Algorithms for multiflows

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most of this work done at the Egerváry Research Group, Eötvös University, Budapest

Motivation

Known result:

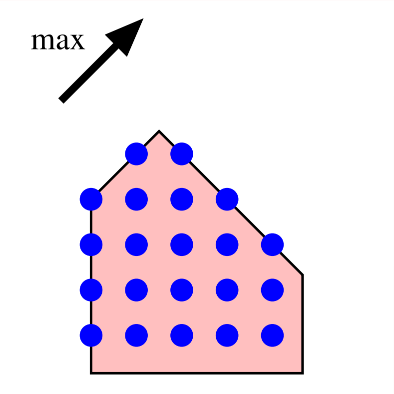
LP for maximum multiflow subject to edge- or node-capacities has as half-integral optimum.

Goal:

Find a half-integral optimum in strongly polynomial time!

Unlucky circumstance:

Very fractional basic optima.

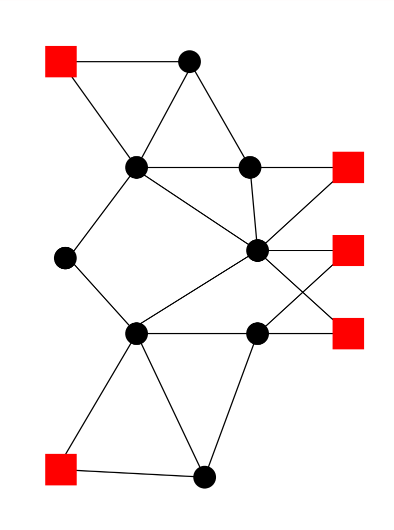


Definition of a multiflow

- We are given an undirected graph $G = (V, E)$ and a set of **terminals** $A \subseteq V$.
- A multiflow, denoted by

  $$x = (x_{ab} : a \neq b, a, b \in A),$$

  is a set of $\binom{|A|}{2}$ flows x_{ab} between all pairs of terminals $a \neq b, a, b \in A$.
- $size(x) := \sum size(x_{ab})$
- Problem: maximize the size of a multiflow subject to certain capacity and/or integrality constraints.



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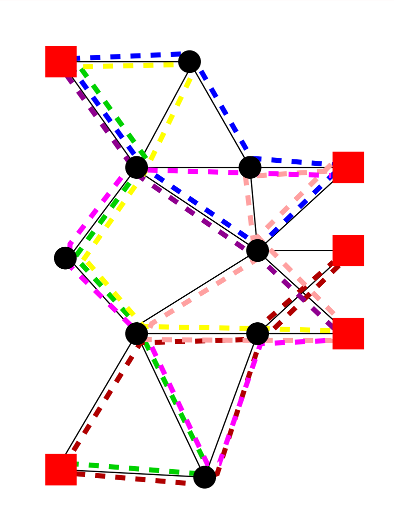
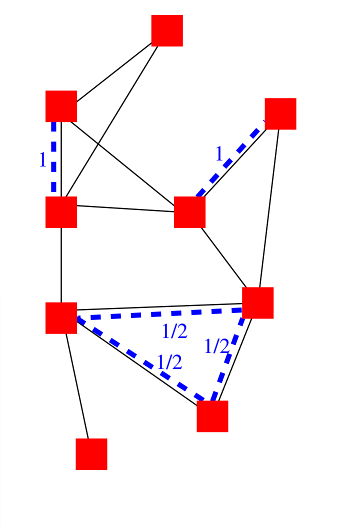
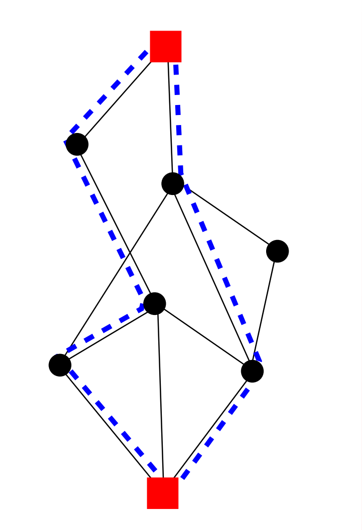


Special Cases

$A = V$:
max (fractional) matching



$|A| = 2$:
max s–t flow



The LP for node-capacities (condensed)

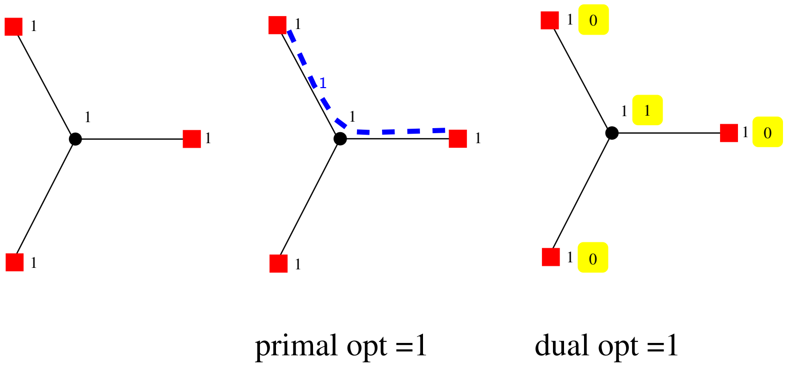
Primal: max $size(x)$ subject to

$$\begin{cases} x & \text{is a multiflow} \\ & \quad \text{(non-negativity, flow-conservation)} \\ total.load_x(v) \le c(v) & \text{for all } v \in V \\ & \quad \text{(capacity constraint)} \end{cases}$$

Dual: min $c \cdot w$ subject to

$$\begin{cases} w \ge 0 & \\ \sum_{v \in P} w(v) \ge 1 & \text{for every path } P \text{ connecting two distinct } a, b \in A \end{cases}$$

