Some new results on node-capacitated packing of $\mathcal{A}$-paths

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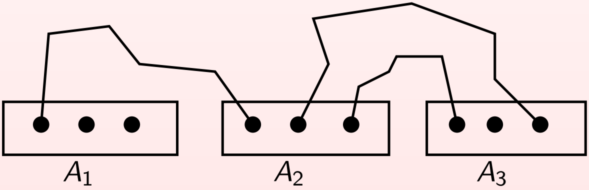
$\mathcal{A}$-paths — definition

- Let $G = (V, E)$ be an undirected graph.



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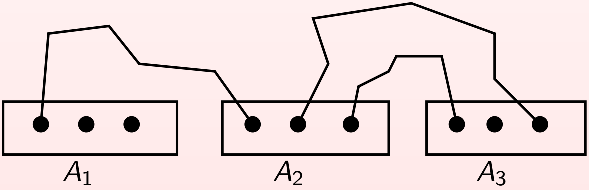
- Let $\mathcal{A} = \{A_1, \cdots, A_k\}$ be a family of disjoint subsets of V, called terminal sets.



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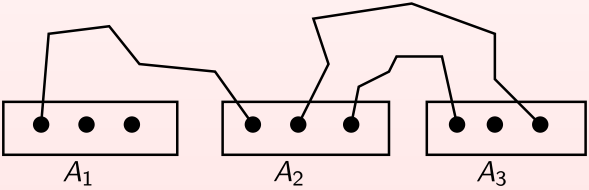
- Let $A := \bigcup \mathcal{A}$, the set of terminal nodes.

A path in G is called an $\mathcal{A}$-**path**, if its ends are in two distinct terminal sets.



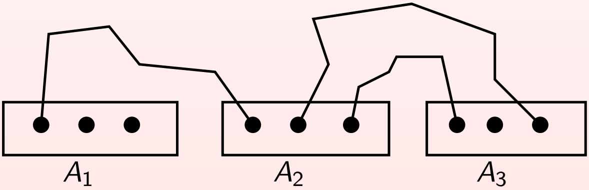
Packing node-disjoint $\mathcal{A}$-paths

Problem:

- find a maximum number of fully node-disjoint $\mathcal{A}$-paths
- notation: $\nu(G, \mathcal{A})$

Special cases:

- disjoint s–t paths
- maximum matching
- edge-disjoint $\mathcal{A}$-paths



Some earlier results:

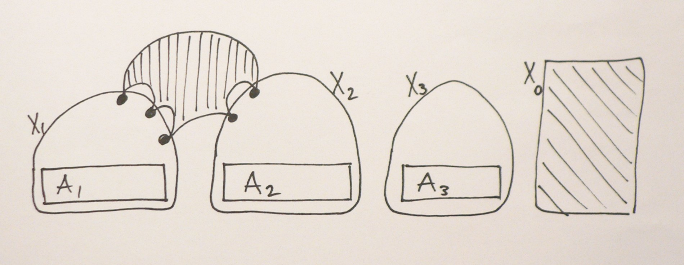
- Gallai, 1961, node-disjoint $\mathcal{A}$-paths for $\mathcal{A} = \{\{a\} : a \in A\}$
- Lovász, Cherkassky, 1976, min-max formula for edge-disjoint A-paths, assuming the Eulerian property
- Mader, 1978, edge-disjoint $\mathcal{A}$-paths, min-max formula
- Mader, 1979, node-disjoint $\mathcal{A}$-paths, min-max formula
- Sebő, Szegő, 2004, structural description
- polytime algorithm: Lovász (1980) + Schrijver (200?)
- generalizations: non-zero A-paths in group-labeled graph (Chudnovsky, Geelen, Gerards, Goddyn, Lohman, Seymour, 2005), non-returning A-paths in permutation-labeled graphs (P, 2006)

Packing node-disjoint $\mathcal{A}$-paths

Mader's Theorem, 1979

$$\nu(G,\mathcal{A}) = \min |X_0| + \sum_{K \in comp(G-X_0-\cup E[X_i])} \left\lfloor \frac{1}{2} |K \cap \ \cup X_i| \right\rfloor$$

where the minimum is taken over disjoint subsets $X_0, X_1, \cdots, X_k \subseteq V$ such that $A_i \subseteq X_0 \cup X_i$ for all i.



