Some recent observations on paths, parity, matroids, and polyhedra

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Topics

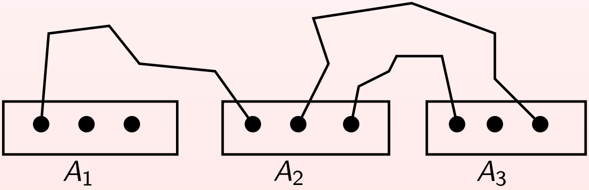
- Edmonds-type blossom-shrinking algorithm for a class of polymatroid matching problems. (Min-max formula: Makai, Szabó, 2006.)
- Fractional packing of $\mathcal{A}$-paths reduces to matroid fractional matching.
- Polynomial time algorithm for node-capacitated packing of $\mathcal{A}$-paths.
- Mader Matroids are Gammoids. (Positive answer to Lex Schrijver's question.)

$\mathcal{A}$-paths

Consider

- a graph $G = (V, E)$,
- a family $\mathcal{A} = \{A_1, \cdots, A_k\}$ of disjoint stable sets,
- let $A := \bigcup \mathcal{A}$.

$\mathcal{A}$-paths are ...



Packing edge-disjoint $\mathcal{A}$-paths

Theorem (Lovász 1976, Cherkassky 1976)

If all nodes in $V - A$ have even degree, then

$$\textit{max integral packing} = \frac{1}{2}\sum_i \lambda(A_i, A - A_i),$$

$\Rightarrow$ following LP is primal and dual half-integral:

$$\begin{aligned} \max\ x \cdot \mathbf{1} \quad & \text{subject to} \\ x &\geq 0 \\ \textstyle\sum_{e \in P} x(P) &\leq 1 \qquad\qquad \text{for all } e \in E \end{aligned}$$

Edge-capacitated fractional packing

- max fractional packing subject to edge-capacities $b \in \mathbb{N}^E$:

$$\max \; x \cdot \mathbf{1} \quad \text{subject to}$$

$$x \geq 0$$

$$\sum_{e \in P} x(P) \leq b(e) \qquad \text{for all } e \in E$$

— min-max formula from Lovász, Cherkassky

• polytime algo via ellipsoid method

Edge-capacitated integral packing

- max integral packing subject to edge-capacities $b \in \mathbb{N}^E$:

  — min-max formula: Mader (1978)
  - polytime algo: Keijsper, Pendavingh, Stougie (1998)

Node-disjoint/node-capacitated packing

- max node-disjoint packing:
  - — min-max formula: Mader (1979)
  - • Lovász (1982) + Schrijver (2001?)
  - • Chudnovsky, Cunningham, Geelen (2005)
  - • P (2005)
- max node-capacitated integral packing:
  - — min-max formula: Mader (1979)
  - • polytime algo: P (2006)
- max node-disjoint fractional packing:
  - — primal and dual half-integral LP
  - — reduction to matroid fractional matching: P (2006)
  - • both imply a polynomial time algorithm

Packing node-disjoint $\mathcal{A}$-paths

Let

$$\nu(G,\mathcal{A}) := \max\{|\mathcal{P}| : \mathcal{P} \subseteq \{\mathcal{A} - \text{paths}\} \text{ a packing }\}$$

$$\nu^*(G,\mathcal{A}) := \max\{x \cdot \mathbf{1} : x \in \mathbb{R}_+^{\{\mathcal{A}-\text{paths}\}} \text{ a fractional packing }\}.$$

Matroid Fractional Matching

Consider

- a finite or infinite matroid $\mathcal{M} = (S, \mathcal{I})$
- a finite set $\mathcal{E}$ of lines, i.e. two-ranked flats of $\mathcal{M}$
- for $K \subseteq S$ we define $d_K \in \{0,1,2\}^{\mathcal{E}}$ by

$$d_K(e) := r\big(K \cap e\big) \quad \text{for } e \in \mathcal{E}.$$

The **Matroid Fractional Matching Polytope** is defined by

$$\mathcal{P}(\mathcal{M}, \mathcal{E}) := \Big\{ x \in \mathbb{R}_+^{\mathcal{E}} \ : \ d_K \cdot x \leq r(K) \quad \text{for all} \quad K \subseteq S \Big\}.$$