Example of a node-capacitated multiflow



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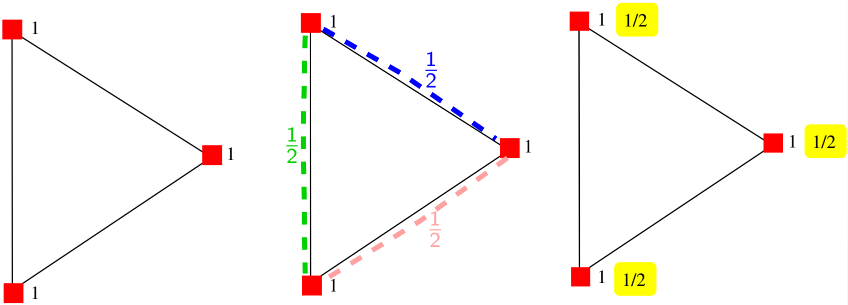
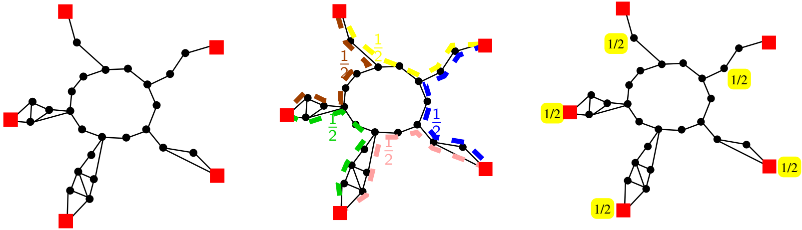


primal opt =1.5

dual opt =1.5

Example of a node-capacitated multiflow



primal opt =2,5

dual opt =2,5

- Gallai '61: node-disjoint A-paths (reduction to matching)
- Lovász, Cherkassky '73: edge-disjoint A-paths in an eulerian graph
- Mader 79': disjoint internally node-disjoint A-paths (only min-max)
- Lovász '81: linear matroid matching (polynomial time, using representation of Schrijver '9?)
- Ibaraki-Karzanov-Nagamochi '98: half-integral, edge-capacitated (strongly polynomial)
- Keijsper-Pendavingh-Stougie '06: integral, edge-capacitated (weakly polynomial, ellipsoid)
- P '07: integral, node-capacitated (weakly polynomial, ellipsoid)
- Babenko-Karzanov '08: half-integral, node-capacitated (weakly polynomial, scaling)
- This talk: half-integral, node-capacitated (strongly polynomial, ellispoid)
- Also implies: integral, node-capacitated (strongly polynomial, ellipsoid)

Further important results by Sebő-Szegő, Schrijver, Chudnovsky-Geelen-Gerards-Goddyn-Lohman-Seymour, Chudnovsky-Cunningham-Geelen, Babenko, and P.

Techniques:

- IP
- LP
- flows
- matroids
- matching
- circulations
- splitting-off
- minimum cuts
- delta matroids
- matroid matching
- graph decomposition

Outline

- Lovász-Cherkassky (min-max, splitting-off algorithm, LP half-integrality)
- Ibaraki-Karzanov-Nagamochi (strongly polynomial, half-integral, edge-c.)
- Negative result:

LP for maximum multiflow may have a "very fractional" basic optimum

- Positive result:

Every basic optimum of LP for shortest maximum multiflow is half-integral

- Corollary – using ellipsoid and Frank-Tardos:
  One can find a maximum half-integral multiflow in strongly polynomial time
- Corollary – using this, and Lovász' matroid matching and P (STOC '07):
  One can find a maximum integral multiflow in strongly polynomial time

All-one edge-capacities, Eulerian graph

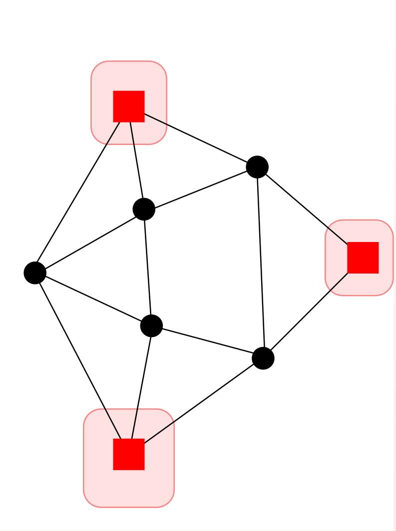
Lovász, Cherkassky, ≈1972

Let $G = (V, E)$ be eulerian, and let $A \subseteq V$. Then the maximum size of an integral multiflow subject to all-one edge-capacities is equal to

$$\frac{1}{2} \sum_{a \in A} \lambda(a, A - a)$$

(Essence: "cut condition" is necessary and sufficient.)

Proof: Split-off. Implies a polynomial time algorithm.



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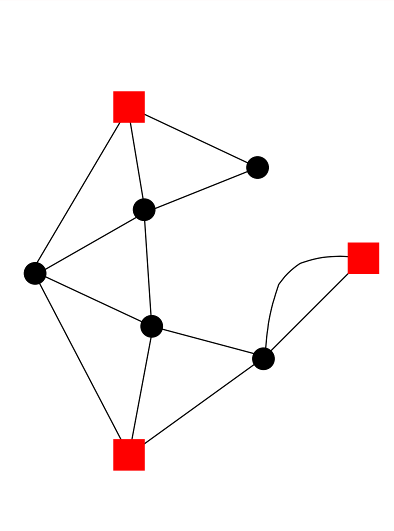

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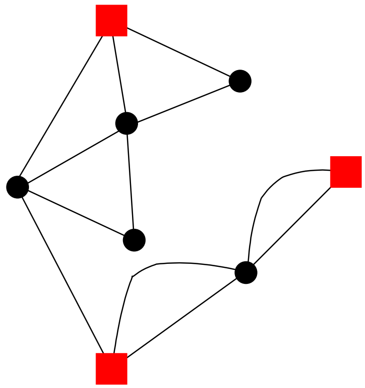

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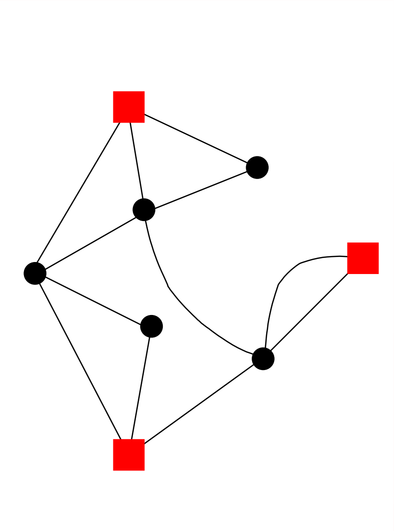

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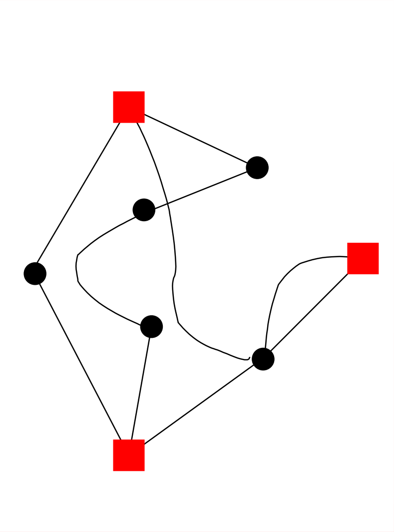