Packing $\mathcal{A}$-paths subject to edge- or node-capacitites

Let $b \in \mathbb{N}^V$ (or $\mathbb{N}^E$) denote the capacities.

b-**packing of** $\mathcal{A}$-**paths** is the following integer programming problem, the optimum value of which is denoted by $\nu_b(G, \mathcal{A})$:

$$\max x \cdot \mathbf{1}$$

$$x : \mathcal{A}\text{-paths} \to \mathbb{N}$$

$$\sum_{v \in P} x(P) \leq b(v) \qquad \text{for all } v \in V \text{ (or } E)$$

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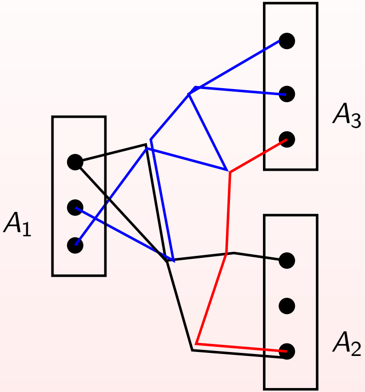
Theorem

The LP, and its dual admit half-integral optima.

- Integrality gap 3/2.

Just an ugly b-packing



Mader's Theorem for b-packing of $\mathcal{A}$-paths

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Special cases:

- maximum flow
- maximum b-matching
- maximum edge-capacitated b-packing of $\mathcal{A}$-paths

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- maximum fractional b-packing via ellipsoid method
- how to construct a maximum integral b-packing from a maximum fractional b-packing?

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- augment value of y one-by-one to reach OPT

Maximum integral flow using Gerards' idea

Proximity Lemma for flows

Let y be an integral flow in G, b. Then the following assertions are equivalent:

- There is an integral flow y' larger than y.
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Corollary

Given G, s, t, b, we can find a maximum integral s-t-flow in polynomial time.

Maximum b-packing of $\mathcal{A}$-paths using Gerards' idea

Proximity Lemma for b-packing of $\mathcal{A}$-paths

Let $(x_{i,j})$ be (the flow-decomposition of) an integral b-packing of $\mathcal{A}$-paths. Then the following two assertions are equivalent:

- There is an integral b-packing $(x'_{i,j})$ larger than $(x'_{i,j})$.
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Corollary

Augmenting an integral b-packing by one can be performed in strongly polynomial time by a reduction to disjoint $\mathcal{A}$-paths.

Theorem, P, 2006

Given $G, \mathcal{A}, b$, we can find a maximum integral b-packing of $\mathcal{A}$-paths in polynomial time.

Matroid Fractional Matching

Consider

- a finite or infinite matroid $\mathcal{M} = (S, \mathcal{I})$
- a finite set $\mathcal{E}$ of lines, i.e. two-ranked flats of $\mathcal{M}$
- for $K \subseteq S$ we define $d_K \in \{0,1,2\}^{\mathcal{E}}$ by

$$d_K(e) := r\big(K \cap e\big) \quad \text{for } e \in \mathcal{E}.$$

The **Matroid Fractional Matching Polytope** is defined by

$$\mathcal{P}(\mathcal{M}, \mathcal{E}) := \Big\{ x \in \mathbb{R}_+^{\mathcal{E}} \ : \ d_K \cdot x \leq r(K) \quad \text{for all} \quad K \subseteq S \Big\}.$$

Vande Vate's Formula

The maximum size of a fractional matching:

$$\nu^*(\mathcal{M},\mathcal{E}) := \max\left\{\mathbf{1}\cdot x : \; x \in \mathcal{P}(\mathcal{M},\mathcal{E})\right\}$$

Theorem (Vande Vate, 1992)

$$\nu^*(\mathcal{M},\mathcal{E}) = \min\left\{\frac{1}{2}r(K) + \frac{1}{2}r(L) : \quad \frac{1}{2}d_K + \frac{1}{2}d_L \geq \mathbf{1}\right\}$$

Observation (Gijswijt, 2006)

The above description of the matroid fractional matching polytope is totally dual half-integral.

Application of Matroid Fractional Matching

Lex Schrijver's construction:

- Let $l_1, \cdots, l_k$ be distinct 1-dimensional subspaces of $\mathbb{R}^2$,
- for $ab \in E$, $L_{ab} := \{x \in (\mathbb{R}^2)^V : x(a) = x(b),\ x|_{V-a-b} \equiv 0\}$,
- let $Q := \{x \in (\mathbb{R}^2)^V : x(a) \in l_i \text{ for } a \in A_i,\ x|_{V-A} \equiv 0\}$,
- $\mathcal{E} := \{L_{ab}/Q : ab \in E\} < \mathcal{Z} := (\mathbb{R}^2)^V/Q$.

Theorem (Lex Schrijver)

$$\nu(G, \mathcal{A}) = \nu(\mathcal{M}_{\mathcal{Z}}, \mathcal{E}) - |V - A|$$

Theorem (P, 2006)

$$\nu^*(G, \mathcal{A}) = \nu^*(\mathcal{M}_{\mathcal{Z}}, \mathcal{E}) - |V - A|$$