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On the maximum even factor in weakly symmetric graphs ☆

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Abstract

As a common generalization of matchings and matroid intersection, Cunningham and Geelen introduced the notion of path-matchings, then they introduced the more general notion of even factor in weakly symmetric digraphs. Here we give a min–max formula for the maximum cardinality of an even factor. Our proof is purely combinatorial. We also provide a Gallai–Edmonds-type structure theorem for even factors.

1. Introduction

Motivated by developing a strongly polynomial separation algorithm for the matchable set polyhedron, Cunningham and Geelen introduced the notion of path-matchings [4]. Their algorithmic approach led them to the notion of even factors [2,5].

In a directed graph, an arc is called symmetric, if the reversed arc is in the arc set of the graph, too. A directed graph is *symmetric*, if all its arcs are symmetric. The directed graph $G=(V,E)$ is said to be *weakly symmetric*, if the arcs in each strongly connected component are symmetric. A set K of edges is called an *even factor* if graph $G_K=(V,K)$ is a collection of node-disjoint directed paths and even directed circuits. The problem is to find an even factor with maximum cardinality. If the

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graph is arbitrary, not necessarily weakly symmetric, then the problem is NP-hard, see [2].

In this paper we give a min–max formula for the maximum cardinality of an even factor and a Gallai–Edmonds-type structure theorem describing the structure of maximum even factors in weakly symmetric graphs.

A set $X \subseteq V$ is said to be a *cut*. We define $N_G^+(X) := \{v \in V - X$: there is a node $u \in X$ such that $uv \in E\}$ and let $G[X]$ be the graph with node set X and arc set $\{uv \in E : u, v \in X\}$. In a directed graph $G = (V, E)$, consider the strongly connected components: a component C having no edge $uv \in E$ such that $u \in V - C$ and $v \in C$, is called a *source component*. Let $odd_G[X]$ denote the number of the source components of $G[X]$ having an odd number of nodes. Let $Odd_G[X]$ denote the union of these components. For an arc (or directed edge) $uv \in E$, u is called the *tail* of arc uv, and v is called the *head*.

$v(G)$ denotes the cardinality of a maximum even factor of G. We prove the following formula for $v(G)$.

Theorem 1.1. *In a weakly symmetric directed graph G one has the following formula for the maximum cardinality of an even factor.*

$$v(G) = |V| + \min_{X \subseteq V} (|N_G^+(X)| - odd_G[X]). \tag{1}$$

Fig. 1 shows an example for which $|V| + |N_G^+(X)| - odd_G[X] = 14 + 2 - 4 = 12$, and an even factor of cardinality 12 is easy to find (the straight lines in the figure correspond to edges oriented in the indicated direction, the curved lines correspond to two edges, one in both directions). If G is not weakly symmetric, then the formula (1) does not necessarily hold, see Fig. 2 for an example.

This formula is a direct extension of the Tutte–Berge formula and also of Kőnig's theorem. The path-matching problem is also a special case of even factors in weakly

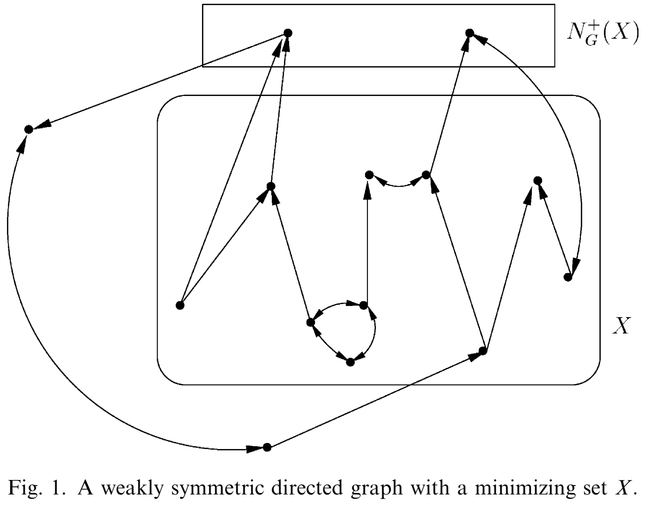

Fig. 1. A weakly symmetric directed graph with a minimizing set X.

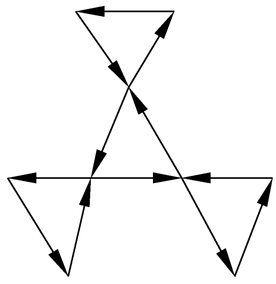

Fig. 2. A directed graph not satisfying equality (1).

symmetric graphs. In [3,4] Cunningham and Geelen gave a min–max formula for the maximum value of a path-matching. In [8] Theorem 1.2, a simplified reformulation of this formula was proved. In the following part, we will discuss corollaries of Theorem 1.1.

