Matching Problems in Polymatroids Without Double Circuits*

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Abstract. According to the present state of the theory of the matroid matching problem, the existence of a good characterization to the size of a maximum matching depends on the behavior of certain substructures, called double circuits. In this paper we prove that if a polymatroid has no double circuits at all, then a partition-type min-max formula characterizes the size of a maximum matching. We provide applications of this result to parity constrained orientations and to a rigidity problem.

A polynomial time algorithm is constructed by generalizing the principle of shrinking blossoms used in Edmonds' matching algorithm [2].

Keywords: matroids and submodular functions.

1 Introduction

Polymatroid matching is a combinatorial optimization problem which is concerned with parity and submodularity. Early well-solved special cases are the matching problem of graphs and the matroid intersection problem, which have in fact motivated Lawler to introduce the matroid and polymatroid matching problems. Jensen, Korte [6], and Lovász [9] have shown that, in general, the matroid matching problem is of exponential complexity under the independence oracle framework. The major breakthrough came when Lovász gave a good characterization to the size of a maximum matching and also a polynomial algorithm for linearly represented matroids [12,9]. Lovász [10], and Dress and Lovász [1] observed that the solvability of the linear case is due to the fact that these matroids can be embedded into a matroid satisfying the so-called *double circuit property*, or *DCP* for short. It was also shown that full linear, full algebraic, full graphic, and full transversal matroids are DCP matroids [1]. The disadvantage of this approach is that, due to the embedding into a bigger matroid, the min-max formula is rather difficult to interpret in a combinatorial way, and often does not even imply a good characterization. However, the diversity and the importance of solvable special cases of the matroid matching problem is a motivation to explore those techniques implying a combinatorial characterization.

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In this paper we investigate the class of those polymatroids having no non-trivial compatible double circuits, called ntcdc-free for short, defined later. We prove that in these polymatroids a partition-type combinatorial formula characterizes the maximum size of a matching. We remark that in the min-max formula for DCP matroids, for example representable matroids, we have to take a partition and a projection into consideration. Contrarily, in ntcdc-free polymatroids, it suffices to consider partitions in the min-max formula. As an application, we show that two earlier results are special cases of this approach. The first application is that the parity constrained orientation problem of Király and Szabó [7] can be formulated as a matching problem in a ntcdc-free polymatroid, which implies the partition-type formula given in [7]. Second, we deduce a result of Fekete [3] on the problem of adding a clique of minimum size to a graph to obtain a graph that is generically rigid in the plane.

1.1 The Partition Formula

To formulate our main result, some definitions are in order. We denote by $\mathbb{R}_+$ and $\mathbb{N}$ the set of non-negative reals and non-negative integers, respectively. Let S be a finite ground set. A set-function $f : 2^S \to \mathbb{Z}$ is called *submodular* if

$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y) \tag{1}$$

holds whenever $X, Y \subseteq S$. b is called *supermodular* if $-b$ is submodular. The set-function f is said to be *non-decreasing* if $f(X) \leq f(Y)$ for every $\emptyset \neq X \subseteq Y \subseteq S$, and we say that f is *non-increasing* if $-f$ is non-decreasing. A non-decreasing submodular set-function $f : 2^S \to \mathbb{N}$ with $f(\emptyset) = 0$ is called a *polymatroid function.* A polymatroid function $f : 2^S \to \mathbb{Z}_+$ induces a *polymatroid* $P(f)$ and a *base polyhedron* $B(f)$ defined by

$$P(f) := \{x \in \mathbb{R}^S : x \geq 0, x(Z) \leq f(Z) \text{ for all } Z \subseteq S\}, \tag{2}$$

$$B(f) := \{x \in \mathbb{R}^S : x(S) = f(S), \text{ and } x \geq 0, x(Z) \leq f(Z) \text{ for all } Z \subseteq S\}, \tag{3}$$

where $x(Z) := \sum_{i \in Z} x_i$ for some $Z \subseteq S$. A vector $m \in \mathbb{Z}^S$ is called *even* if m_i is even for every $i \in S$. The even vectors $m \in P(f)$ are called the *matchings* of f. The *size* of a matching is $m(S)/2$. The *polymatroid matching problem* is to find a *maximum matching*, i.e. a matching of maximum size

$$\nu(f) = \max\{m(S)/2 : \ m \text{ is a matching of } f\}.$$

We will investigate the polymatroid matching problem in ntcdc-free polymatroids, defined below. Our main result goes as follows.