Vande Vate's Formula

The maximum size of a fractional matching:

$$\nu^*(\mathcal{M},\mathcal{E}) := \max\left\{\mathbf{1}\cdot x :\ x \in \mathcal{P}(\mathcal{M},\mathcal{E})\right\}$$

Theorem (Vande Vate, 1992)

$$\nu^*(\mathcal{M},\mathcal{E}) = \min\left\{\frac{1}{2}r(K)+\frac{1}{2}r(L) :\quad \frac{1}{2}d_K+\frac{1}{2}d_L \geq \mathbf{1}\right\}$$

Observation (Gijswijt, 2006)

The above description of the matroid fractional matching polytope is totally dual half-integral.

Application of Matroid Fractional Matching

Lex Schrijver's construction:

- Let $l_1, \cdots, l_k$ be distinct 1-dimensional subspaces of $\mathbb{R}^2$,
- for $ab \in E$, $L_{ab} := \{x \in (\mathbb{R}^2)^V : x(a) = x(b),\ x|_{V-a-b} \equiv 0\}$,
- let $Q := \{x \in (\mathbb{R}^2)^V : x(a) \in l_i \text{ for } a \in A_i,\ x|_{V-A} \equiv 0\}$,
- $\mathcal{E} := \{L_{ab}/Q : ab \in E\} < \mathcal{Z} := (\mathbb{R}^2)^V/Q$.

Theorem (Lex Schrijver)

$$\nu(G, \mathcal{A}) = \nu(\mathcal{M}_{\mathcal{Z}}, \mathcal{E}) - |V - A|$$

Theorem (P, 2006)

$$\nu^*(G, \mathcal{A}) = \nu^*(\mathcal{M}_{\mathcal{Z}}, \mathcal{E}) - |V - A|$$

The Partition Formula

- maximum genus graph embedding (Nebeský, 1981) (polytime algorithm: Furst, Gross, McGoech, 1988)
- parity-constrained rooted-k-arc-connected orientation (Frank, Jordán, Szigeti, 2001)
- parity constrained orientation covering an intersecting submodular function (Király, Szabó, 2003)
- pinning-down a minimum number of nodes in $\mathbb{R}^2$ to obtain a generically rigid framework (Fekete, 2005)
- polymatroid parity in "ntcdc-free" polymatroids (Makai, Szabó, 2006)
- Edmonds-type blossom-shrinking algorithm (P, 2006)

Polymatroid Matching

- consider a polymatroid function $b : 2^S \to \mathbb{N}$
- inducing the polymatroid
  $$P(b) := \{x \in \mathbb{R}^S_+ : x(U) \leq b(U) \text{ for all } U \subseteq S\}$$
- matchings are the even vectors in $P(b)$
- $\text{size}(x) := \frac{1}{2}x(S) := \frac{1}{2}\sum_{i \in S} x(i)$
- $\nu(b) := \max\{\ \text{size}(x) : \text{matchings } x \text{ of } b\ \}$

Projection of a polymatroid function

For some $B \subseteq S$, the polymatroid function $b^B : 2^{S-B} \to \mathbb{N}$ obtained from the projection of B is defined by

$$b^B(X) := \min\{\, b(X),\ b(X \cup B) - b(B) + 1 \,\}.$$

1. Consider x, b, where $x \in P(b)$ is an even vector.
2. Check, if $x + \chi_s \in P(b)$ for some $s \in S$.
3. If not, then x is **MAXIMUM**.
4. Next, suppose $x + \chi_s \in P(b)$.
5. If, moreover, $x + 2\chi_s \in P(b)$, then step to x', b, which is an **AUGMENTATION**.
6. Otherwise, let $B :=$ the unique inclusionwise minimal deficient set with respect to $x + 2\chi_s$.
7. Step to x', b', where $b' := b^B$ and $x' := x|_{S-B} \in P(b')$, which is a **CONTRACTION**.