Cunningham and Geelen defined a path-matching as follows. Let $G' = (V', E')$ be an undirected graph and $T_1, T_2 \subseteq V'$ disjoint stable sets of G'. We denote $V' - (T_1 \cup T_2)$ by R. A *path-matching* with respect to T_1, T_2 is a set M of edges such that every component of the subgraph $G_M = (V', M)$ having at least one edge is a simple path from $T_1 \cup R$ to $T_2 \cup R$, all of whose internal nodes are in R. The one-edge-components in R are called the *matching edges* of M. The *value* of a path-matching M is defined to be the number $val(M) = |M| + |M'|$, where M' denotes the set of the matching edges of M. (That is, the matching edges count twice.)

We define a *cut for path-matchings* separating the terminal sets T_1 and T_2 to be a subset $Y \subseteq V'$ for which there is no path between $T_1 - Y$ and $T_2 - Y$ in $G' - Y$.

From now on we denote by $odd_{G'}(Y)$ the number of connected components of $G' - Y$ which are disjoint from $T_1 \cup T_2$ and have an odd number of nodes. In [8] the following was proved.

Theorem 1.2. *For the maximum value of a path-matching one has the following formula*:

$$\max_{M \text{ a path-matching}} val(M) = |R| + \min_{Y \text{ a cut}} (|Y| - odd_{G'}(Y)). \tag{2}$$

Now we show how Theorem 1.1 implies Theorem 1.2. Given an instance G', T_1, T_2 of the maximum path-matching problem we construct an instance of the maximum even factor problem as follows: We replace each edge in $i_{G'}(R) := \{uv \in E' : u, v \in R\}$ by a pair of oppositely directed arcs; and orient each other edge, such that nodes in T_1 become source nodes and nodes in T_2 become sink nodes. The resulting digraph G is weakly symmetric.

A path-matching in G' corresponds to an even factor in G, when we replace each matching edge by the two-arc dicircuit. An even factor in G corresponds to a path-matching in G', when we replace the even dicircuits by a matching (a dicircuit can only be in R). Corresponding solutions have same size and value. It is easy to see that a path-matching is maximum in G' if and only if the corresponding even factor is maximum in G.

By Theorem 1.1 we have a cut $X \subseteq V$ in G so that $|V| + |N_G^+(X)| - odd_G[X] = \max_{\text{even factor}}|K| = \max_{\text{path-matching}} val(M)$. It is easy to see that $Y := N_G^+(X - T_2) \cup (T_1 - X)$ is a cut for path-matchings in G'. We have

$$\begin{aligned}
&|R| + |Y| - odd_{G'}(Y)\\
&\quad \leqslant |R| + (|N_G^+(X - T_2)| + |T_1 - X|) - (odd_G[X] - |T_1 \cap X|\\
&\qquad - |T_2 - N_G^+(X - T_2)|)\\
&\quad = (|R| + |T_1| + |T_2|) + |N_G^+(X)| - odd_G[X]\\
&\quad = |V| + |N_G^+(X)| - odd_G[X].
\end{aligned}$$

Hence the maximum value of a path-matching equals to $|R| + |Y| - odd_{G'}(Y)$ for cut Y, which finishes the proof.

We also mention, that Menger's theorem on node-connectivity follows by a reduction to path-matchings.

When considering only acyclic digraphs, source components can only be source nodes. A single node is considered to be a dipath of length zero.

Theorem 1.3. *Let $D = (V, A)$ be a directed acyclic graph. The minimum number of dipaths covering all the nodes of D equals to*

$$\max_{X \subseteq V}(|A| - |B|), \tag{3}$$

where the maximum is taken over the disjoint sets $A, B \subseteq V$ such that no dipath in $G - B$ connects nodes of A.

Proof. For each dipath P in D, $|V(P) \cap A| \leqslant |V(P) \cap B| + 1$, hence the maximum is a lower bound for the number of dipaths.

An even factor of a directed acyclic graph is a dipath-cover. There exist k dipaths covering all the nodes of D if and only if D has an even factor of cardinality $|V| - k$. By Theorem 1.1 we have a set $X \subseteq V$ such that $k = odd_D[X] - |N_D^+(X)|$, and there exist k dipath covering the nodes. Then the choice of $A := Odd_D[X]$ and $B := N_D^+(X)$ finishes our proof. □

We mention the following well-known consequence without defining the notions which are used.