Theorem 1. *Let $f : 2^S \to \mathbb{N}$ be a ntcdc-free polymatroid function. Then*

$$\nu(f) = \min \sum_{j=1}^{t} \left\lfloor \frac{f(U_j)}{2} \right\rfloor,$$

where the minimum is taken over all partitions $U_1, U_2, \ldots, U_t$ of S.

We propose two different proofs. In the first proof we exploit a theorem of Lovász, and a couple of polymatroid operations. The second proof relies on a (semi-strongly) polynomial time algorithm, which is based on a generalization of the contraction of blossoms in Edmonds' matching algorithm [2].

1.2 Circuits and Compatible Double Circuits in Polymatroids

Consider a polymatroid function $f : 2^S \to \mathbb{N}$, and a vector $x \in \mathbb{N}^S$. For a set $Z \subseteq S$, we call $\mathrm{def}_{f,x}(Z) := x(Z) - f(Z)$ the *deficiency of set* Z with respect to f, x. A set is called k*-deficient* with respect to f, x if $\mathrm{def}_{f,x}(Z) = k$. The *deficiency of a vector* x is defined by $\mathrm{def}_f(x) := \max_{Z\subseteq S} \mathrm{def}_{f,x}(Z)$, which is non-negative. Notice that $\mathrm{def}_{f,x}(\cdot)$ is a supermodular set-function, hence the family of sets Z such that $\mathrm{def}_{f,x}(Z) = \mathrm{def}_f(x)$ is closed under taking unions and intersections.

Consider a 1-deficient vector x. x is called a *circuit* if $supp(x)$ is equal to the unique inclusionwise minimal 1-deficient set.

Consider a 2-deficient vector $x \in \mathbb{N}^S$, and let $W := supp(x)$. x is called a *compatible double circuit* (or *cdc*, for short), if W is the unique inclusionwise minimal 2-deficient set, and there is a partition $\pi = \{W_1, \cdots, W_k\}$ of W such that $k \geq 2$ and $\{W - W_i : i = 1, \cdots, k\}$ is equal to the family of all inclusionwise minimal 1-deficient sets. We remark that if x is a cdc, then π is uniquely determined – let it be called the *principal partition* of x. If $k = 2$, then x is called a *trivial cdc*. If $k \geq 3$, then x is called a *non-trivial compatible double circuit*, or *ntcdc*, for short.

A polymatroid is called *ntcdc-free* if there is no ntcdc.

2 First Proof of the Partition Formula

For some well-known notions and results on the theory of matroids, polymatroids and matroid matching, see [14]. We need some more preparation.

2.1 Preliminaries

There is a close relation between polymatroid functions and matroids. First, if $M = (T, r)$ is a matroid and $\varphi : T \to S$ is a function then $f : 2^S \to \mathbb{N}$, $X \mapsto r(\varphi^{-1}(X))$ is a polymatroid function, the *homomorphic image of* M *under* φ. Second, for any polymatroid function f it is possible to define a matroid M, the homomorphic image of which is f, in such a way that M is "most independent" in some sense. The ground set T of M is the disjoint union of sets T_i for $i \in S$ of size $|T_i| \geq f(\{i\})$. If $X \subseteq T$ then we define the vector $\chi^X \in \mathbb{N}^S$ with $\chi^X_i = |X \cap T_i|$ for $i \in S$. With this notation, a set $X \subseteq T$ is defined to be independent in M if $\chi^X \in P(f)$. It is routine to prove that M is indeed a matroid with rank function $r(X) = \min_{Y\subseteq X}(|Y| + f(\varphi(X - Y)))$, where $\varphi : T \to S$ maps t to i if $t \in T_i$. This M is called a *prematroid* of f. Note that a prematroid M is uniquely determined by f and by the sizes $|T_i|$, $i \in S$. If M is a matroid with rank function r then the prematroids of r are the parallel extensions of M. If we consider a prematroid M then we tacitly assume that $M = (T, r)$ and that the function $\varphi : T \to S$ is given with $t \mapsto i$ if $t \in T_i$.