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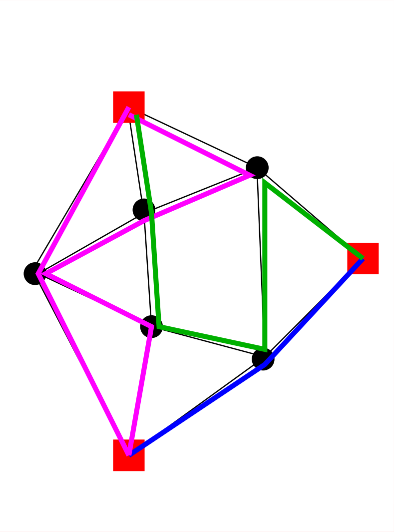


Half-Integral Edge-Capacitated Multiflows

Replacing edges $pq \in E$ by a number of $2c(pq)$ paralel edges, and applying Lovász-Cherkassky, we get the following Corollary:

Corollary — Half-Integrality of Edge-Capacitated Multiflows

For any $G = (V, E), A \subseteq V, c : E \to \mathbb{N}$,
the edge-capacitated multiflow LP has a half-integral optimum.

Find a half-integral optimum in strongly polynomial time!

Half-Integral Node-Capacitated Multiflows

Analogous half-integrality result holds in the node-capacitated case.

Theorem — Half-Integrality of Node-Capacitated Multiflows

For any $G = (V, E), A \subseteq V, c : V \to \mathbb{N}$,
the node-capacitated multiflow LP has a half-integral optimum.

— Dual half-integrality: elegant proof seen in Vazirani's book.
— Primal half-integrality: follows from Mader's min-max
(and there's a self-contained proof in my thesis)

Find a half-integral optimum in strongly polynomial time!

Approach 1: LP + play with fractions

Solve LP and then "play with the fractions" until half-integral.

- Implies a polynomial time algorithm by applying the uncapacitated algorithm for the fractional part.
- You can solve the LP in strongly polynomial time by a general result of Frank and Tardos. But, ...
- ... by definition, you mustn't take the fractional part of numbers in a strongly polynomial time algorithm ...
- ... and actually, it is not obvious at all how to "play with the fractions".

Approach 2: capacity scaling

Capacity scaling for edge-capacities: (only weakly polynomial time)

Capacity scaling + Lovász-Cherkassky $\Rightarrow$

weakly polynomial time algorithm to find a maximum half-integral edge-capacitated multiflow. Naive time using Goldberg-Rao for min cut:

$$O(m^3 n \min\{n^{2/3}, m^{1/2}\} \log(n^2/m) \log^2 n \log U) \approx O(n^8 \log U)$$

Capacity scaling for node-capacities: (only weakly polynomial time)

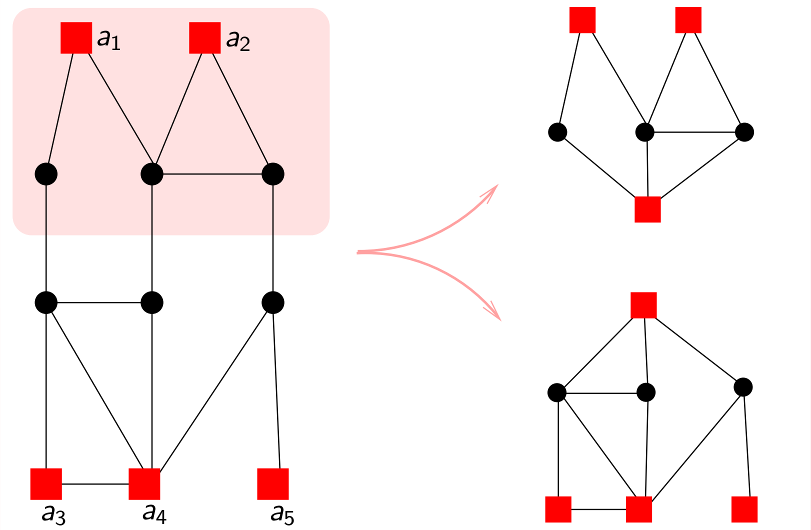
Babenko, Karzanov (ESA '08) constructed a faster scaling algorithm for node-capacities, achieving a running time of

$$O(mn^2 \min\{n^{2/3}, m^{1/2}\} \log(n^2/m) \log^2 n \log^2 U) \approx O(n^5 \log^2 U),$$

where U denotes the maximum capacity of a node.

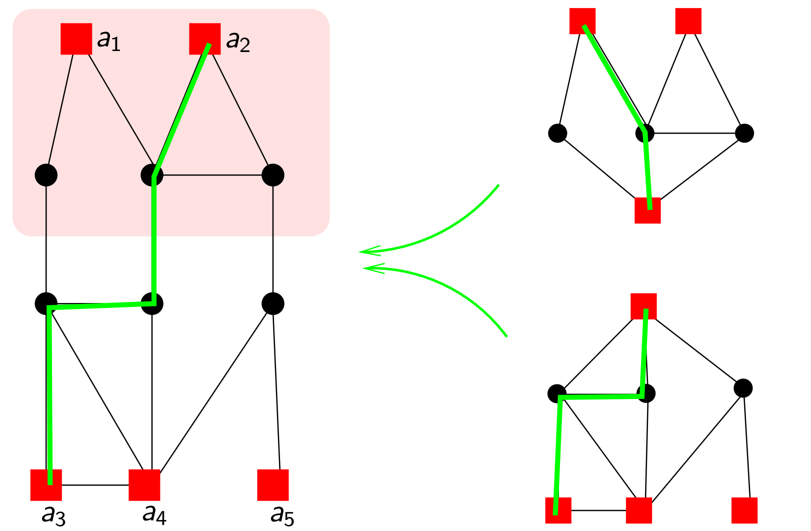
Approach 3: Karzanov, Ibaraki, Nagamochi

Decompose the graph by a minimum $\{a_1, a_2\}$ - $\{a_3, .., a_k\}$ cut:



Approach 3: Karzanov, Ibaraki, Nagamochi

Put together the multiflow like this:



Corollary – Reduction to 3 Terminals

Enough to solve for three terminals.

