A survey of results on packing $\mathcal{A}$-paths

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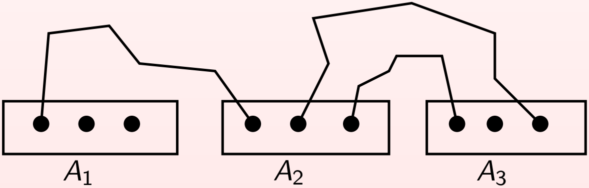
- Overview of well-known results on edge-disjoint packing of $\mathcal{A}$-paths.
- LP-relaxation of node-disjoint packing of $\mathcal{A}$-paths, its primal and dual half-integrality, and its reduction to matroid fractional matching.
- A polynomial time algorithm for node-capacitated packing. (A strengthening of Keijsper, Pendavingh, and Stougie's result concerning edge-capacities.)
- Mader Matroids are Gammoids. (Positive answer to Lex Schrijver's question.)

$\mathcal{A}$-paths

Consider

- a graph $G = (V, E)$,
- a family $\mathcal{A} = \{A_1, \cdots, A_k\}$ of disjoint stable sets,
- let $A := \bigcup \mathcal{A}$.

$\mathcal{A}$-paths are ...



Packing edge-disjoint $\mathcal{A}$-paths

Theorem (Lovász 1976, Cherkassky 1976)

If all nodes in $V - A$ have even degree, then

$$\text{max integral packing} = \frac{1}{2}\sum_i \lambda(A_i, A - A_i),$$

$\Rightarrow$ following LP is primal and dual half-integral:

$$\begin{aligned} \max\ x \cdot \mathbf{1} \quad & \text{subject to} \\ x &\geq 0 \\ \textstyle\sum_{e \in P} x(P) &\leq 1 \qquad \text{for all } e \in E \end{aligned}$$

Edge-capacitated fractional packing

- max fractional packing subject to edge-capacities $b \in \mathbb{N}^E$:

$$\max\ x \cdot \mathbf{1} \quad \text{subject to}$$

$$x \geq 0$$

$$\textstyle\sum_{e \in P} x(P) \leq b(e) \qquad \text{for all } e \in E$$

— min-max formula from Lovász, Cherkassky

$*$ polytime algo via ellipsoid method

Edge-capacitated integral packing

- max integral packing subject to edge-capacities $b \in \mathbb{N}^E$:

  — min-max formula: Mader (1978)
  $*$ polytime algo: Keijsper, Pendavingh, Stougie (1998)

Node-disjoint/node-capacitated packing

- max node-disjoint packing:
  — min-max formula: Mader (1979)
  ∗ Lovász (1982) + Schrijver (2001?)
  ∗ Chudnovsky, Cunningham, Geelen (2005)
  ∗ P (2005)

- max node-capacitated integral packing:
  — min-max formula: Mader (1979)
  ∗ polytime algo: P (2006)

- max node-disjoint fractional packing:
  — primal and dual half-integral LP
  — reduction to matroid fractional matching: P (2006)
  ∗ both imply a polynomial time algorithm

Packing node-disjoint $\mathcal{A}$-paths

Let

$$\nu(G,\mathcal{A}) := \max\{|\mathcal{P}| : \mathcal{P} \subseteq \{\mathcal{A}-\text{paths}\} \text{ a packing }\}$$

$$\nu^*(G,\mathcal{A}) := \max\{x \cdot \mathbf{1} : x \in \mathbb{R}_+^{\{\mathcal{A}-\text{paths}\}} \text{ a fractional packing }\}.$$

Matroid Fractional Matching

Consider

- a finite or infinite matroid $\mathcal{M} = (S, \mathcal{I})$
- a finite set $\mathcal{E}$ of lines, i.e. two-ranked flats of $\mathcal{M}$
- for $K \subseteq S$ we define $d_K \in \{0,1,2\}^{\mathcal{E}}$ by

$$d_K(e) := r\big(K \cap e\big) \quad \text{for } e \in \mathcal{E}.$$

The **Matroid Fractional Matching Polytope** is defined by

$$\mathcal{P}(\mathcal{M}, \mathcal{E}) := \Big\{ x \in \mathbb{R}^{\mathcal{E}}_{+} \ : \ d_K \cdot x \leq r(K) \quad \text{for all} \quad K \subseteq S \Big\}.$$

Vande Vate's Formula

The maximum size of a fractional matching:

$$\nu^*(\mathcal{M},\mathcal{E}) := \max\left\{\mathbf{1}\cdot x :\ x \in \mathcal{P}(\mathcal{M},\mathcal{E})\right\}$$

Theorem (Vande Vate, 1992)

$$\nu^*(\mathcal{M},\mathcal{E}) = \min\left\{\frac{1}{2}r(K) + \frac{1}{2}r(L) :\quad \frac{1}{2}d_K + \frac{1}{2}d_L \geq \mathbf{1}\right\}$$

Observation (Gijswijt, 2006)

The above description of the matroid fractional matching polytope is totally dual half-integral.