If f is a polymatroid function and $x \in \mathbb{Z}^S$ then we define the *rank* of x as $r_f(x) = \min_{U \subseteq S}(x(S - U) + f(U))$. If $x \in \mathbb{N}^S$ then $r_f(x) = x(S)$ if and only if $x \in P(f)$. Besides, if $M = (T, r)$ is a prematroid of f and $X \subseteq T$ then $r_f(\chi^X) = r(X)$. The *span* of $x \in \mathbb{N}^S$ is defined by $\mathrm{sp}_f(x) = \{i \in S : r_f(x + \chi_i) = r_f(x)\}$. If M is a prematroid of f and $X \subseteq T$ then $\mathrm{sp}_f(\chi^X) = \{i \in S : T_i \subseteq \mathrm{sp}_M(X)\}$.

2.2 Circuits and Double Circuits in Matroids

Let $M = (T, r)$ be a matroid. A set $C \subseteq T$ is said to be a *circuit* if $r(C - x) = r(C) = |C| - 1$ for every $x \in C$. A set $D \subseteq T$ is a *double circuit* if $r(D - x) = r(D) = |D| - 2$ for every $x \in D$. If D is a double circuit then the dual of $M|D$ is a matroid of rank 2 without loops, that is a line, showing that there exists a *principal partition* $D = D_1 \dot\cup D_2 \dot\cup \ldots \dot\cup D_d$, $d \geq 2$, such that the circuits of D are exactly the sets of the form $D - D_i$, $1 \leq i \leq d$. We say that D is *non-trivial* if $d \geq 3$, and *trivial* otherwise. A trivial double circuit is simply the direct sum of two circuits.

Analogously, we define circuits and double circuits of the polymatroid function $f : 2^S \to \mathbb{N}$. For a vector $x \in \mathbb{R}^S_+$ let $\mathrm{supp}(x) = \{i \in S : x_i > 0\}$. A vector $c \in \mathbb{N}^S$ is a *circuit* of f if $r_f(c - \chi_i) = r_f(c) = c(S) - 1$ for every $i \in \mathrm{supp}(c)$. A vector $w \in \mathbb{N}^S$ is a *double circuit* of f if $r_f(w - \chi_i) = r_f(w) = w(S) - 2$ for every $i \in \mathrm{supp}(w)$. It is also easy to see the exact relation between matroidal and polymatroidal double circuits, which is given as follows.

Lemma 1. *Let M be a prematroid of f, $D \subseteq T$ and $\chi^D = w$. Then D is a double circuit of M if and only if w is a double circuit of f.*

Recall that we have already defined cdc's and ntcdc's. Next we add another definition, which is easily seen to be equivalent with those above. For $x \in \mathbb{R}^S$ and $U \subseteq S$ we introduce the notation $x|_U$ for the vector by $(x|_U)_i := x_i$ for $i \in U$ and $(x|_U)_i := 0$ for $i \in S - U$. Let M be a prematroid of f and w be a double circuit of f such that there is a set $D \subseteq T$ with $\chi^D = w$. By Lemma 1, D is a double circuit of M, thus it has a principal partition $D = D_1 \dot\cup D_2 \dot\cup \ldots \dot\cup D_{d'}$. We define the principal partition of w as follows. Due to the structure of prematroids it is easy to check that $\mathrm{supp}(w)$ has a partition $W_0 \dot\cup W_1 \dot\cup \ldots \dot\cup W_d$ with the property that each set D_j is either a singleton belonging to some T_i with $w_i \geq 2$ and $i \in W_0$, or is equal to $D \cap \bigcup_{i \in W_h} T_i$ for some $1 \leq h \leq d$. Note that a partition $W_0 \dot\cup W_1 \dot\cup \ldots \dot\cup W_d$ of $\mathrm{supp}(w)$ is the principal partition of w if and only if $w - \chi_i$ is a circuit of f and $w_i \geq 2$ whenever $i \in W_0$, moreover, $w|_{W - W_i}$ is a circuit of f for each $1 \leq i \leq d$. A double circuit w is said to be *compatible* if $W_0 = \emptyset$, and it is *trivial* if D is trivial. We remark that these definitions are easily see equivalent with the above ones.

We shortly mention what is the double circuit property, or DCP, for short. If $M = (T, r)$ is a prematroid of the polymatroid function f and $Z \subseteq T$ then $\varphi(M/Z)$ is called a *contraction* of f. A polymatroid function f is said to have the

DCP if whenever w is a non-trivial compatible double circuit in a contraction f' of f with principal partition $W_1\dot{\cup}\dots\dot{\cup}W_d$ then $f'(\bigcap_{1\le i\le d}\operatorname{sp}(w|_{W-W_i}))>0$, [1]. A polymatroid function without non-trivial compatible double circuits has not necessarily the DCP, as its contractions may have many non-trivial compatible double circuits.

