sqrt[4]{N}$ we get:

$$|\mathcal{A}| \le \rho(p_1^2)\rho(p_2^2)\left\lceil \frac{N}{p_1^2p_2^2} \right\rceil \le 2\rho(p_1^2)\rho(p_2^2) < 2(p_1p_2)^c \le 2N^{\frac{c}{2}} = |\mathcal{A}|\,.$$

But this is a contradiction.

To prove Theorem 1 we may assume that $|\mathcal{A}| \ge 2\sqrt{N}(\log N)^2$ and $|\mathcal{B}| \ge 2\sqrt{N}(\log N)^2$ because otherwise the theorem is trivial. Let $c_1 > 0$ be a constant, we will specify its value later. Let $N > e^{c_1^2} + 1$. If $\sqrt[4]{\frac{N}{2}} \le p \le \sqrt[4]{N}$ then we have:

$$2\frac{\log|\mathcal{A}| - \log 2}{\log N} \ge 2\frac{\log\left(\sqrt{N}(\log N)^2\right)}{\log N} = 1 + \frac{4\log\log N}{\log N} \ge$$

$$\ge 1 + \frac{\log c_1}{\log \sqrt[4]{\frac{N}{2}}} \ge \frac{\log(pc_1)}{\log p}.$$

Using Lemma 1 we get that there is at most one prime $p_{\mathcal{A}}$, such that $\sqrt[4]{\frac{N}{2}} \le p_{\mathcal{A}} \le \sqrt[4]{N}$ and $\rho(p_{\mathcal{A}}^2) < c_1 p_{\mathcal{A}}$. Therefore if $\sqrt[4]{\frac{N}{2}} \le p \le \sqrt[4]{N}$ is a prime and $p \ne p_{\mathcal{A}}$ then we have $f(p^2) \le p^2 - c_1p$. We can prove similarly: there is at most one prime $p_{\mathcal{B}}$, such that if $\sqrt[4]{\frac{N}{2}} \le p \le \sqrt[4]{N}$ is a prime and $p \ne p_{\mathcal{B}}$ then we have $g(p^2) \le p^2 - c_1p$.

$f(p^2)+g(p^2)\ge p^2-p$ for all primes p because if x is a residue class mod p^2 which contains an element of $\mathcal{A}$ and $(x,p^2)=1$ then the residue class x^* where $xx^*\equiv -1 \bmod p^2$ contains no element of $\mathcal{B}$. Therefore $\rho(p^2)-p\le g(p^2)$ so $p^2-f(p^2)-p\le g(p^2)$.

G. N. Sárközy [7] (see especially formula (3) on p. 274) proved the following variant of the large sieve: if $\mathcal{A},\mathcal{B}\subseteq\{1,2,\dots,N\}$, $\mathcal{M}\subseteq\{1,2,\dots,[\sqrt{N}]\}$ such that the elements of $\mathcal{M}$ are pairwise coprime, then we have

$$|\mathcal{A}|\le\frac{2N}{\sum\limits_{m\in\mathcal{M}}\frac{f(m)}{m-f(m)}},\ \ |\mathcal{B}|\le\frac{2N}{\sum\limits_{m\in\mathcal{M}}\frac{g(m)}{m-g(m)}}.$$

Multiplying these equations:

$$|\mathcal{A}|\,|\mathcal{B}|\le\frac{4N^2}{\sum\limits_{m\in\mathcal{M}}\frac{f(m)}{m-f(m)}\sum\limits_{m\in\mathcal{M}}\frac{g(m)}{m-g(m)}}. \tag{1}$$

Let $\mathcal{M}=\{p^2:\sqrt[4]{\frac{N}{2}}\le p\le\sqrt[4]{N},\ p \text{ prime},\ p\ne p_{\mathcal{A}} \text{ and } p\ne p_{\mathcal{B}}\}$.

In order to estimate the denominator of the right hand side of inequality (1) we shall need two lemmas.