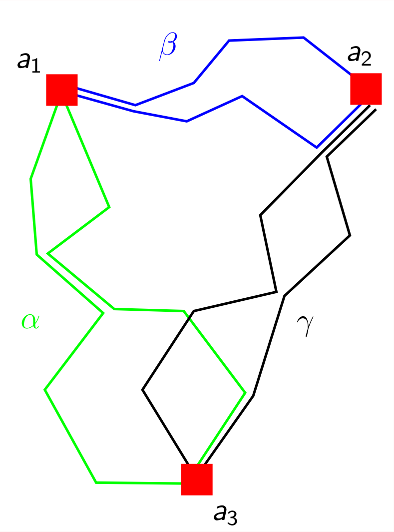
Idea: from well-known proof of Hu's and Rothschild-Whinston's Theorem on two-commodity flow, applying

$$|y+z|+|y-z| \leq \max\{2|y|, 2|z|\}$$

Approach 3: Karzanov, Ibaraki, Nagamochi

$$\alpha + \beta = \lambda_c(\{a_1\}, \{a_2, a_3\})$$
$$\beta + \gamma = \lambda_c(\{a_2\}, \{a_3, a_1\})$$
$$\gamma + \alpha = \lambda_c(\{a_3\}, \{a_2, a_1\})$$

We can compute α, β, γ!



Approach 3: Karzanov, Ibaraki, Nagamochi

There is a circulation z satisfying

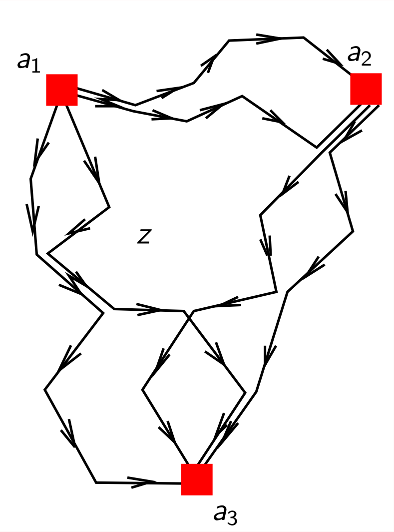
$excess_z(a_1) = \alpha + \beta$

$excess_z(a_2) = \gamma - \beta$

$excess_z(a_3) = -\alpha - \gamma$

$excess_z(v) = 0$
for $v \in V - A$.

We construct an integral z by a flow algorithm.



Approach 3: Karzanov, Ibaraki, Nagamochi

And there is a circulation y satisfying

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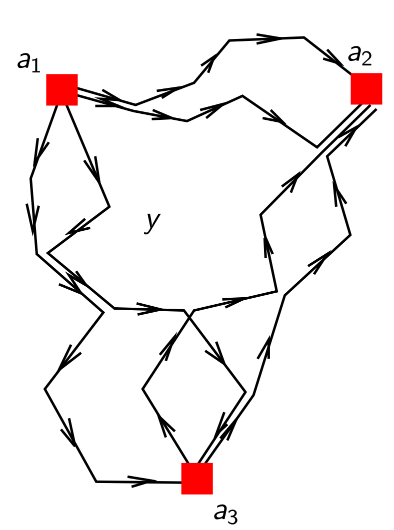
$excess_y(a_2) = -\gamma - \beta$

$excess_y(a_3) = \gamma - \alpha$

$excess_y(v) = 0$ for $v \in V - A$.

We construct an integral y by a flow algorithm.



Then $\frac{1}{2}(z+y)$ satisfies

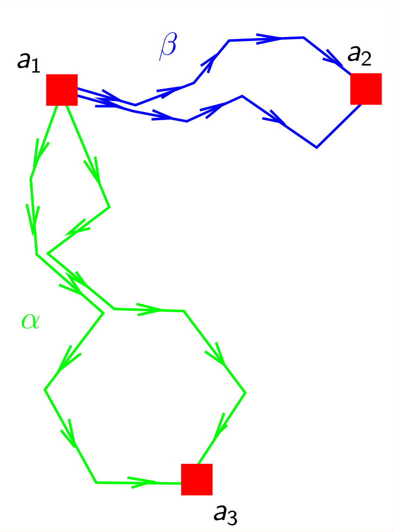
$excess_{\frac{1}{2}(z+y)}(a_1) = \alpha + \beta$

$excess_{\frac{1}{2}(z+y)}(a_2) = -\beta$

$excess_{\frac{1}{2}(z+y)}(a_3) = -\alpha$

$excess_{\frac{1}{2}(z+y)}(v) = 0$
for $v \in V - A$,

thus $\frac{1}{2}(z+y)$ is a flow of value $\alpha+\beta$ from a_1 to a_2, a_3.



Then $\frac{1}{2}(z-y)$ satisfies

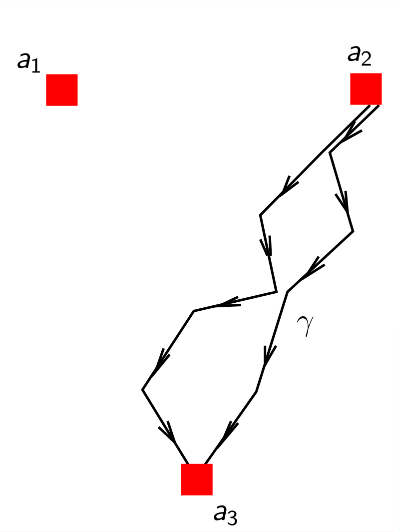
$$excess_{\frac{1}{2}(z-y)}(a_1) = 0$$

$$excess_{\frac{1}{2}(z-y)}(a_2) = \gamma$$

$$excess_{\frac{1}{2}(z-y)}(a_3) = -\gamma$$

$$excess_{\frac{1}{2}(z-y)}(v) = 0 \quad \text{for } v \in V - A,$$

thus $\frac{1}{2}(z-y)$ is a flow of value γ from a_3 to a_2.