Note that every polymatroid function has double circuits, say $(f(\{i\})+2)\chi_i$ for some $i\in S$. However, these are not compatible, as $W_0=\{i\}$.

Lemma 2. *If $w\in\mathbb{N}^S$ is a double circuit of the polymatroid function $f:2^S\to\mathbb{N}$ with principal partition $W=W_0\dot{\cup}W_1\dot{\cup}\dots\dot{\cup}W_d$ then $f(W)=w(W)-2$ and $f(W-W_i)=w(W-W_i)-1$ for $1\le i\le d$.*

Proof. We prove that if $x\in\mathbb{N}^S$ is a vector with the property that $r_f(x)=r_f(x-\chi_i)$ for all $i\in\operatorname{supp}(x)$ then $f(\operatorname{supp}(x))=r_f(x)$. By definition, $r_f(x)=x(S-Y)+f(Y)$ for some $Y\subseteq S$. Note that $r_f(x-\chi_i)\le(x-\chi_i)(S-Y)+f(Y)=r_f(x)-1$ for all $i\in\operatorname{supp}(x)-Y$. Thus $\operatorname{supp}(x)\subseteq Y$. Finally, $f(Y)=r_f(x)\le f(\operatorname{supp}(x))\le f(Y)$, since f is non-decreasing. If x is a circuit or a double circuit then $r_f(x)=r_f(x-\chi_i)$ for all $i\in\operatorname{supp}(x)$, we are done.

2.3 Polymatroid Operations

Next we investigate how two polymatroid operations (translation, deletion) effect double circuits. If $f:2^S\to\mathbb{N}$ is a function and $n\in\mathbb{Z}^S$ then define $f+n:2^S\to\mathbb{N}$ by $X\mapsto f(X)+n(X)$. If f is a polymatroid function and $n\in\mathbb{N}^S$ then $f+n$ is clearly a polymatroid function, too.

Lemma 3. *If $n\in\mathbb{Z}^S$ and f and $f+n$ are polymatroid functions then a vector w is a double circuit of f with $W=\operatorname{supp}(w)$ if and only if $w+n|_W$ is a double circuit of $f+n$. In this case their principal partition coincide.*

Proof. Clearly, $r_{f+n}(x+n)-(x+n)(S)=r_f(x)-x(S)$ for all $x\in\mathbb{Z}^S$. Thus by symmetry, it is enough to prove that if w is a double circuit of f with support W then $w_i+n_i>0$ for every $i\in W$. Otherwise by Lemma 2 we would have $w(W-i)-n_i\ge w(W)=f(W)+2\ge f(W-i)-n_i+2$, which is impossible.

Let $u\in\mathbb{N}^S$ be a bound vector and define $f\backslash u=\varphi(r_{M|Z})$ where M is a prematroid of f and $Z\subseteq T$ with $\chi^Z=u$. The matroid union theorem asserts that $(f\backslash u)(X)=\min_{Y\subseteq X}(u(Y)+f(X-Y))$. If M is a matroid with rank function r then $r\backslash u$ is the rank function of $M|\operatorname{supp}(u)$.

Lemma 4. *Let $u\in\mathbb{N}^S$. If $w\in\mathbb{N}^S$ is a double circuit of $f':=f\backslash u$ then w is either a double circuit of f with the same principal partition, or trivial, or non-compatible.*

Proof. Let $M=(T,r)$ be a prematroid of f and $Z\subseteq T$ with $\chi^Z=u$. If $w\le\chi^Z$ then w is a double circuit of f with the same principal partition by Lemma 1. Observe that $w_i\le f'(\{i\})+2$ and $f'(\{i\})\le u_i$ for every $i\in S$. Thus if $w\not\le\chi^Z$ then there exists an $i\in S$ such that $w_i-f'(\{i\})\in\{1,2\}$. If $w_i=f'(\{i\})+2$

then $r_{f'}(w_i\chi_i) = w_i - 2$, thus $W_0 = \mathrm{supp}(w) = \{i\}$, implying that w is non-compatible. If $w_i = f'(\{i\}) + 1$ then $w_i\chi_i$ is a circuit of f' thus if $W_0 \neq \emptyset$ then w is non-compatible and if $W_0 = \emptyset$ then w is trivial.