Lemma 2 *If $f,g\ge 1$ and $f+g\ge p^2-p$ then*

$$\left(\frac{f}{p^2-f}+\frac{1}{p+1}\right)\left(\frac{g}{p^2-g}+\frac{1}{p+1}\right)\ge\left(\frac{p}{p+1}\right)^2.$$

Lemma 3 *If $0\le f,g\le p^2-c_1p$ then*

$$\frac{f}{p^2-f}+\frac{g}{p^2-g}\le\frac{2}{c_1}p.$$

Proof of Lemma 2

Let $a = \frac{f}{p^2-f}$ and $b = \frac{g}{p^2-g}$. So: $\frac{p^2 a}{a+1} = f$, $\frac{p^2 b}{b+1} = g$. By $f + g \geq p^2 - p$ we have

$$\frac{a}{a+1} + \frac{b}{b+1} \geq \frac{p-1}{p}.$$

From this:

$$2ab + a + b \geq \frac{p-1}{p}(ab + a + b + 1),$$

which is equivalent to

$$\left(a + \frac{1}{p+1}\right)\left(b + \frac{1}{p+1}\right) \geq \left(\frac{p}{p+1}\right)^2.$$

This completes the proof of the lemma.

Proof of Lemma 3

$$\frac{f}{p^2-f} + \frac{g}{p^2-g} = -2 + p^2\left(\frac{1}{p^2-f} + \frac{1}{p^2-g}\right) \leq \frac{2}{c_1}p.$$

Thus the lemma is proved.

Now we are ready to complete the proof of Theorem 1. The denominator of the right hand side of inequality (1) is:

$$S \stackrel{\text{def}}{=} \sum_{m\in\mathcal{M}} \frac{f(m)}{m-f(m)} \sum_{m\in\mathcal{M}} \frac{g(m)}{m-g(m)} = \left(\sum_{p^2\in\mathcal{M}} \left(\frac{f(p^2)}{p^2-f(p^2)} + \frac{1}{p+1}\right) - \right.$$

$$\left. - \sum_{p^2\in\mathcal{M}} \frac{1}{p+1}\right)\left(\sum_{p^2\in\mathcal{M}} \left(\frac{g(p^2)}{p^2-g(p^2)} + \frac{1}{p+1}\right) - \sum_{p^2\in\mathcal{M}} \frac{1}{p+1}\right) =$$

$$= \sum_{p^2\in\mathcal{M}} \left(\frac{f(p^2)}{p^2-f(p^2)} + \frac{1}{p+1}\right) \sum_{p^2\in\mathcal{M}} \left(\frac{g(p^2)}{p^2-g(p^2)} + \frac{1}{p+1}\right) -$$

$$- \left(\sum_{p^2\in\mathcal{M}} \frac{1}{p+1}\right)\left(\sum_{p^2\in\mathcal{M}} \left(\frac{f(p^2)}{p^2-f(p^2)} + \frac{g(p^2)}{p^2-g(p^2)}\right)\right) - \left(\sum_{p^2\in\mathcal{M}} \frac{1}{p+1}\right)^2.$$

Then for $N \geq N_0$ we have:

$$\sum_{p^2 \in \mathcal{M}} \frac{1}{p+1} < \sum_{\substack{\sqrt[4]{\frac{N}{2}} \leq p \leq \sqrt[4]{N}, \\ p\ prime}} \frac{1}{\sqrt[4]{\frac{N}{2}}} < \frac{\pi(\sqrt[4]{N})}{\sqrt[4]{\frac{N}{2}}} < \frac{2}{\log \sqrt[4]{N}}.$$