Theorem 1.4 (Dilworth [6]). *Let $P = (V, \leqslant)$ be a partially ordered set. The minimum number of chains covering all the elements of X is equal to the cardinality of a maximum antichain.*

Proof. Let $D=(V,E)$ be a directed graph so that uv is an edge iff $u\leqslant v$. D is acyclic.

Take a pair A,B where the maximum is attained in Theorem 1.3. Trivially, $B=\emptyset$ and A is an antichain. By Theorem 1.3 we also have a dipath cover of $|A|-|B|=|A|$ dipaths. □

In [9] Felsner gave a min–max result for the maximum number of nodes that can be covered by l directed paths in a directed acyclic graph. Theorem 1.3 can be proved from his result.

In the proof of Theorem 1.1, we will use the following well-known facts about factor-critical graphs. An undirected graph $G'=(V',E')$ is said to be *factor-critical* if it is connected and each node is missed by a maximum matching. A symmetric directed graph is defined to be factor-critical, if the underlying undirected graph is factor-critical.

Lemma 1.5 (Gallai's lemma [10]). *If $G'=(V',E')$ is factor-critical, then $|V'|$ is an odd number and a maximum matching of G' has cardinality* $(|V'|-1)/2$.

Recall the definition of $odd_{G'}(Y)$ which denotes the number of components of $G'-Y$ having an odd number of nodes. It follows from Tutte's theorem, that

a connected G' is factor-critical if and only if

$$odd_{G'}(Y)\leqslant|Y|-1 \;\; \forall \;\; Y\subseteq V', |Y|\geqslant 1. \tag{4}$$

The following is an easy corollary of Gallai's lemma for a factor-critical symmetric digraph $G=(V,E)$. Here $\varrho_G(v)$ and $\delta_G(v)$ denote the set of directed edges in G having their head, resp. tail at node v.

$$\begin{aligned} s,t\in V \Rightarrow & \text{ there exists an even factor } K \text{ of cardinality } |V|-1 \\ & \text{such that } |K\cap\varrho_G(s)|=|K\cap\delta_G(t)|=0, \text{ and} \\ & K \text{ consists of an even length } s-t \text{ path and two-arc dicircuits.} \end{aligned} \tag{5}$$

Our proof of Theorem 1.1 is a direct extension of the one of Theorem 1.2 appeared in [8] which mimicked Anderson's simple proof [1] on Tutte's theorem on perfect matchings. In Section 3 an extension of Theorem 1.1, a Gallai–Edmonds-type structure theorem is given for even factors and its proof is based on a thorough investigation of the proof of Theorem 1.1.

2. Proof

First we state some essential properties of weakly symmetric digraphs which are used along the proofs.

Claim 2.1. *Let $G = (V, E)$ be a weakly symmetric digraph. The following operations result in a weakly symmetric digraph.*

- *Delete any node with all of its adjacent edges.*
- *Let C be a strongly connected component of G. Delete an edge having its tail in C and head in $V - C$.*
- *For a node $v \in V$, delete the directed edges having tail at v.*
- *For a node $v \in V$, delete the directed edges having head at v.*
- *Let C be a strongly connected component of G. Shrink C into a single node, delete the loops arising.*

Proof. By definition, a graph is weakly symmetric if and only if its edge-set includes the edges in the opposite direction of each directed cycle. In the first four cases all the directed cycles of the resulting graphs are directed cycles in G as well, and any of the edges of their oppositely directed versions are not deleted.

In the last case if we consider a directed cycle in the resulting graph, then it can be completed to a directed cycle of G since there are directed paths between any pairs of nodes in C. Hence its oppositely oriented cycle is also included. □

A cut X is defined to be *tight* if the minimum in (1) is equal to $|N_G^+(X)| - odd_G[X]$. A cut X is called *trivial* if one of the following holds:

(i) The source components of $G[X]$ are single nodes, $V = (X \cup N_G^+(X))$ and there is no arc uv such that $u \in N_G^+(X)$.
(ii) X is a stable set in G, and there is no arc uv such that $u \in X$ and $v \in V - X$.

The concept of the definition is the following. Having a nontrivial cut contributes to running the inductive proof in Case 2. The forthcoming dividing procedure of Case 2 does not necessarily result in graphs with smaller number of edges than of G, but in case of a nontrivial cut, it does.

Observation 2.2. *$X = V$ is the only tight cut of type* (i).

Proof. If $X \neq V$ is a tight cut of type (i), then $|V| + |N_G^+(X)| - odd_G[X] > |V| + |N_G^+(X \cup N_G^+(X))| - odd_G[X \cup N_G^+(X)] = |V| - odd_G[V]$, a contradiction. □

Let G be a symmetric digraph and let G_u denote the underlying undirected graph of G.