Application of Matroid Fractional Matching

Lex Schrijver's construction:

- Let $l_1, \cdots, l_k$ be distinct 1-dimensional subspaces of $\mathbb{R}^2$,
- for $ab \in E$, $L_{ab} := \{x \in (\mathbb{R}^2)^V : x(a) = x(b),\ x|_{V-a-b} \equiv 0\}$,
- let $Q := \{x \in (\mathbb{R}^2)^V : x(a) \in l_i \text{ for } a \in A_i,\ x|_{V-A} \equiv 0\}$,
- $\mathcal{E} := \{L_{ab}/Q : ab \in E\} < \mathcal{Z} := (\mathbb{R}^2)^V/Q$.

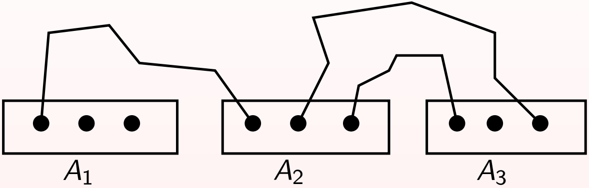
Theorem (Lex Schrijver)

$$\nu(G, \mathcal{A}) = \nu(\mathcal{M}_{\mathcal{Z}}, \mathcal{E}) - |V - A|$$

Theorem (P, 2006)

$$\nu^*(G, \mathcal{A}) = \nu^*(\mathcal{M}_{\mathcal{Z}}, \mathcal{E}) - |V - A|$$

Mader Matroid and Delta-Matroid



- Mader Matroid:

$$\{I \subseteq A : \ \exists \text{ packing } \mathcal{P} \text{ of } \mathcal{A}\text{-paths s.t. } I \subseteq A \cap V(\mathcal{P})\}.$$

- Mader Delta-Matroid:

$$\{V(\mathcal{P}) \cap A : \ \ \mathcal{P} \text{ a packing of } \mathcal{A}\text{-paths } \}.$$

Mader Matroids are Gammoids

- Is each Mader matroid a gammoid? [c.f. Schrijver, his book.]

Theorem (P, 2006)

Each Mader matroid is a gammoid.

- Is each Mader delta-matroid representable? [Open.]

- max fractional packing subject to node-capacities $b \in \mathbb{N}^V$:

$$\max \ x \cdot \mathbf{1} \quad \text{subject to}$$

$$\begin{aligned} x &\geq 0 \\ \textstyle\sum_{v \in P} x(P) &\leq b(v) \qquad \text{for all } v \in V \end{aligned}$$

Considering node-capacities $b \in \mathbb{N}^V$, we define

$$\nu_b(G, \mathcal{A}) := \max\{x \cdot \mathbf{1} : x \in \mathbb{N}^{\{\mathcal{A}-\text{paths}\}} \text{ a } b\text{-packing }\}.$$

$$\nu_b^*(G, \mathcal{A}) := \max\{x \cdot \mathbf{1} : x \in \mathbb{R}_+^{\{\mathcal{A}-\text{paths}\}} \text{ a fractional } b\text{-packing }\}.$$

INPUT: $G, \mathcal{A}, b \in \mathbb{N}^E$.
OUTPUT: maximum integral b-packing.

1. $y :=$ max fractional packing such that $supp(x) \leq \binom{k}{2}|E|$.
   (Via ellipsoid method for "sum-of-A_i–A_j-flows" LP.)
2. $x := \lfloor y \rfloor$
3. Perform AUGMENT(x), until optimality is reached.

AUGMENT(x):

1. Find $x' \leq x$ such that $\mathbb{1} \cdot x' \leq 2|V|^4$, and $x'(\{P : v_{1,2,3,4} \in P\}) \geq \min\{2, x(\{P : v_{1,2,3,4} \in P\})\}$ for all quadruples $v_1, v_2, v_3, v_4 \in V$.
2. $b'(v) := \min\{x'(\{P : v \in P\}) + 2,\ b(v) + x'(\{P : v \in P\}) - x(\{P : v \in P\})\}$
3. Find a maximum integral b'-packing y'.
4. Return integral b-packing $x - x' + y'$.

Theorem (Proximity)

If x' is a maximum b'-packing, then x is a maximum b-packing.