Packing non-returning A-paths algorithmically

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"Packing" = "Fully node-disjoint family"

Background

- maximum non-bipartite matching (Berge '58; Tutte '47)
- packing A-paths, $\mathcal{A}$-paths (Gallai '61; Mader '78)
- packing non-zero A-paths:
  formula (Chudnovsky,Geelen,Gerards,Goddyn,Lohman,Seymour'04)
  algorithm and structure (Chudnovsky,Cunningham,Geelen '05)
- Packing non-returning A-paths (P '05)

Goal

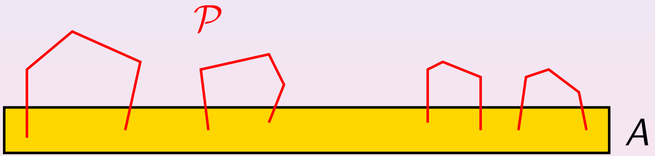
- alternative algorithm
- "odd ear-decomposition" of criticals
- fractional packings, 2-packings: WHY SIMPLER?

Crucial concept

- the so-called "3-Way Lemma" for A-paths

$G = (V, E)$ an undirected graph, $A \subseteq V$ a set of terminals.
$\mathcal{H} = \{$family of **allowed** A-**paths**$\}$ which is "nice".

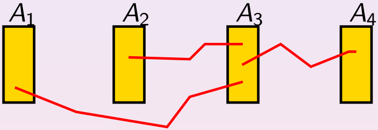
Packing of allowed A-paths: $\mathcal{P} = \{P_1, \cdots, P_k\}$ with $P_i \in \mathcal{H}$.



Maximize $|\mathcal{P}|$**.**

Special cases.

1. $A = S \cup T, \quad \mathcal{H} := \{\ S\text{–}T \text{ paths}\}$. VERTEX CUT
2. Case of $A = V$ is equivalent to (non-bipartite) matching.
   ODD COMPONENTS
3. .



Mader '73

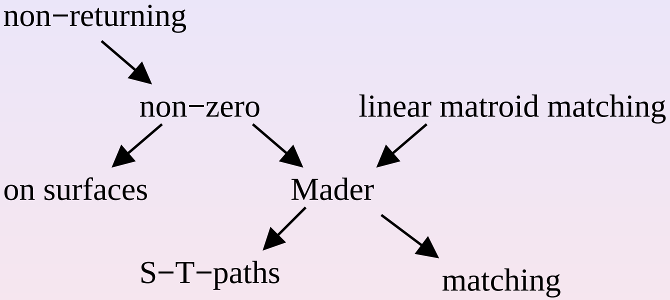
4. Packing non-zero A-paths in group-labeled graphs.
   (Chudnovsky et al. '04)
5. Packing non-returning A-paths. (P '05)

Theorem Mader

Consider a graph $G = (V, E)$, and a sub-partition $\mathcal{A} = \{A_1, \cdots, A_k\}$ of V. Then the maximum cardinality of a packing of $\mathcal{A}$-paths is determined by

$$\min_{\mathcal{A}'} \ |X| + \sum_{C \in \mathcal{C}} \left\lfloor \frac{|A' \cap C|}{2} \right\rfloor,$$

where $\mathcal{A}' = \{A'_1, \cdots, A'_k\}$ is a sub-partition such that $A_i \subseteq A'_i$, $A' := \bigcup \mathcal{A}'$, and $\mathcal{C}$ is the family of components of the subgraph $G - X - \bigcup E[A'_i]$.


non−returning
non−zero
linear matroid matching
on surfaces
Mader
S−T−paths
matching

Fractional Matchings

$$x(ab) \geq 0 \quad \text{for all } ab \in E$$
$$\sum_{b \in \Gamma(a)} x(ab) \leq 1 \quad \text{for all } a \in V$$

$\exists$ Half-Integer Opt

$$\max \ \mathbf{1} \cdot x$$

$$=$$

$$\min \ \mathbf{1} \cdot w$$

$$w(a) \geq 0 \quad \text{for all } a \in V$$
$$w(a) + w(b) \geq 1 \quad \text{for all } ab \in E$$

$\exists$ Half-Integer Opt

Fractional Packings of A-Paths (P '05)