Claim 2.3. *Let K be a maximum even factor of G and M be a maximum matching of G_u. $|K| = 2|M|$.*

Proof of Theorem 1.1. First we prove that for any even factor K and cut X we have $|K| \leqslant |V| + |N_G^+(X)| - odd_G[X]$. The sum of the following three observations gives this.

$$|i_G(X) \cap K| \leqslant |X| - odd_G[X], \tag{6}$$

$$|\delta_G(X) \cap K| \leqslant |N_G^+(X)|, \tag{7}$$

$$|(i_G(V - X) \cup \delta_G(V - X)) \cap K| \leqslant |V| - |X|, \tag{8}$$

where $i_G(X)$ denotes the set of the arcs of G with both ends in X and $\delta_G(X)$ denotes the set of the arcs of G with tail in X and head in $V - X$.

The proof that there is a cut X and an even factor K such that $|K| = |V| + |N_G^+(X)| - odd_G[X]$ goes by an induction on $|E|$. If $|E| \leqslant 1$, then the theorem is obviously true. If G is strongly connected, then a maximum even factor corresponds to a maximum matching by Claim 2.3, formula (1) follows from Berge–Tutte formula. Hence we assume that there is at least one arc uv in G so that arc vu does not exist. It is easy to see that if $X = \emptyset$ is a tight cut, then there exists a nonempty tight cut: for example, a strongly connected sink component.

Case 1: Every tight cut is trivial.

We use $\tau_G(X) := |V| + |N_G^+(X)| - odd_G[X]$ as the *value* of cut X in G. Let $\tau_G := \min_{X \text{ a cut}} \tau_G(X)$ be the value of a tight cut in G. Let $uv = e \in E$ be an arc having its tail u in a source component C of G and having its head v in $V - C$. $G - e$ is a weakly symmetric digraph by Claim 2.1.

For any cut X we have

$$\tau_{G-e}(X) \leqslant \tau_G(X) \leqslant \tau_{G-e}(X) + 1. \tag{9}$$

In (9) we have $\tau_G(X) = \tau_{G-e}(X) + 1$ if and only if for $e = uv$ either

(A) $u \in X$ and $v \in V - X - N_{G-e}^+(X)$ or

(B) $u \in X$ and $v \in Odd_{G-e}[X]$ and u, v are in different strongly connected components of $G[X] - e$.

If $\tau_{G-e} = \tau_G$, then we are done by induction. Otherwise (9) implies, that for a tight cut X in G

$$\tau_G = \tau_G(X) = \tau_{G-e}(X) + 1.$$

Take a tight cut X in G. By assumption, X is a trivial cut in G. Arc e accords to (A) or (B), so X cannot be of type (ii). Thus X is a trivial tight cut of type (i). By Observation 2.2 $X = V$.

Hence $C = \{u\}$ is a single node source component in G. Arc $e = uv$ cannot be of type (A), because $X = V$. Arc $e = uv$ is of type (B), thus $V - u$ is a tight cut in G. By Observation 2.2, $V - u$ can only be a cut of type (ii) in G. Then the arc set E consists of some arcs with tail in u. In this case $\tau_G(V - u) = 1$, and $K = e$ is an even factor of size 1, this completes the proof in Case 1.

Case 2: There exists a nontrivial tight cut. Let us consider a minimal nontrivial nonempty tight cut X.

Claim 2.4. *Each source component of $G[X]$ is factor-critical.*

Proof. If a source component C has an even number of nodes, then for any $v \in C$ the following holds: $\tau_G(X - v) \leqslant \tau_G(X)$, contradicting the minimality of X. Suppose $|C|$ is an odd number. If a subset $\emptyset \neq Y \subseteq C$ gives $odd_{G_u[C]}(Y) \geqslant |Y| + 1$, then we would have $\tau_G(X - Y) \leqslant \tau_G(X)$, a contradiction. Thus by parity, for each $\emptyset \neq Y \subseteq C$ $odd_{G_u[C]}(Y) \leqslant |Y| - 1$, and $G_u[C]$ is factor-critical by (4). □

Let $G_Q = (V_Q, E_Q)$ denote the weakly symmetric graph we get by contracting each component of $Odd_G(X)$ to a node. Let Q denote the set of new nodes, $X_Q \coloneqq X - Odd_G[X] \cup Q$. Remark $|Q| = odd_G[X]$ and $V_Q = X - Odd_G[X] \cup Q \cup (V - X) = X_Q \cup (V - X)$. G_Q is weakly symmetric by Claim 2.1. Now we define two subgraphs of G_Q.

