

A new approach to alternating paths

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The Hungarian Method is 50

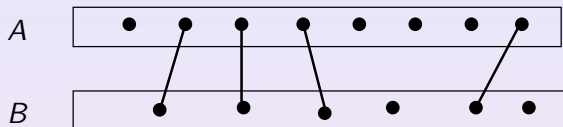
“Relax – Reduce”

A black-box approach to combinatorial algorithms

“3-Way Lemma” and an “Equivalent Reduction”

- König's Theorem
- Square-free simple 2-matchings
- Further applications:
 - non-bipartite matching
 - path-matching, even factor
 - packing node-disjoint \mathcal{A} -paths
 - hypo-matching in directed graphs

Bipartite Matching

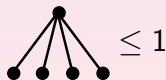


König's Theorem

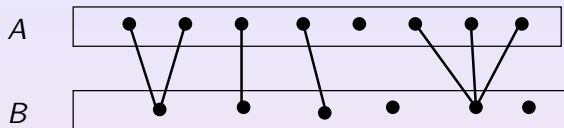
If $G = (A, B; E)$ is a bipartite graph, then

the maximum cardinality of a matching
=
the minimum total capacity of a cover.

capacity:



The “Relax – Reduce” approach for bipartite matching

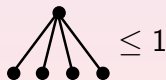


Observation

If $G = (A, B; E)$ is a bipartite graph, then

the maximum cardinality of an A -matching
=
the minimum total capacity of a cover.

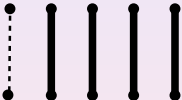
capacity:



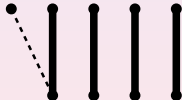
The “Relax – Reduce” approach for bipartite matching

Given a matching M . Is M a max A -matching?

YES \Rightarrow we are done.

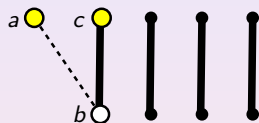
NO \Rightarrow \exists  \Rightarrow **augment**

or

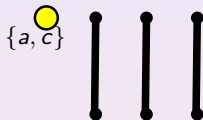
\exists  \Rightarrow **reduce ...**

The “Relax – Reduce” approach for bipartite matching

...the reduction:



G, M



$(G - b) / \{a, c\}, M - bc$

$N + ab$ or $N + bc$



N a larger matching

$X + b$



X a verifying cover

The “Relax – Reduce” approach for bipartite matching

“3-Way Lemma”

Given a matching M in bipartite graph $G = (A, B; E)$, then at least one of the following alternatives holds:

- a) M is a maximum A -matching.
- b) There is a matching N with $|N| = |M| + 1$.
- c) There are nodes $a, c \in A, b \in B$ with $bc \in M$ and $a \notin V(M)$.

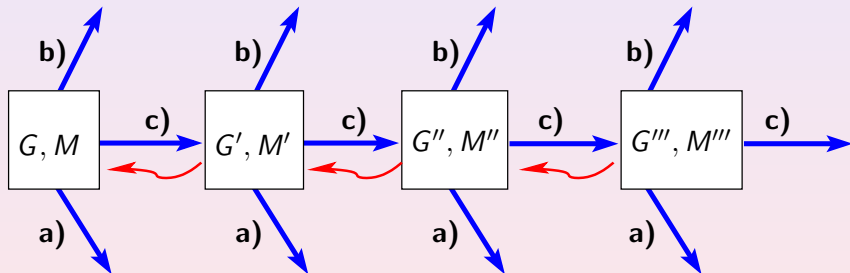
“Equivalent Reduction”

Suppose $a, c \in A, b \in B$ with $bc \in M$ and $a \notin V(M)$. Then

$$M \text{ is max in } G \iff M - bc \text{ is max in } (G - b) / \{a, c\}.$$

The “Relax – Reduce” approach for bipartite matching

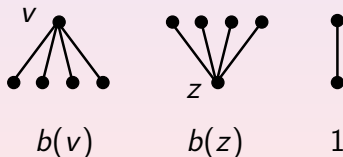
The algorithmic scheme:



Theorem

$G = (A, B; E)$ a simple bipartite graph with $b : A \cup B \rightarrow \{0, 1, 2\}$.
Then

the maximum cardinality of a simple b -matching
=
the minimum total capacity of a cover.



capacity:

$b(v)$

$b(z)$

1

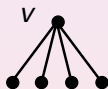
Square-free simple 2-matchings

Theorem

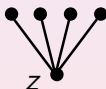
Z. Király, 1999

$G = (A, B; E)$ a simple bipartite graph with $b : A \cup B \rightarrow \{0, 1, 2\}$.
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the maximum cardinality of a square-free simple b -matching
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$b(v)$



$b(z)$



1



3

capacity:

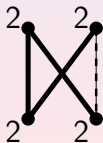
“Relax – Reduce” for square-free simple 2-matchings

“3-Way Lemma”

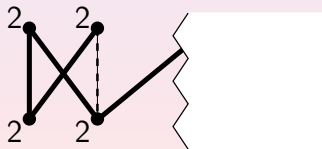
Given a square-free b -matching M in G , then at least one of the following alternatives holds:

- a) M is a maximum b -matching.
- b) There is a square-free b -matching N with $|N| = |M| + 1$.
- c) There is a square-free b -matching N with $|N| = |M|$ s.t. there is square which is “nice with N ”.

nice square:



or

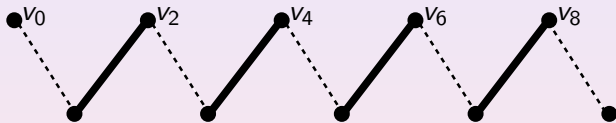


“Relax – Reduce” for square-free simple 2-matchings

Proof.