Using the Cauchy-Schwarz inequality and Lemma 2 we have:

$$\sum_{p^2 \in \mathcal{M}} \left(\frac{f(p^2)}{p^2 - f(p^2)} + \frac{1}{p+1} \right) \sum_{p^2 \in \mathcal{M}} \left(\frac{g(p^2)}{p^2 - g(p^2)} + \frac{1}{p+1} \right) \geq$$

$$\geq \left(\sum_{p^2 \in \mathcal{M}} \sqrt{\left(\frac{f(p^2)}{p^2 - f(p^2)} + \frac{1}{p+1} \right) \left(\frac{g(p^2)}{p^2 - g(p^2)} + \frac{1}{p+1} \right)} \right)^2 \geq$$

$$\geq \left(\sum_{\substack{\sqrt[4]{\frac{N}{2}} \leq p \leq \sqrt[4]{N}, \\ p \neq p_{\mathcal{A}}, p_{\mathcal{B}},\ p\ prime}} \frac{p}{p+1} \right)^2 = \left(\left(1 - \sqrt[4]{\frac{1}{2}} + o(1) \right) \frac{\sqrt[4]{N}}{\log \sqrt[4]{N}} \right)^2.$$

By Lemma 3:

$$\sum_{p^2 \in \mathcal{M}} \left(\frac{f(p^2)}{p^2 - f(p^2)} + \frac{g(p^2)}{p^2 - g(p^2)} \right) \leq \sum_{p^2 \in \mathcal{M}} \frac{2}{c_1} p = \frac{1 + o(1)}{c_1} \int_{\frac{\sqrt[4]{\frac{N}{2}}}{\log \sqrt[4]{\frac{N}{2}}}}^{\frac{\sqrt[4]{N}}{\log \sqrt[4]{N}}} 2x \log x dx =$$

$$= \frac{1 + o(1)}{c_1} \left(1 - \sqrt{\frac{1}{2}} \right) \frac{\sqrt{N}}{\log \sqrt[4]{N}}.$$

It follows that

$$S \geq \left(\left(1 - \sqrt[4]{\frac{1}{2}} \right)^2 - \frac{2}{c_1} \left(1 - \sqrt{\frac{1}{2}} \right) + o(1) \right) \frac{\sqrt{N}}{\left(\log \sqrt[4]{N} \right)^2}.$$

Now we fix the value of c_1 so that $c_1 > \frac{2\left(1 - \sqrt{\frac{1}{2}}\right)}{\left(1 - \sqrt[4]{\frac{1}{2}}\right)^2}$. Then $S \gg \frac{\sqrt{N}}{\left(\log \sqrt[4]{N}\right)^2}$. Therefore $|\mathcal{A}| \, |\mathcal{B}| \ll N^{\frac{3}{2}} (\log N)^2$. This completes the proof.

Let p_i denote the i-th prime number. Let N be a large positive integer, define the positive integer K by

$$\prod_{i=1}^{K-1} p_i < N^{\frac{1}{6}} \leq \prod_{i=1}^{K} p_i, \tag{2}$$

and put $P = \prod_{i=1}^{K} p_i$. Then by the prime number theorem we have

$$\log P = \sum_{i=1}^{K} \log p_i \sim \sum_{n \leq p_K} \Lambda(n) \sim p_K \tag{3}$$

so that, in view of (2), for large N

$$N^{\frac{1}{6}} \leq P < N^{\frac{1}{5}}. \tag{4}$$

Let $f(a, b)$ be $a + b$ in case a) and $ab + 1$ in case b) in the proof of Theorem 2 and Theorem 3.

Proof of Theorem 2

Lemma 4 *Assume that the simple graph G with n vertices and m edges contains no $K_{r,r}$. Then:*

$$2m < \sqrt[r]{r-1} \cdot n^{2-\frac{1}{r}} + (r-1)n.$$

Proof of Lemma 4

See [5, ch.10, problem 37].

We will need the following consequence of the lemma:

Lemma 5 *There exists an n_0 such that if $n \geq n_0$ and G is a graph with n vertices and m edges and $r \leq min \left\{ \frac{2m}{n}, \frac{\binom{n}{2}-m}{n}, \frac{n^2 logn}{7(\binom{n}{2}-m)} - \frac{3logn}{7} \right\}$, then G contains a $K_{r,r}$.*