Finally we cite Lovász's deep and important theorem on 2-polymatroids, which can be translated to arbitrary polymatroids as follows. This theorem will be a key to our first proof below.

Theorem 2 (Lovász [10]). *If $f : 2^S \to \mathbb{N}$ is a polymatroid function then at least one of the following cases holds.*

1. $f(S) = 2\nu(f) + 1$.
2. *There exists a partition $S = S_1 \dot\cup S_2$, $S_i \neq \emptyset$, s.t. $\nu(f) = \nu(f|_{2^{S_1}}) + \nu(f|_{2^{S_2}})$.*
3. *There exists an $i \in S$, $f(i) \geq 2$ such that for each maximum matching m we have $i \in \mathrm{sp}_f(m)$.*
4. *There exists a certain substructure, called ν-*double flower *in f, which we do not define here, but which always contains a non-trivial compatible double circuit.*

Proof (First proof of Theorem 1). It is easy to see that $\nu(f) \leq \sum_{j=1}^{t} \left\lfloor \frac{f(U_j)}{2} \right\rfloor$ holds for every partition $U_1, U_2, \ldots, U_t$ of S. For the other direction we argue by induction on the pair $(|S|, |K(f)|)$, where $K(f) = \{s \in S : s \in \mathrm{sp}_f(m)$ for each maximum matching m of $f\}$. If $S = \emptyset$ then the statement is trivial. If $K(f) = \emptyset$ then either *1.* or *2.* holds in Theorem 2. If *1.* holds then the trivial partition will do, while if *2.* holds then we can use our induction hypothesis applied to $f|_{2^{S_1}}$ and $f|_{2^{S_2}}$.

Next, let $K(f) \neq \emptyset$. We prove that if m is a maximum matching of $f + 2\chi_s$ then $m(s) \geq 2$. Indeed, assume that $m(s) = 0$. As m is a maximum matching, there exists a set $s \in U \subseteq S$ with $m(U) \geq (f + 2\chi_s)(U) - 1$. Thus $m(U - s) = m(U) \geq (f + 2\chi_s)(U) - 1 \geq f(U - s) + 1$, which is a contradiction. It is also clear that $m + 2\chi_s$ is a matching of $f + 2\chi_s$ for each matching m of f. Therefore, m is a maximum matching of f if and only if $m + 2\chi_s$ is a maximum matching of $f + 2\chi_s$.

Let $s \in K(f)$. Clearly, $\nu(f) \leq \nu(f + \chi_s) \leq \nu(f + 2\chi_s) = \nu(f) + 1$ and we claim that in fact, $\nu(f + \chi_s) = \nu(f)$ holds. Indeed, if $\nu(f + \chi_s) = \nu(f) + 1$ and m is a maximum matching of $f + \chi_s$ then m is also a maximum matching of $f + 2\chi_s$, thus $m(s) \geq 2$. Then $m - 2\chi_s$ is a maximum matching of f and, as $s \in \mathrm{sp}_f(m - 2\chi_s)$, there exists a set $s \in U \subseteq S$ with $(m - 2\chi_s)(U) = f(U)$. This implies $m(U) = f(U) + 2$, contradicting to that m is a matching of $f + \chi_s$.

So if m is a maximum matching of f then m is a maximum matching of $f + \chi_s$, too, and clearly, $\mathrm{sp}_f(m) = \mathrm{sp}_{f+\chi_s}(m) - s$. Thus we have $K(f + \chi_s) \subseteq K(f) - s$. By Lemma 3, $f + \chi_s$ has no non-trivial compatible double circuits, so we can apply induction to $f + \chi_s$. This gives a partition $U_1, U_2, \ldots, U_t$ of S such that $\nu(f + \chi_s) = \sum_{j=1}^{t} \left\lfloor \frac{1}{2}(f + \chi_s)(U_j) \right\rfloor$. But then, $\nu(f) = \nu(f + \chi_s) = \sum_{j=1}^{t} \left\lfloor \frac{1}{2}(f + \chi_s)(U_j) \right\rfloor \geq \sum_{j=1}^{t} \left\lfloor \frac{f(U_j)}{2} \right\rfloor$.