$x : \{\text{ allowed } A\text{-walks }\} \longrightarrow \mathbb{R}_+$

$$\sum_{P \text{ trav. } a} x(P) \leq 1 \quad \text{for all } a \in V$$

$\exists$ Half-Integer Opt

$$\max \ \mathbf{1} \cdot x$$

$=$

$$\min \ \mathbf{1} \cdot w$$

$$w(a) \geq 0 \quad \text{for all } a \in V$$
$$w(V(P)) \geq 1 \quad \text{for allowed } A\text{-walks } P$$

$\exists$ Half-Integer Opt

Following Lemma implicit in Edmonds' Matching Algorithm:

Lemma "3-Way Lemma" for matching

Consider a matching M in graph G. Then at least one of the following alternatives holds:

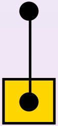
1. there is a matching $|N| = |M| + 1$, or
2. χ_M is a maximum fractional matching, or
3. there is a matching $|N| = |M|$ for which there is an N-alternating odd cycle.

Lemma — "3-Way Lemma" for A-paths — P '05

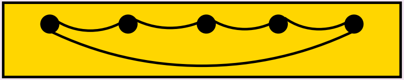
Consider a packing $\mathcal{P}$ of A-paths in G. Then at least one of the following alternatives holds:

1. there is a packing $|\mathcal{R}| = |\mathcal{P}| + 1$, or
2. $\chi_{\mathcal{P}}$ is a maximum fractional packing, or
3. there is a packing $|\mathcal{R}| = |\mathcal{P}|$ for which there is an "$\mathcal{R}$-alternating odd cycle", or a "rod".

a "**ROD**"

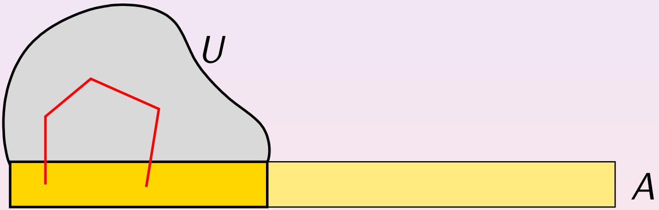


an "**ODD CYCLE**"



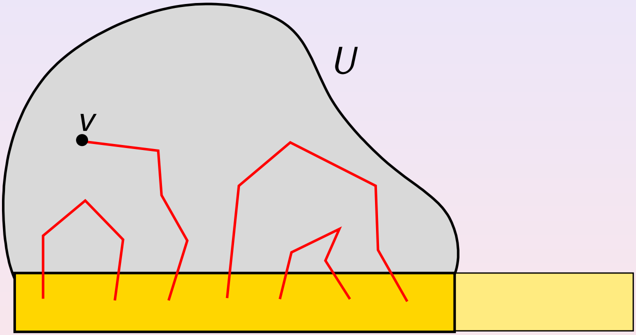
DEF A set $U \subseteq V$ **induces** a problem:

$$G[U], \qquad A \cap U, \qquad \mathcal{H}' := \{P \in \mathcal{H} : V(P) \subseteq U\}$$



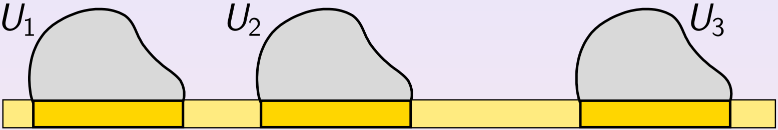
definition

$G[U]$ is a called a **"dragon"** if all the nodes $v \in U$ are "reachable inside U".

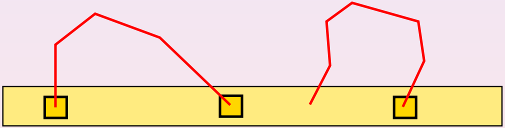


I.e. for all $v \in U$ there is a packing inside U covering all but one of the nodes in $A \cap U$ such that there is an additional disjoint path joining v to the remaining node of $A \cap U$.

DEF **Contraction of a family of disjoint dragons.**

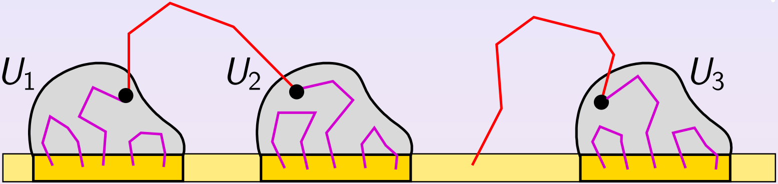


$G/U_1/U_2/U_3$

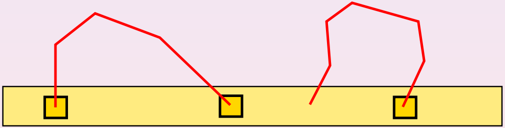


A path in $G/U_1/U_2/\cdots/U_m$ is allowed if its pre-image can be extended by “inside reaching paths”.