Let $G_1 = (V_1, E_1)$ denote the graph having node set $V_1 \coloneqq X_Q \cup N_G^+(X)$ and arc set $E_1 \coloneqq \{uv \in E_Q : u \in X_Q\}$.

Let $G_2 = (V_2, E_2)$ denote the graph having node set $V_2 \coloneqq Q \cup (V_Q - X_Q)$ and arc set $E_2 \coloneqq \{uv \in E_Q : v \in V_2 - N_G^+(X)\}$.

Both G_1 and G_2 are weakly symmetric by Claim 2.1. These two graphs may have nodes in common, but have disjoint arc sets. Since X is nontrivial, $|E_1| < |E|$ and $|E_2| < |E|$.

We are going to show that G_Q has an even factor K_Q with cardinality $|K_Q| = |V_Q| + |N_G^+(X)| - odd_G[X]$, which finishes the proof by the following claim.

Claim 2.5. *If K_Q is an even factor of G_Q, then G has an even factor K with cardinality $|K| \coloneqq |K_Q| + (|Odd_G[X]| - |Q|)$.*

Proof. Let K' denote the set of arcs of G corresponding to K_Q. We claim that K' can be completed in G so that it has the desired cardinality. To this end let C denote a component of $Odd_G[X]$, and let c denote its corresponding node in G_Q. By Claim 2.4, C is factor-critical. K' has at most one arc in $\delta_G(C)$: choose $t \in C$ as the tail of this arc if present, otherwise choose t arbitrarily. K' has at most one arc in $\varrho_G(C)$: choose $s \in C$ as the head of this arc if present, otherwise choose s arbitrarily. ($t = s$ may happen.) By (5), there is an even factor K_C in $G[C]$ of size $|C| - 1$.

$K \coloneqq K' \cup \bigcup_{c \in Q} K_C$ is an even factor K with cardinality $|K_Q| + (|Odd_G[X]| - |Q|)$. □

Claim 2.6. *G_1 has an even factor K_1 with cardinality $|V_1| - odd_G[X]$.*

Proof. $\tau_{G_1}(V_1) = |V_1| - odd_{G_1}[V_1] = |V_1| - odd_G[X]$, hence it is enough to prove, that $\tau_{G_1}(Y) \geqslant |V_1| - odd_G[X]$ holds for all $Y \subseteq V_1$, and we are done by induction.

$\tau_{G_1}(Y) \geqslant \tau_{G_1}(Y \cup N_G^+(X))$, hence we suppose that $N_G^+(X) \subseteq Y \subseteq V_1$. Let $S \coloneqq \{v \in N_G^+(X)$: there is no arc uv with $u \in Y - N_G^+(X)\}$.

We have $N_{G_1}^+(X_Q \cap Y) = N_{G_1}^+(Y) \cup (N_G^+(X) - S)$, thus

$$|N_{G_1}^+(X_Q \cap Y)| \leqslant |N_{G_1}^+(Y)| + |N_G^+(X)| - |S|, \tag{10}$$

$$odd_{G_1}[Y] - |S| = odd_{G_1}[X_Q \cap Y]. \tag{11}$$

Let Y_G denote the resulting set after replacing the nodes of $Y \cap Q$ by the corresponding components of $Odd_G[X]$. Since X is a tight cut in G,

$$|V| + |N_G^+(X)| - odd_G[X] \leqslant |V| + |N_G^+(X \cap Y_G)| - odd_G[X \cap Y_G]. \tag{12}$$

It is easy to see that $odd_G[X] = |Q| = odd_{G_1}[X_Q]$, $N_G^+(X \cap Y_G) = N_{G_1}^+(X_Q \cap Y)$, and $odd_G[X \cap Y_G] = odd_{G_1}[X_Q \cap Y]$. Then by inequality (12) we get

$$|N_G^+(X)| - odd_{G_1}[X_Q] \leqslant |N_{G_1}^+(X_Q \cap Y)| - odd_{G_1}[X_Q \cap Y]. \tag{13}$$

By adding up (10), (11) and (13)

$$odd_{G_1}[Y] - odd_{G_1}[X_Q] \leqslant |N_{G_1}^+(Y)|. \tag{14}$$

Thus,

$$|V_1| - odd_G[X] = |V_1| - odd_{G_1}[X_Q] \leqslant |V_1| + |N_{G_1}^+(Y)| - odd_{G_1}[Y]$$
$$= \tau_{G_1}(Y). \qquad \square \tag{15}$$

Claim 2.7. *G_2 has an even factor K_2 with cardinality $|V_Q| - |X_Q|$.*

Proof. $\tau_{G_2}(Q) = |V_2| - odd_{G_2}[Q] = |V_2| - |Q| = |V_Q| - |X_Q|$. Hence it is enough to prove, that $\tau_{G_2}(Z) \geqslant |V_Q| - |X_Q|$ holds for all $Z \subseteq V_2$, and we are done by induction.