3 Second, Constructive Proof of the Partition Formula

The second proof is based on projections of blossoms, which is the generalization of the principle in Edmonds' matching algorithm [2]. For this, of course, we need some more definitions and direct observations concerning projections.

3.1 Projections

Consider a polymatroid function f on groundset S, as above. For a subset $B \subseteq S$ we define the *projection* $f^B : 2^{S-B} \to \mathbb{N}$ by $f^B(X) := \min\{ f(X), f(X \cup B) - f(B) + 1 \}$ for $X \subseteq S - B$. It is easy to see that f^B is a polymatroid function, and its induced polymatroid is equal to

$$P(f^B) = \{y \in \mathbb{R}^{S-B} : \text{ there is } [z, y] \in P(f) \text{ s.t. } z(B) = f(B) - 1\}. \quad (4)$$

For $x \in \mathbb{R}^S$, $Z \subseteq S$ we introduce the notation $x||_Z \in \mathbb{R}^Z$ for the vector such that $(x||_Z)_i = x_i$ for all $i \in Z$.

Consider a family $\mathcal{H} = \{H_1, \cdots, H_m\}$ of disjoint subsets of S. Assume that there is a vector $x \in P(f)$ such that for all $i = 1, \cdots, m$, we have $x(H_i) = f(H_i) - 1$, and there is an element $h_i \in H_i$ such that $x + \chi_{h_i} \in P(f)$. By (4) we get that $x||_{S-H_i} \in P(f^{H_i})$, thus $f^{H_i}(H_j) = f(H_j)$ for all $i \neq j$. This implies that we obtain the same polymatroid function on groundset $S - \cup\mathcal{H}$ no matter which order the sets H_i are projected. Let $f^{\mathcal{H}}$ denote the unique polymatroid function obtained by projecting all the members of $\mathcal{H}$. Then

$$P(f^{\mathcal{H}}) = \{y \in \mathbb{R}^{S-\cup\mathcal{H}} : \text{ there is } [z, y] \in P(f) \text{ s.t. } z(H_i) = f(H_i) - 1\}, \quad (5)$$

and we get that for any $X \subseteq S - \bigcup\mathcal{H}$,

$$f^{\mathcal{H}}(X) = \min\left\{f(X \cup \bigcup\mathcal{H}') - x(\bigcup\mathcal{H}') : \mathcal{H}' \subseteq \mathcal{H}\right\}. \quad (6)$$

We remark without proof that $f^{\mathcal{H}}$ may be evaluated in strongly polynomial time.

3.2 Blossoms

The notion of blossoms comes from an algorithmic point of view, which is the analogue of Edmonds' blossoms in the matching algorithm. An ear-decomposition of a matching is constructed by finding a circuit induced in the matching, and iterating this procedure after the projection. More precisely, the definition is the following. If $y \in P(f)$, $y + \chi_u \in P(f)$, $y + 2\chi_u \notin P(f)$, $u \in C \subseteq S$, and C is the unique inclusionwise minimal 1-deficient set for $y + 2\chi_u$, then we say that *"u induces a circuit on C in y"*.

Consider a matching x with respect to a polymatroid function $f : 2^S \to \mathbb{N}$. Consider a laminar family $\mathcal{F} = \{B_1, \cdots, B_k\}$ of subsets of S, that is, any two members of $\mathcal{F}$ are either disjoint or one contains the other. For indices $i = 1, \cdots, k$, let $\mathcal{F}_i$ denote the family of inclusionwise maximal proper subsets of B_i in $\mathcal{F}$, and let $G_i := B_i - \cup\mathcal{F}_i$. Consider a set $U = \{u_1, \cdots, u_k\} \subseteq S$ such that $u_i \in G_i$. Hence $\mathcal{F}, U$ is called an *x-ear-decomposition* if

(a) $x(B_i) = f(B_i) - 1$, and
(b) u_i induces a circuit on G_i in $x||_{S-\cup\mathcal{F}_i}$ with respect to $f^{\mathcal{F}_i}$.

Notice that the above definition implies that $x+\chi_{u_i} \in P(f)$ holds whenever B_i is an inclusionwise minimal member of $\mathcal{F}$. This implies that the projection of $\mathcal{F}$, or $\mathcal{F}_i$ satisfies the assumption in the previous section, and thus the projection may be performed in arbitrary order. Notice, if we drop an inclusionwise maximal member $B_i \in \mathcal{F}$ together with u_i, we retain another ear-decomposition. A set B appearing in the family $\mathcal{F}$ of some ear-decomposition is called an x*-blossom*. An ear-decomposition of a blossom B is an ear-decomposition $\mathcal{F}, U$ such that B is the unique inclusionwise maximal member of $\mathcal{F}$.