DEF **Contraction of a family of disjoint dragons.**



$G/U_1/U_2/U_3$



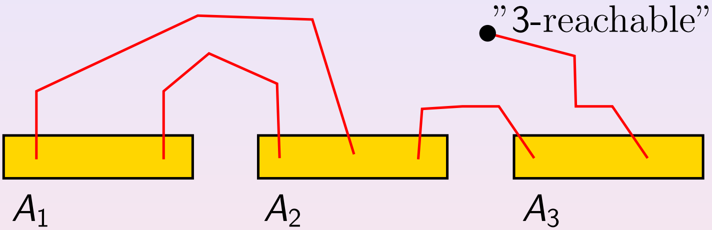
A path in $G/U_1/U_2/\cdots/U_m$ is allowed if its pre-image can be extended by "inside reaching paths".

Sketch of the Algorithm.

Input: G, packing $\mathcal{P}$.
Output: augmentation or certificate of optimality.

1. Set $G_0 := G$, $\mathcal{P}_0 := \mathcal{P}$, $\mathcal{Z}_0 := \emptyset$.
2. Consider $\mathcal{Z}_i = \{$ family of disjoint dragons $\}$.
3. Let $G_i := G/\mathcal{Z}_i$, suppose $\mathcal{P}_i$ is a packing in G_i.
4. 3-Way Lemma for $G_i, \mathcal{P}_i \quad \Rightarrow$
   1. augmentation in $G_i \quad \Rightarrow$ augmentation in G, or
   2. $\chi_{\mathcal{P}_i}$ maximum fractional packing $\quad \Rightarrow$ certificate, or
   3. odd cycle or rod $\quad \Rightarrow$ finer family $\mathcal{Z}_{i+1}$.

Criticals. — Example for Mader $\mathcal{A}$-paths.



Definition

$G, \mathcal{A}$ is called **critical** if

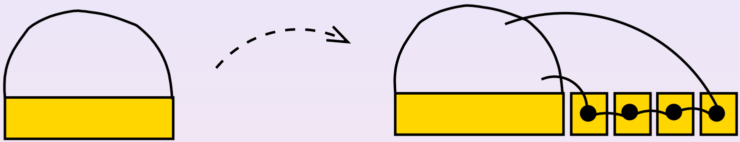
— nodes in T are **reachable**, and

— nodes in $V - T$ are **"multi-reachable"**.

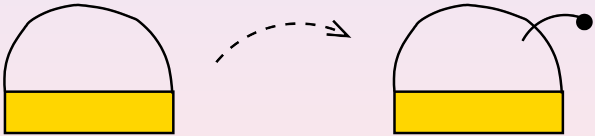
Theorem

Criticals can be constructed by the below operations.

A.

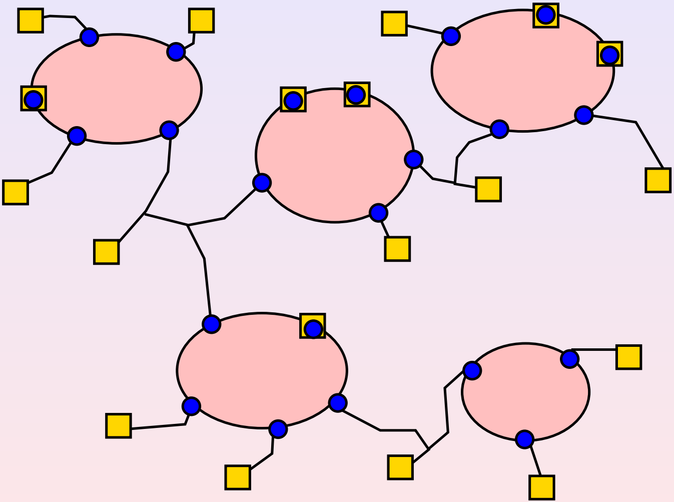


B.



C. Splitting or adding edges.

A dragon decomposes into a tree and some criticals.



Open Questions.

- Why are fractional packings easier?
- What is the reason for having a half-integer optimum fractional packing?
- Does half-integer packing reduce to matroid matching, or other well-known concepts?
- Strongly polynomial algorithm for b-packings — where b denotes node-capacities?