$\tau_{G_2}(Z) \geqslant \tau_{G_2}(Z \cup Q)$, hence we suppose that $Q \subseteq Z \subseteq V_2$. Let Z_G denote the resulting set after replacing the nodes of Q by the corresponding $Odd_G[X]$ components in Z.

We have $N_G^+(X \cup Z_G) = (N_G^+(X) - (Z \cap N_G^+(X))) \cup N_{G_2}^+(Z)$, thus

$$|N_G^+(X \cup Z_G)| = |N_G^+(X)| - |Z \cap N_G^+(X)| + |N_{G_2}^+(Z)|. \tag{16}$$

Since X is tight in G,

$$|V| + |N_G^+(X)| - odd_G[X] \leqslant |V| + |N_G^+(X \cup Z_G)| - odd_G[X \cup Z_G]. \tag{17}$$

Now we prove inequality (18). Consider the odd source components of $G_2[Z]$. These are all the nodes in $Z \cap N_G^+(X)$ as single node components and some other components disjoint from $N_G^+(X)$. The latter type components are odd source components of $G[X \cup Z_G]$, too. This proves

$$odd_{G_2}[Z] - |Z \cap N_G^+(X)| \leqslant odd_G[X \cup Z_G]. \tag{18}$$

By adding up (16), (17) and (18)

$$odd_{G_2}[Z] - |Q| = odd_{G_2}[Z] - odd_G[X] \leqslant |N_{G_2}^+(Z)|. \tag{19}$$

Thus,

$$|V_Q| - |X_Q| = |V_2| - |Q| \leqslant |V_2| + |N_{G_2}^+(Z)| - odd_{G_2}[Z].$$

Claim 2.8. *If K_1, K_2 are even factors in G_1, G_2, respectively, then $K_Q := K_1 \cup K_2$ is an even factor in G_Q.*

Proof. Since the in-degree $\varrho_{K_Q}(v) \leqslant 1$ and the out-degree $\delta_{K_Q}(v) \leqslant 1$ for all $v \in V_Q$, we have to prove that there is no odd cycle in K_Q. Let C denote a cycle of K_Q which is not a cycle in K_1 nor in K_2. Since a cycle cannot have an arc uv with $u \in V - (X \cup N_G^+(X))$ and $v \in X \cup N_G^+(X)$, the nodes of C are in $X_Q \cup N_G^+(X)$. By the definition of G_1 and G_2, C contains a node u in Q. For any node $v \in Q$ of C let arc $e = vw$ be the arc leaving v in C. By the weakly symmetry, arc wv is in G, hence $w \in N_G^+(X)$. Hence C is a cycle in the bipartite subgraph of G_Q on $Q \cup N_G^+(X)$. □

Now we have $|K_Q| = |V_Q| + |N_G^+(X)| - odd_G[X]$, thus we have finished the proof of Theorem 1.1 by Claim 2.5. □

3. A Gallai–Edmonds-type structure theorem

The following theorem plays an important role in Matching Theory. It asserts, that there is a canonical set that attains minimum in the Berge–Tutte formula, and this set has special properties.

Theorem 3.1 (Gallai–Edmonds structure theorem [7,11] See [12]). *Let $G = (V, E)$ be an undirected graph. D denotes the set of nodes which are not covered by at least one maximum matching of G. Let A be the set of nodes in $V - D$ adjacent to at least one node in D. Let $C = V - A - D$. Then*:

1. *The number of the covered nodes by a maximum matching in G equals to $|V| + |A| - c(D)$, where $c(D)$ denotes the number of components of the graph spanned by D.*
2. *The components of the subgraph induced by D are factor-critical.*
3. *The subgraph induced by C has a perfect matching.*
4. *The bipartite graph obtained from G by deleting C and the edges in A and by contracting each component of D to a single node has the following property: there is a matching covering A after deleting any node coming from D.*
5. *If M is any maximum matching of G, then $E(D) \cap M$ covers all the nodes except one of any component of D, $E(C) \cap M$ is a perfect matching and M matches all the nodes of A with nodes in distinct components of D.*