Proof of Lemma 5

Assume that the statement is not true. By Lemma 4 we have:

$$2m < \sqrt[r]{r-1} \cdot n^{2-\frac{1}{r}} + (r-1)n.$$

Using $r \leq \frac{2m}{n}$ we obtain:

$$(2m - (r-1)n)^r < (r-1)n^{2r-1}.$$

From this and $r - 1 \leq \sqrt{\frac{\binom{n}{2}-m}{n} \frac{n^2 \log n}{7\left(\binom{n}{2}-m\right)}} \leq n^{\frac{4}{7}}$, $rn \leq \binom{n}{2} - m$ and $1 + x < e^x$ we get:

$$n^{\frac{3}{7}} \leq \frac{n}{r-1} < \left(\frac{n^2}{2m-(r-1)n}\right)^r = \left(1 + \frac{2\left(\binom{n}{2}-m\right)+rn}{n^2 - 2\left(\binom{n}{2}-m\right)-rn}\right)^r <$$

$$< \left(1 + \frac{3\left(\binom{n}{2}-m\right)}{n^2 - 3\left(\binom{n}{2}-m\right)}\right)^r < e^{r\frac{3\left(\binom{n}{2}-m\right)}{n^2-3\left(\binom{n}{2}-m\right)}}.$$

Therefore:

$$\frac{n^2 \log n}{7\left(\binom{n}{2}-m\right)} - \frac{3 \log n}{7} < r,$$

which is a contradiction.

In order to prove Theorem 2 we consider the graph G whose vertices are $1, 2P+1, 4P+1, \ldots, \left[\frac{N-1}{2P}\right] 2P+1$. Let the edge e join the vertices x and y if and only if $f(x,y)$ is squarefree. If $x, y \in \{1, 2P+1, 4P+1, \ldots, \left[\frac{N-1}{2P}\right] 2P+1\}$, then assign the value $f(x,y)$ to the pair (x,y).

By the definition of P, there is no pair (x,y) whose value is divisible by a prime square $q^2 \leq p_K^2$. The number of pairs whose value is divisible by a prime square $q^2 > p_K^2$ is

$$\sum_{k=0}^{\left[\frac{N-1}{2P}\right]} \sum_{\substack{l=0 \\ q^2 | f(2Pk+1, 2Pl+1)}}^{\left[\frac{N-1}{2P}\right]} 1 \leq \sum_{k=0}^{\left[\frac{N-1}{2P}\right]} \left(\frac{\left[\frac{N-1}{2P}\right]}{q^2} + 1\right) = \frac{\left[\frac{N-1}{2P}\right]^2 + \left[\frac{N-1}{2P}\right]}{q^2} + \left[\frac{N-1}{2P}\right] + 1. \tag{5}$$

Since $x+y \le 2N$ in case a) there is no pair (x,y) whose value has a divisor greater than $\frac{N^2}{P^2}$.

In case b) at most $\left(\pi(N) - \pi\left(\frac{N}{P}\right)\right) P^2 < P^2 N$ numbers $xy+1$ exist which have a prime square divisor exceeding $\frac{N^2}{P^2}$. By Wigert's theorem [9] if N is large enough and $xy+1$ is given then x can take at most $\tau(xy) \le \max_{m \le N^2} \tau(m) \le N^{\frac{2}{\log\log N}}$ numbers. Therefore in case b) at most $P^2 N^{1+\frac{2}{\log\log N}}$ pairs (x,y) exist whose value is divisible by a prime square greater than $\frac{N^2}{P^2}$. By (4) we have $P^2 N^{1+\frac{2}{\log\log N}} < \frac{N^2}{P^2 \log N}$.

Thus by (3), (4) and (5) the number of pairs (x,y) whose value is not squarefree is less than

$$\frac{N^2}{P^2 \log N} + \sum_{\substack{p_{K+1} \le q \le \frac{N}{P} \\ q \ prime}} \left(\frac{\left[\frac{N-1}{2P}\right]^2}{q^2} + 3\left[\frac{N-1}{2P}\right] \right) \le \frac{N^2}{P^2 \log N} + \sum_{p_{K+1} \le q} \frac{\left[\frac{N-1}{2P}\right]^2}{q(q-1)} +$$

$$+3\left[\frac{N-1}{2P}\right] \pi\left(\frac{N}{P}\right) \ll \frac{N^2}{P^2 \log N} + \frac{\left[\frac{N-1}{2P}\right]^2}{p_{K+1}-1} \ll \frac{\left[\frac{N-1}{2P}\right]^2}{\log\left[\frac{N-1}{2P}\right]}.$$