The following Lemma 5 will be our crucial inductive tool to deal with ear-decompositions by extending a matching with respect to $f^{\mathcal{F}}$ to a matching with respect to f.

Lemma 5. *Suppose we are given a matching x, an x-blossom B together with an x-ear-decomposition, and a vector $y \in P(f^B)$. There is a polynomial time algorithm to find either*

(A) a ntcdc, or
(B) an even vector $z \in (2\mathbb{N})^B$ such that $z(B) = f(B) - 1$ and $[z, y] \in P(f)$.

Proof. Let us use notation from above. The algorithm is recursive on the number k of ears. Firstly, notice that $\operatorname{def}_f([x||_B, y]) \leq 1$. If $\operatorname{def}_f([x||_B, y]) = 0$, then (B) holds for $z = x||_B$, and we are done. Henceforth we suppose that $\operatorname{def}_f([x||_B, y]) = 1$, and let D denote the inclusionwise minimal 1-deficient set for $[x||_B, y]$. Say $B = B_k$ and $G = G_k$.

We claim that either $[x||_G, y] \in P(f^{\mathcal{F}_k})$, or $D \subseteq (S - B) \cup G$. Suppose $[x||_G, y] \notin P(f^{\mathcal{F}_k})$. By (4), there is a set Q such that $\operatorname{def}_{f,[x||_B,y]}(Q) \geq 1$, and for all $B_i \in \mathcal{F}_k$ we have $Q \cap B_i = \emptyset$ or $Q \supseteq B_i$. Clearly, $\operatorname{def}_{f,[x||_B,y]}(B) = -1$. Since $y \in P(f^B)$, we get that $\operatorname{def}_{f,[x||_B,y]}(B \cup Q) \leq 0$. Thus, by supermodularity of deficiency, $0 \leq \operatorname{def}_{f,[x||_B,y]}(B \cap Q) = \operatorname{def}_{f,x}(B \cap Q)$. Recall that for every inclusionwise minimal set $B_i \in \mathcal{F}$ we have $x + \chi_{u_i} \in P(f)$ for $u_i \in B_i$. Thus, $u_i \notin B \cap Q$, which implies that $D \subseteq Q \subseteq (S - B) \cup G$.

Now suppose that $[x||_G, y] \in P(f^{\mathcal{F}_k})$. Thus, by (4), there is a (not necessarily even) vector $z' \in \mathbb{N}^{\cup\mathcal{F}_k}$ such that $[z', x||_G, y] \in P(f)$, and $z'(B_i) = b(B_i) - 1$ for all $B_i \in \mathcal{F}_k$. Then we apply the algorithm recursively for $B_i \in \mathcal{F}_k$ and $[z', x||_G, y]$, that is, we replace $z'||_{B_i}$ step-by-step by an even vector retaining the above properties – or we find a ntcdc.