In [13] a Gallai–Edmonds-type structure theorem was proved for path-matchings as a generalization of Theorem 3.1. By the same reduction principle as in Section 1, it can easily be deduced from the even factor structure theorem, which is the following:

Theorem 3.2 (Structure theorem). *Let $G = (V, E)$ be a weakly symmetric digraph. Let $D := \{v \in V$: there exists a maximum even factor K such that $\delta_K(v) = 0\}$. Let $A := N_G^+(D)$, and $C := V - D - A$.*

1. $v(G) = |V| + (|N_G^+(D)| - odd_G[D])$,
2. *The strongly connected source components of $G[D]$ are factor-critical*,

3. *For any maximum even factor* K, *the following properties hold*:
   - *For all the nodes* v *of* D *except one of any source component of* $G[D]$, $\varrho_F(v) = 1$, *where* $F := i_G(D) \cap K$.
   - A *is covered by edges of* K *coming out of* D.
   - $\delta_K(v) = 1$ *for any* $v \in C \cup A$, *furthermore the head of any arc of* K *coming out of* v *is in* $C \cup Odd_G[D]$.

Remark that set X in Fig. 1 is equal to D defined in the previous theorem.

Proof of Theorem 3.2. Let X be a tight cut such that $|X|$ is minimum. We are going to prove that $X = D$.

Claim 3.3. *Each source component of* $G[X]$ *is factor-critical.*

Proof. Since X is also minimal tight this is straightforward from Claim 2.4. □

First we prove that $D \subseteq X$. Take any node $v \in D$. Let K_v be an even factor of size $|K_v| = \tau_G = \tau_G(X)$, with $\delta_{K_v}(v) = 0$. By formula (1), for $K = K_v$, we must have equality in (6), (7), and (8). From equality in (8) we get that $v \notin V - X$.

Now we prove $X \subseteq D$. Consider G_Q, G_1 and G_2 which were defined for any tight cut in the proof of Theorem 1.1.

Claim 3.4. *For any* $v \in X_Q$, G_1 *has an even factor* K_1 *with cardinality* $|V_1| - odd_G[X]$, *such that* $\delta_{K_1}(v) = 0$.

Proof. Let G_1' denote the graph obtained from G_1 by deleting the arcs coming out of v. G_1' is weakly symmetric by Claim 2.1. We have to prove that there is an even factor in G_1' of cardinality $|V_1| - odd_G[X]$.

We are going to prove, that $\tau_{G_1}(Y) \geqslant |V_1| - odd_G[X] + 1$ for any $Y \subseteq V_1 - v$. By Theorem 1.1 it is enough, because $\tau_{G_1}(Y + v) \leqslant \tau_{G_1}(Y) - 1$ for any set $Y \subseteq V_1 - v$.

If $Y \subseteq V_1 - v$, then $\tau_{G_1}(Y) \geqslant \tau_{G_1}(Y \cup N_G^+(X))$, hence we suppose that $N_G^+(X) \subseteq Y \subseteq V_1 - v$. Let $S := \{w \in N_G^+(X)$: there is no arc uw with $u \in Y - N_G^+(X)\}$. We have $N_{G_1}^+(X_Q \cap Y) = N_{G_1}^+(Y) \cup (N_G^+(X) - S)$, thus

$$|N_{G_1}^+(X_Q \cap Y)| \leqslant |N_{G_1}^+(Y)| + |N_G^+(X)| - |S|, \tag{20}$$

$$odd_{G_1}[Y] - |S| = odd_{G_1}[X_Q \cap Y]. \tag{21}$$

Let Y_G denote the resulting set after replacing the nodes of $Y \cap Q$ by the corresponding $Odd_G[X]$ components in Y. Since X is a minimum tight cut in G,

$$|V| + |N_G^+(X)| - odd_G[X] + 1 \leqslant |V| + |N_G^+(X \cap Y_G)| - odd_G[X \cap Y_G]. \tag{22}$$

It is easy to see that $odd_G[X] = |Q| = odd_{G_1}[X_Q]$, and $odd_G[X \cap Y_G] = odd_{G_1}[X_Q \cap Y]$. Then by inequality (22) we get

$$|N_G^+(X)| - odd_G[X_Q] + 1 \leqslant |N_{G_1}^+(X_Q \cap Y)| - odd_{G_1}[X_Q \cap Y]. \tag{23}$$

By adding up (20), (21) and (23)

$$odd_{G_1}[Y] - odd_G[X_Q] + 1 \leqslant |N^+_{G_1}(Y)|.$$

Thus,

$$|V_1| - odd_G[X_Q] + 1 \leqslant |V_1| + |N^+_{G_1}(Y)| - odd_{G_1}[Y] = \tau_{G_1}(Y).$$