From this we get that the graph G has $n = \left[\frac{N-1}{2P}\right] + 1$ vertices and $m > \binom{n}{2} - c\frac{n^2}{\log n}$ edges where c is a constant. Now we remove some edges until the graph G has exactly $\left[\binom{n}{2} - c\frac{n^2}{\log n}\right] + 1$ edges. Using Lemma 5 we get that G contains $K_{r,r}$ for $r = \left[\frac{(\log n)^2}{8c}\right]$. Finally we choose $\mathcal{A}$ and $\mathcal{B}$ as the two vertex sets of $K_{r,r}$, which completes the proof of Theorem 2.

Proof of Theorem 3

We consider the graph G whose vertices are

$$\left\{ 2lP+1 : f(2lP+1, 2lP+1) \text{ is squarefree, } 1 \le l \le \left[\frac{N-1}{2P}\right] \right\}.$$

The edges are the same as in the proof of Theorem 2. It is easy to see that in case a) the number of vertices of the graph G is $n \gg \left[\frac{N-1}{2P}\right]$. In order to see that the same also holds in case b) we will study how many numbers l exist where $(2lP+1)^2+1$ is not squarefree. We distinguish two cases depending on whether $(2lP+1)^2+1$ has a prime square divisor greater than $\left(\frac{N}{12P}\right)^2$ or not.

Let p be a prime. If $p^2 \le p_K^2$ there is no integer $1 \le l \le \left[\frac{N-1}{2P}\right]$ such that $p^2 \mid (2lP+1)^2+1$. If $p_K^2 < p^2$ there are at most $\frac{N-1}{p^2P}+2$ integers $1 \le l \le \left[\frac{N-1}{2P}\right]$ where $p^2 \mid (2lP+1)^2+1$. From this we get that at most

$$\frac{N}{6P} + \frac{N-1}{p_K^2 P} + \frac{N-1}{p_{K+1}^2 P} + \frac{N-1}{p_{K+2}^2 P} \cdots < \frac{N}{6P} + \frac{2N}{P\log P} < \frac{N}{5P}$$

integers $1 \le l \le \left[\frac{N-1}{2P}\right]$ exist such that $(2lP+1)^2+1$ has a prime square divisor not greater than $\left(\frac{N}{12P}\right)^2$. In the other case when $(2lP+1)^2+1$ has a prime square divisor $p^2 > \left(\frac{N}{12P}\right)^2$ we have

$$(2lP+1)^2+1 = p^2 D$$

for some $D \le \frac{(2lP+1)^2+1}{\left(\frac{N}{12P}\right)^2} \le \frac{N^2+1}{\left(\frac{N}{12P}\right)^2} < 150P^2$. For a fixed D the number of solutions of the Pell equation $a^2 - p^2 D = -1$ in $a \in \{1,2,\dots,N\}$, $p \in \mathbb{N}$ is less than $\frac{\log N}{\log 2}$. Therefore the number of solutions of

$$(2lP+1)^2 - p^2 D = -1$$

in $2lP+1 \in \{1,2,\dots,N\}$, $D \in \{1,2,\dots 150P^2\}$ is less than $150P^2 \frac{\log N}{\log 2} < 500P^2 \log N$. So there are at most $\frac{N}{5P} + 500P^2\log N < \frac{1}{2}\left[\frac{N-1}{2P}\right]$ integers $1 \le l \le \left[\frac{N-1}{2P}\right]$ such that $(2lP+1)^2+1$ is not squarefree. From this we get that the number of vertices of the graph G is $n \gg \left[\frac{N-1}{2P}\right]$.

By the proof of Theorem 2 we get that the number of edges of the graph G is $m > \binom{n}{2} - c\frac{n^2}{\log n}$ where c is a constant. By using Turán's theorem [8] in place of Lemma 4 we obtain the statement of Theorem 3.

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