Approach 3: Karzanov, Ibaraki, Nagamochi

To show that the multiflow constructed from $\frac{1}{2}(z+y)$ and $\frac{1}{2}(z-y)$ satisfies the capacity constraint for an edge e, we need to see that

$$|\tfrac{1}{2}(z+y)(e)| + |\tfrac{1}{2}(z-y)(e)| \leq c(e).$$

Proof: using the elemetary inequality for all $n, m \in \mathbb{R}$

$$|n+m| + |n-m| \leq \max\{2|n|, 2|m|\}$$

we get that

$$|\tfrac{1}{2}(z+y)(e)| + |\tfrac{1}{2}(z-y)(e)| \leq \max\{|y(e)|, |z(e)|\} \leq c(e).$$

Approach 3: Karzanov, Ibaraki, Nagamochi

Theorem (based on Ibaraki-Karzanov-Nagamochi)

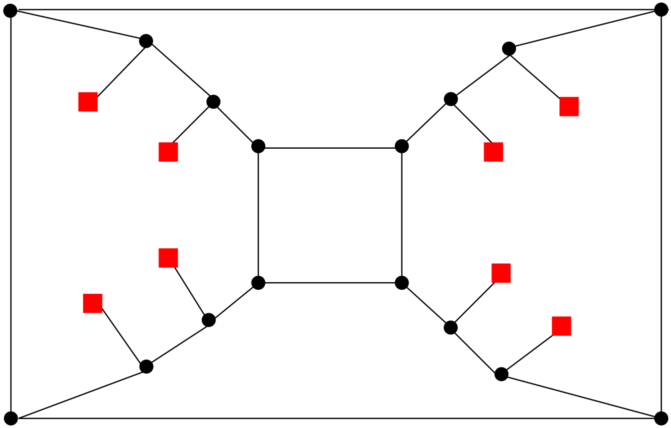
There is a strongly polynomial time algorithm to find a maximum half-integral multiflow subject to edge-capacities.

Approach 4: find a basic optimum

Maybe a basic optimum of the LP is guaranteed to be half-integral!

- This – by Frank and Tardos' result – would imply a strongly polynomial time algorithm.
- Unluckily, some basic optima may not half-integral. Some entries could be as bad as $\lfloor\sqrt{n}\rfloor!^{-1}$. Example follows next.

The LP may have a fractional basic optimum:



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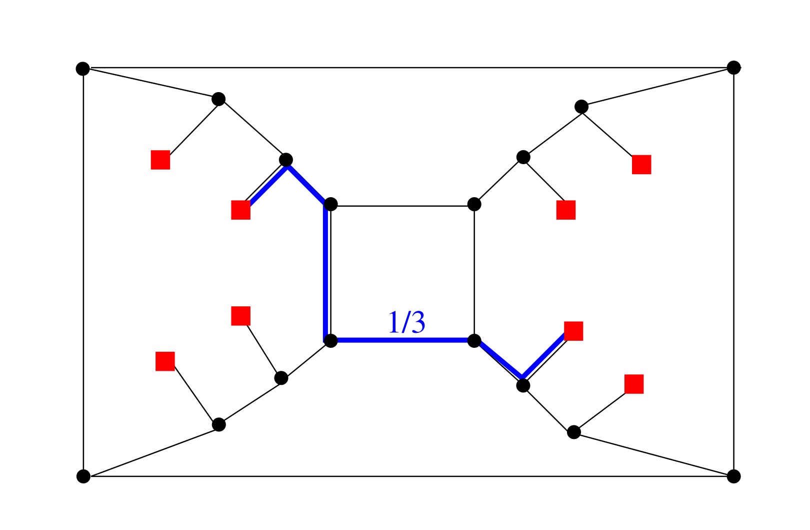

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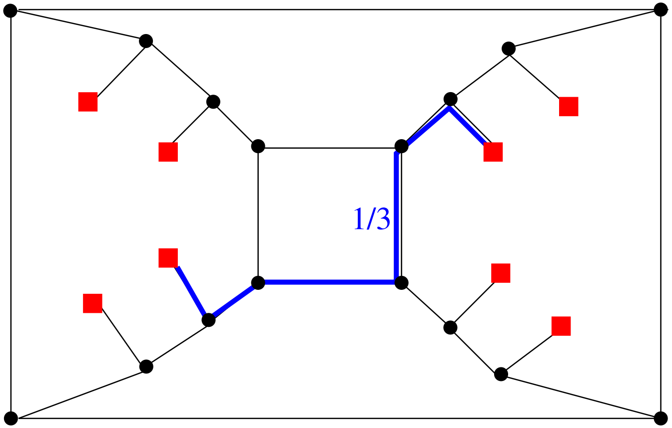

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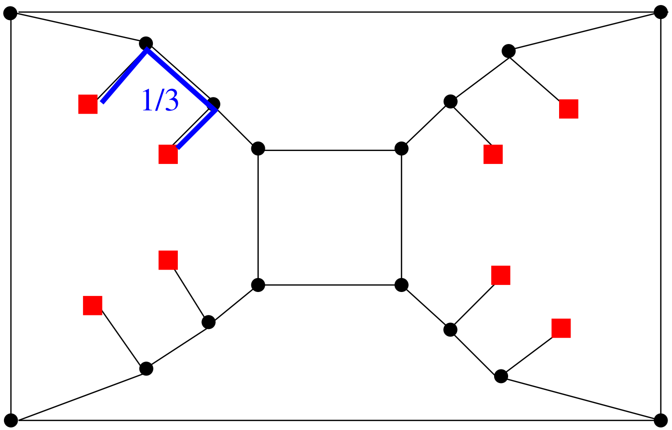


Node-Capacitated Multiflows

– LP of maximum node-capacitated multiflow has half-integral optima.

– LP of maximum node-capacitated multiflow may have bad basic optima.

Theorem

Every basic optimum of the LP for "shortest" maximum multiflow is half-integral.

This Lemma also implies a strongly polynomial time algorithm!

Node-Capacitated Multiflows

LP for **maximum multiflows**:

$$M := \max \; size(x) \quad \text{subject to}$$

$$\begin{cases} x & \text{is a multiflow} \\ \text{total.load}_x(p) \le c(p) & \text{for all } p \in V \end{cases}$$

LP for **shortest maximum multiflows**:

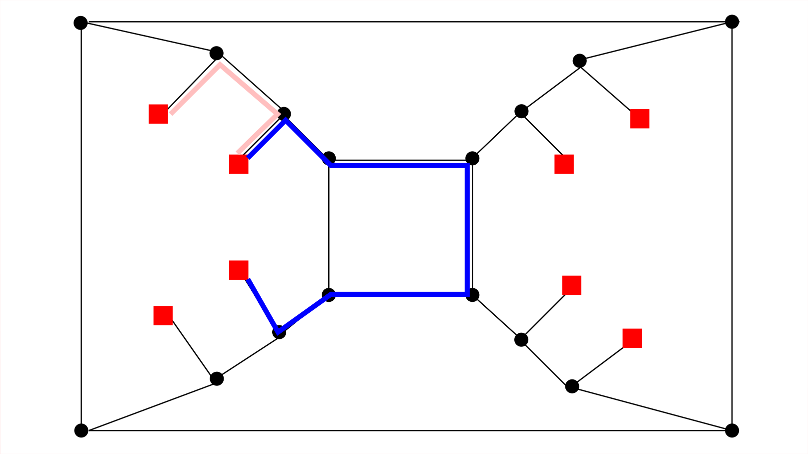
$$L := \min \; \mathbf{1} \cdot x \quad \text{subject to}$$

$$\begin{cases} x & \text{is a multiflow} \\ \text{total.load}_x(p) \le c(p) & \text{for all } p \in V \\ size(x) = M & \end{cases}$$

Thus we want to show that all the vertices of the following **polytope of shortest maximum multiflows** is half-integral:

$$\begin{cases} x & \text{is a multiflow} \\ \text{total.load}_x(p) \leq c(p) & \text{for all } p \in V \\ size(x) = M & \\ \mathbf{1} \cdot x = L & \end{cases}$$

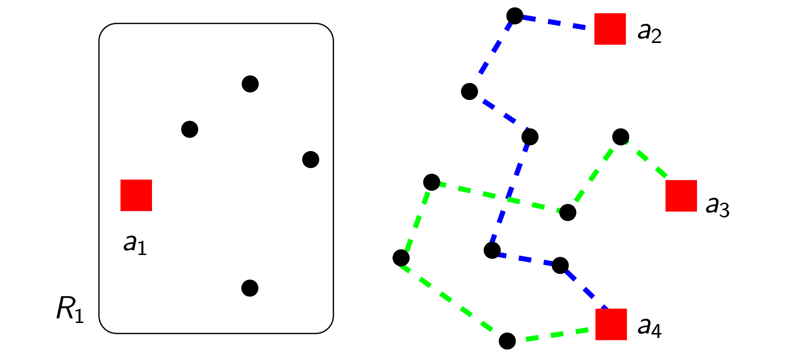
The point is that we incourage the multiflow to use the pink path instead of the blue path!



Proof Sketch.

1. W.l.o.g. delete nodes v not traversed by any s.m.m.

2. Let $R_i \subseteq V$ be the "region", i.e. the set of nodes $v \in V$ such that there is no s.m.m. x and $r, q \neq i$ such that $x_{rq}(\delta(v)) > 0$.

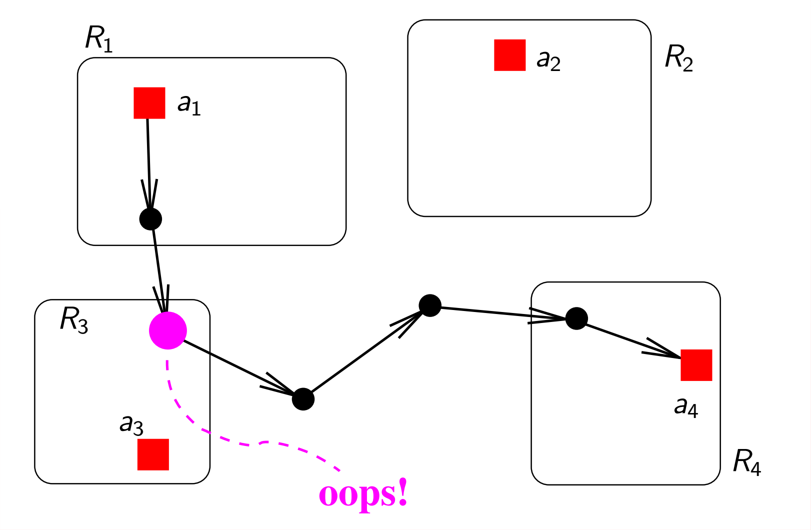


3. Then every s.m.m. x satisfies the following "region constraints":

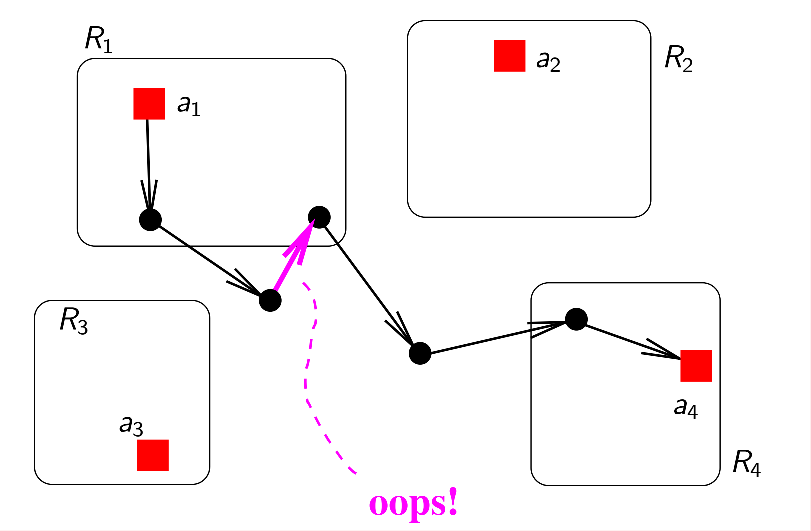
Region Constraints

$$x_{ij}(vz) = 0 \qquad \text{if } vz \in E,\ v \in V - R_i,\ z \in V - R_j.$$

By the definition of the regions, a shortest maximum multiflow MUST NOT contain the following path with positive weight:



The region-constraint says that a shortest maximum multiflow MUST NOT contain the following path with positive weight:



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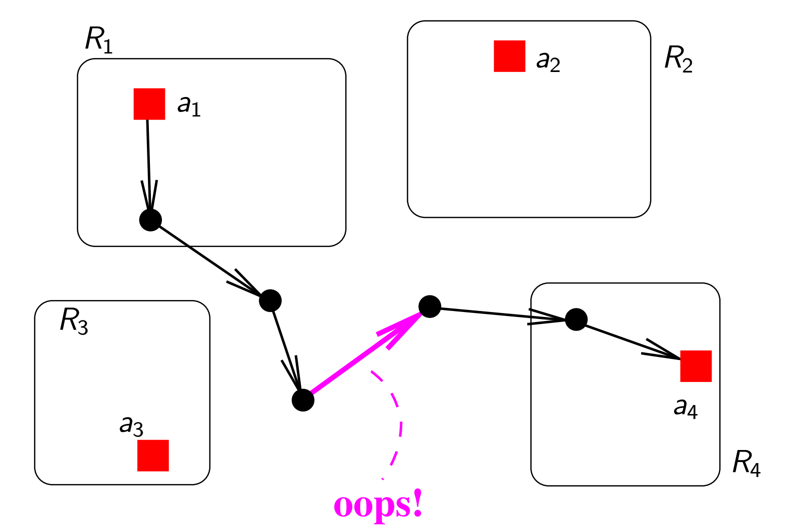