If M a maximum b -matching \implies a)

Otherwise there is an augmenting M -alternating path P :



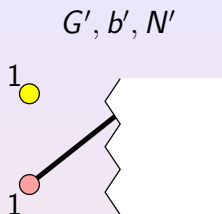
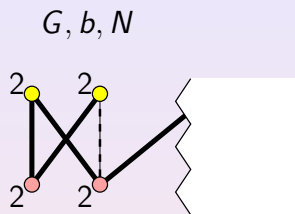
If $M \Delta P$ is square-free \implies b)

Otherwise, $P_{2i} :=$ the sub-path from v_0 to v_{2i} . $M_i := M \Delta P_{2i}$

Some M_i is square-free, but M_{i+1} is not square-free.

Then $N := M_i$ is nice with the square in M_{i+1} . \implies c)

“Relax – Reduce” for square-free simple 2-matchings



“Equivalent Reduction”

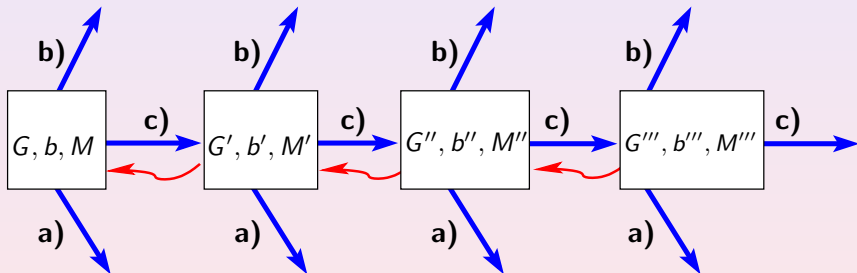
N is a maximum \square -free b -matching in G



N' is a maximum \square -free b' -matching in G' .

The “Relax – Reduce” square-free simple 2-matchings

The algorithmic scheme:

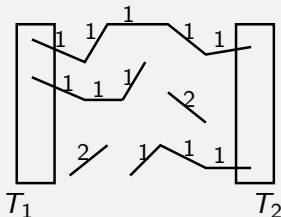


3-Way Lemma for Path-Matching

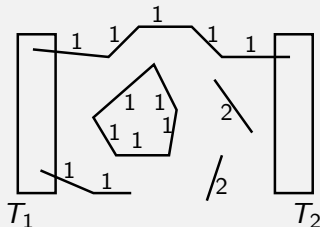
Given a path-matching x with respect to G, T_1, T_2 . Then at least one of the following alternatives holds:

- x is a maximum path-cycle-matching.
- There is a path-matching x' with $\|x'\| = \|x\| + 1$.
- There is a path-matching x' with $\|x'\| = \|x\|$ s.t. there is an odd cycle C which is nice with x' .

Path-Matching



Path-Cycle-Matching

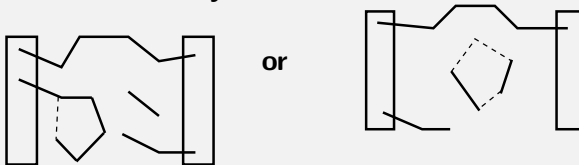


3-Way Lemma for Path-Matching

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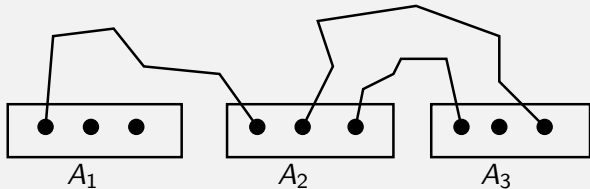
a nice odd cycle:



3-Way Lemma for \mathcal{A} -paths

Given a packing \mathcal{P} of \mathcal{A} -paths in G . Then at least one of the following alternatives holds:

- \mathcal{P} is a maximum fractional packing of \mathcal{A} -paths.
- There is a packing \mathcal{P}' with $|\mathcal{P}'| = |\mathcal{P}| + 1$.
- There is a packing \mathcal{R} with $|\mathcal{R}| = |\mathcal{P}|$ s.t. there is an odd cycle, or a rod which is nice with \mathcal{R} .

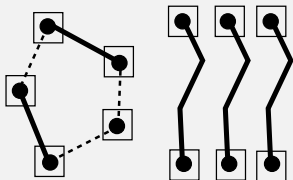


3-Way Lemma for \mathcal{A} -paths

Given a packing \mathcal{P} of \mathcal{A} -paths in G . Then at least one of the following alternatives holds:

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- \mathcal{P} is a maximum fractional packing of \mathcal{A} -paths.
- There is a packing \mathcal{R} with $|\mathcal{R}| = |\mathcal{P}|$ s.t. there is an odd cycle, or a rod which is nice with \mathcal{R} .

a nice odd cycle:



a nice "rod"