Take any $v \in X_Q$. By Claim 2.7 there is an even factor K_2 in G_2 of cardinality $|V_Q| - |X_Q|$, and by Claim 3.4 there is an even factor K_1 of cardinality $|V_1| - odd_G[X]$ such that $\delta_{K_1}(v) = 0$. As was shown, $K_Q := K_1 \cup K_2$ is an even factor in G_Q, and it is easy to see that $\delta_{K_Q}(v) = 0$. If $v \notin Q$, by (the proof of) Claim 2.8 we get a maximum even factor K in G with $\delta_K(v) = 0$. If $v \in Q$, then the construction gives a maximum even factor K in G which has no edges coming out of the factorcritical component C corresponding to v. Take any $v' \in C$, then for the construction of K_C we can choose $t = v'$. The maximum even factor K will have $\delta_K(v') = 0$.

We have proved, that $X \subseteq D$, which implies 1. Hence 2. follows from Claim 3.3, the further statements follow from equality in (6), (7), and (8). □

If we reverse the orientation of the edges, we get the following structural result. $N^-_G(X) := \{v \in V - X :$ there is a node $u \in X$ such that $vu \in E\}$. Let $odd^*_G[X]$ denote the number of the strongly connected components of $G[X]$ with no leaving arc (i.e. sink components) having an odd number of nodes.

Theorem 3.5. *Let $D^* := \{v \in V$: there exists a maximum even factor K such that $\varrho_K(v) = 0\}$. Let $A^* := N^-_G(D^*)$, and $C^* := V - D^* - A^*$.*

1. $\nu(G) = |V| + (|N^-_G(D^*)| - odd^*_G[D^*])$,
2. *The strongly connected sink components of $G[D^*]$ are factor-critical,*
3. *For any maximum even factor K, the following properties hold*
   - *For all the nodes v of D^* except one of any sink component of $G[D]$, $\delta_F(v) = 1$, where $F := i_G(D^*) \cap K$.*
   - *A^* is covered by edges of K entering D^*.*
   - *$\varrho_K(v) = 1$ for any $v \in C^* \cup A^*$, furthermore the tail of any arc of K entering v is in $C^* \cup D^*$.*

The following result gives the connection between the two—possibly different—canonical tight cuts.

Proposition 3.6. *Let W be a component of $D \cap D^*$. Then W is a source component of D and a sink component of D^*. Furthermore, $D \cap D^* = \{v \in V$: there exists a maximum even factor K such that $\varrho_K(v) = \delta_K(v) = 0\}$.*

Proof. Let $v \in D \cap D^*$. After deleting the directed edges entering v the minimum in (1) does not decrease. Hence, by part 3. of Theorem 3.2, v is in a source component of D. Similarly, v is in a sink component of D^*. Let W' denote the source component of $G[D]$ containing v.

Let K be a maximum even factor for which $|\varrho_K(v)| = 0$. By part 3. of Theorem 3.2, K covers the nodes of W' by a path P (perhaps consisting only of the single node v) and by even circuits. Since $|\varrho_K(v)| = 0$, K has at most one arc (of P) in $\delta_G(W')$, and no arc in $\varrho_G(W')$. Let w be an arbitrary node of W'. Since W' is factor-critical, K can be modified using (5) so, that for the obtained maximum even factor K', $|\varrho_{K'}(w)| = 0$. Hence $W' \subseteq D^*$, we have proved that $D \cap D^*$ is the union of some components of $Odd_G[D]$. By symmetry, $D \cap D^*$ is the union of some components of $Odd^*_G[D^*]$, the first part of the proposition follows.

Let $v \in D \cap D^*$, and let $v_Q \in Q$ be the corresponding node in G_Q. By the definition of D, G has a maximum even factor K_1' such that $\delta_{K_1'}(v) = 0$. $K_1 = K_1' \cap E[G_1]$ is an even factor K_1 such that $\delta_{K_1}(v_Q) = 0$. By the definition of D^*, G has a maximum even factor K_2^* such that $\varrho_{K_2^*}(v) = 0$. $K_2 = K_2^* \cap E[G_2]$ is an even factor K_2 such that $\varrho_{K_2}(v_Q) = 0$. Then $K = K_1 \cup K_2$ is a maximum even factor of G_Q such that $\varrho_K(v_Q) = \delta_K(v_Q) = 0$. Then choosing $s = t = v$, the construction of Claim 2.5 gives a maximum even factor K in G such that $\varrho_K(v) = \delta_K(v) = 0$. □

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