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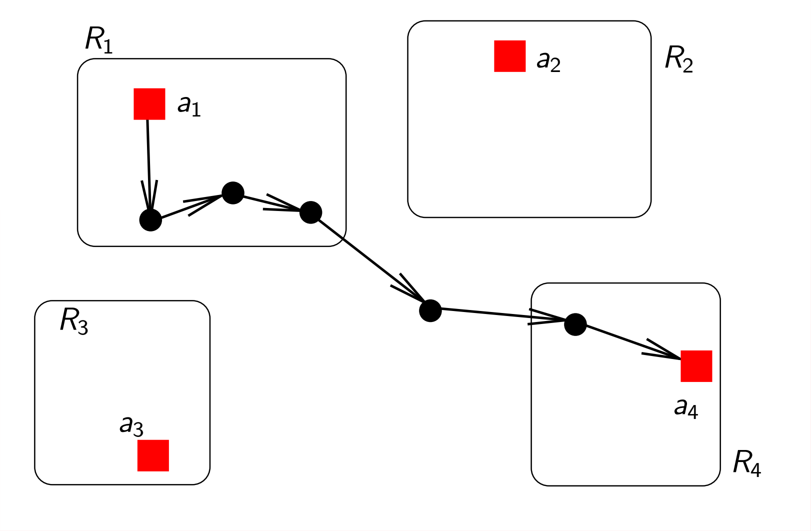

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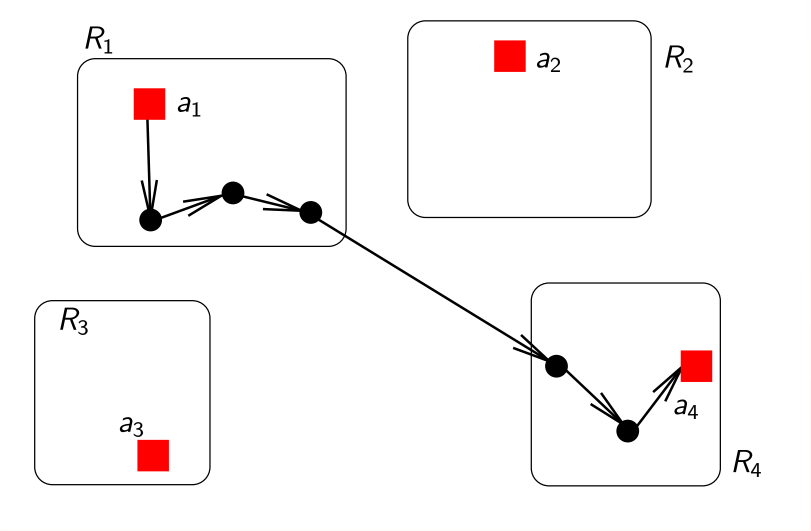
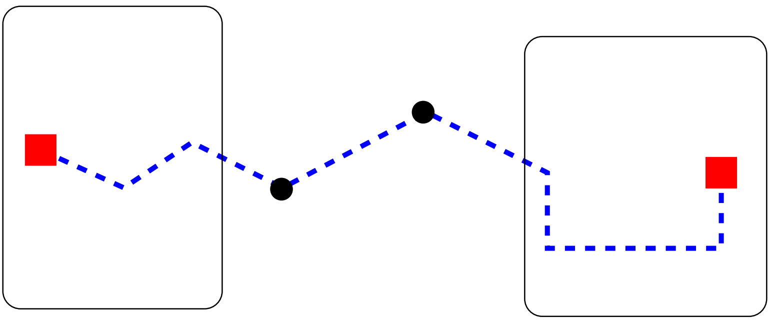
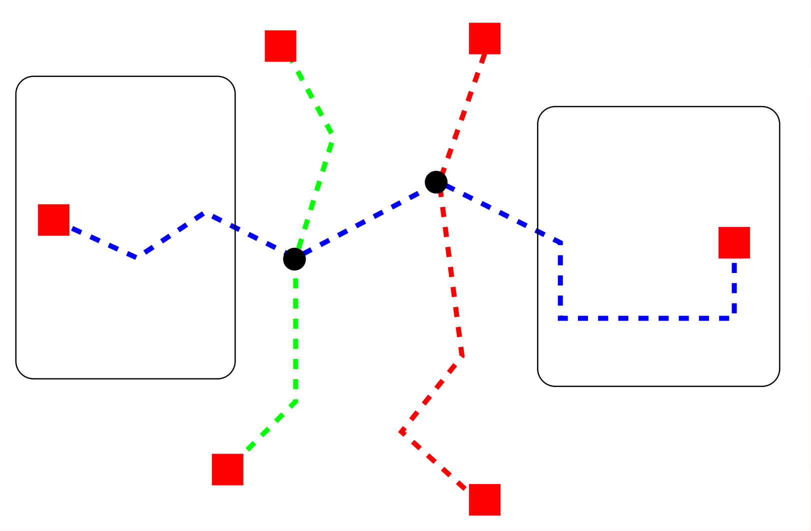


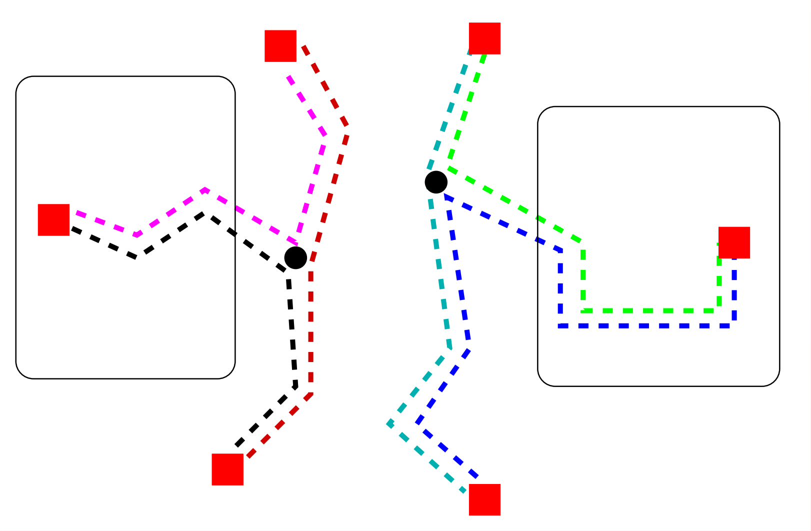
Suppose – for contradiction – that s.m.m. x contains this path with weight $\varepsilon > 0$:



Then, by definition, s.m.m. x' contains the following configuration with weight $\varepsilon' > 0$:



We replace that configuration with weight ε' by the following with weight $\varepsilon'/2$: (contradicting x being "shortest")



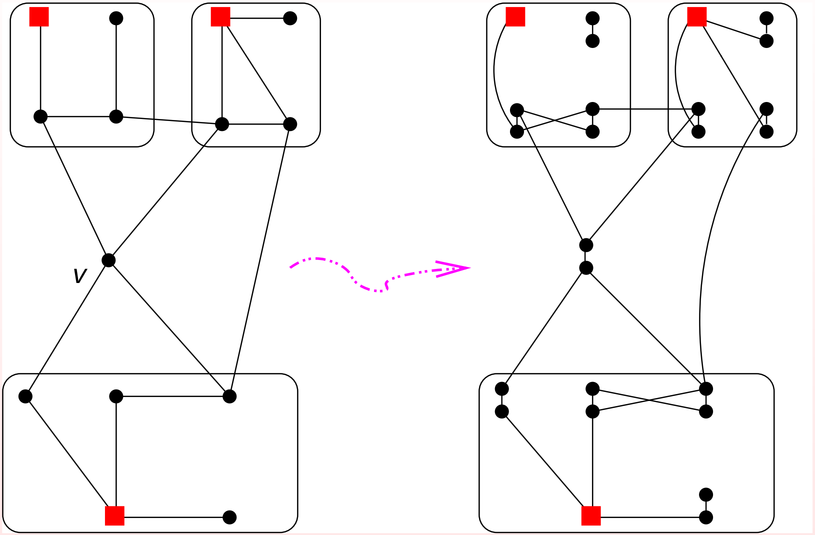
... and a similar argument shows the other case of the region constraint.

4. Thus from this point on, we can argue based on the region-constraints.

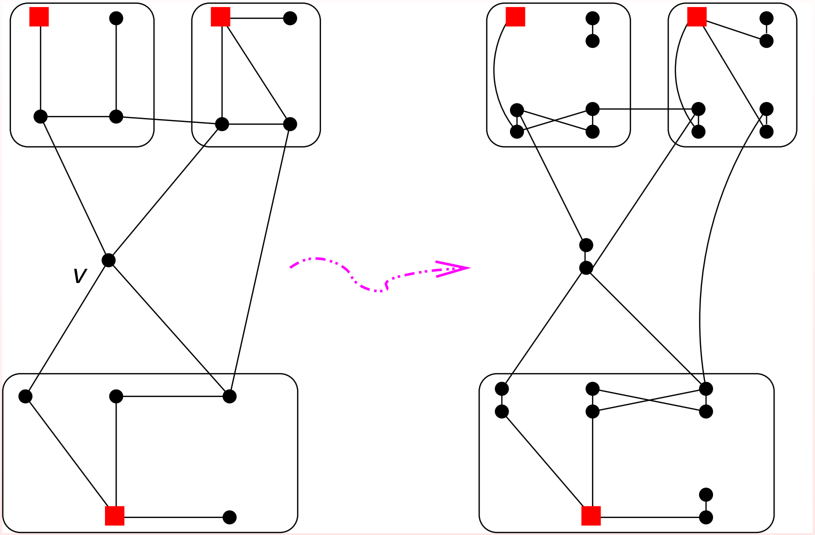
5. Let x be an extreme point of the shortest max multiflow polytope.

6. Based on x, we construct an auxiliary graph G' with node-weights b', and consider b'-matchings in G'. The construction goes as follows.

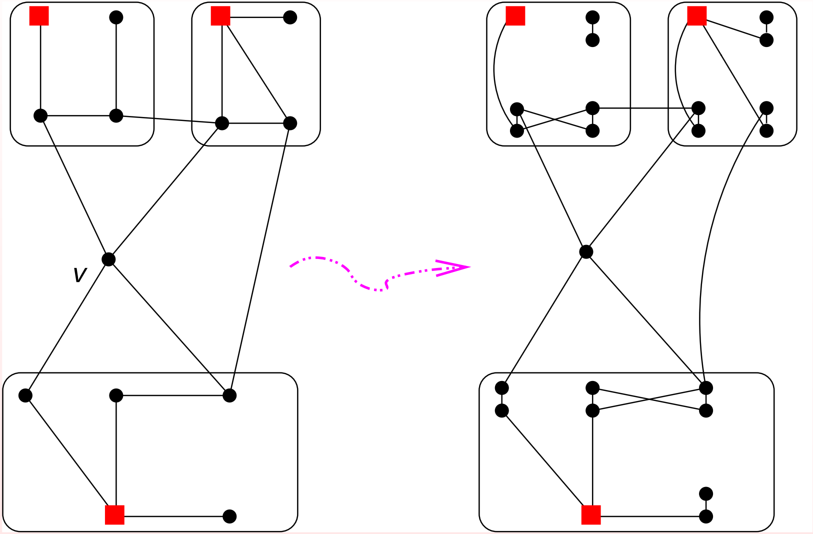
If all flow through v comes from a_3:



If all flow through v comes from a_2:



Otherwise:



7. Every b'-matching in G' can be converted into a maximum multiflow in G, A, c.

8. If x is not half-integral, then x' is not half-integral, and then x' is not a vertex of the b'-matching polytope, and then x' is the convex combination of two b'-matchings x'' and x''', which can be converted into maximum multiflows x'''' and x''''', implying that x is the convex combination of x'''' and x''''', which contradicts that x is a vertex.

Concluding Remarks

- Implies a strongly polynomial time algorithm to find a maximum integral multiflow via ellipsoid, using Frank-Tardos's approximation technique.
- Implies a strongly polynomial time algorithm to find a maximum integral multiflow by combining this, and a proximity lemma proved by P '07, and an algorithm for all-one capacities, say by using Lovász' linear matroid matching algorithm.
- Open Question: combinatorial algorithm without ellipsoid?