Finally suppose that $D \subseteq (S - B) \cup G$. Notice that $y \in P(f^B)$ implies $D \cap B \neq \emptyset$. Also, $x \in P(f)$ implies $D - B \neq \emptyset$. Moreover, $y \in P(f^B)$ implies $\operatorname{def}_{f,[x|_B,y]}(B \cup D) \leq 0$. Recall that $\operatorname{def}_{f,[x||_B,y]}(D) = 1$ and $\operatorname{def}_{f,[x||_B,y]}(B) = -1$. By supermodularity of deficiency, $\operatorname{def}_{f,[x||_B,y]}(B \cap D) \geq 0$. Thus, by (b) we get that $u_k \notin D$. Consider an arbitrary element $d \in D \cap B$. By (b), $[x||_G + 2\chi_{u_k} - \chi_d, 0] \in P(f^{\mathcal{F}_k})$. By applying the algorithm recursively for $[x||_G + 2\chi_{u_k} - \chi_d, 0]$ one can find either a ntcdc, or an even vector $q \in (2\mathbb{N})^{\cup\mathcal{F}_k}$ such that $[q, x||_G + 2\chi_{u_k} - \chi_d, 0] \in P(f)$. Next, we will find out whether there is an element e such that $z = [q, x||_G + 2\chi_{u_k} - 2\chi_e]$ satisfies (B). Clearly, all these vectors are even. It is easy to see that $\text{def}_f([q, x||_G + 2\chi_{u_k}, y])$ is 1 or 2. If $\text{def}_f([q, x||_G + 2\chi_{u_k}, y]) = 1$, then for some element e we get that $[q, x||_G + 2\chi_{u_k} - 2\chi_e, y]$, and we are done. If $\text{def}_f([q, x||_G + 2\chi_{u_k}, y]) = 2$, then let W denote the unique minimal 2-deficient set. If there is an element $e \in W$ such that all the 1-deficient sets contain e, then $[q, x||_G + 2\chi_{u_k} - 2\chi_e, y] \in P(f)$, and we are done. Otherwise, if for every element e there is a 1-deficient set $e \notin W_e$, then $[q, x||_G + 2\chi_{u_k}, y]||_W \in \mathbb{N}^S$ is a cdc. Notice that B and D are circuits in $[q, x||_G + 2\chi_{u_k}, y]$, thus $W - B \in \pi$ and $W - D \in \pi$. Since $d \in B \cap D \neq \emptyset$, this implies $|\pi| \geq 3$.

3.3 A Semi-strongly Polynomial Time Algorithm

We construct a semi-strongly polynomial time algorithm which either returns a ntcdc, or returns a maximum matching x and a partition certifying its maximality. The algorithm maintains a matching, and iteratively augments its size by one, until it either finds a certifying partition, or a ntcdc. We may initiate x as a basis of $P(f)$, rounded down to the closest even vector. This initialization may be performed in semi-strongly polynomial time, where "semi-" comes only from the fact that we have to take lower integer part to detect parity. The remaining part of the algorithm may be performed in strongly polynomial time.

The idea behind the algorithm is the following. If our matching x is a basis in the polymatroid, then we are done. Thus we find an element $u \in S$ such that $x + \chi_u \in P(f)$. If $x + 2\chi_u \in P(f)$, then that gives a larger matching, and we are done. Otherwise, we may assume that $x + \chi_u \in P(f)$ and $x + 2\chi_u \notin P(f)$, i.e. u induces a circuit in x, which can be used building blossoms and projections. If we find a larger matching in the projection, then we use Lemma 5 to expand blossoms and retain a larger matching over the original groundset. This idea is developed in detail below.

Consider a matching x. Define $\mathcal{C} := \emptyset$. In a general step of the algorithm, $\mathcal{C} = \{B_1, \cdots, B_k\}$ is a family of disjoint x-blossoms. This implies that $x||_{S-\cup\mathcal{C}} \in P(f^{\mathcal{C}})$. We distinguish three cases on how close $x||_{S-\cup\mathcal{C}}$ is to a basis of $P(f^{\mathcal{C}})$.

Case 1. Suppose that $x(S - \bigcup\mathcal{C}) = f^{\mathcal{C}}(S - \bigcup\mathcal{C})$. Then, by claim (6), there is a set $\mathcal{C}' \subseteq \mathcal{C}$ such that $f(S - \bigcup\mathcal{C} + \bigcup\mathcal{C}') = x(S - \bigcup\mathcal{C}' + \bigcup\mathcal{C}')$. Then $\mathcal{C}' = \emptyset$, since for all blossoms $B_i \in \mathcal{C}$ there is an element $t \in B_i$ such that $x + \chi_t \in P(f)$. We conclude that x is a maximum matching, certified by the partition $\mathcal{C} \cup \{S - \bigcup\mathcal{C}\}$.

Case 2. Suppose that $x||_{S-\cup\mathcal{C}} + \chi_u \in P(f^{\mathcal{C}})$, but $x||_{S-\cup\mathcal{C}} + 2\chi_u \notin P(f^{\mathcal{C}})$. Then there is a set $u \in Z \subseteq S - \cup\mathcal{C}$ such that u induces a circuit on Z in $x||_{S-\cup\mathcal{C}}$ with respect to $f^{\mathcal{C}}$. By claim (6) there is a set $\mathcal{C}' \subseteq \mathcal{C}$ such that $f(Z \cup \bigcup\mathcal{C}') = x(Z \cup \bigcup\mathcal{C}') + 1$. Thus, $\mathcal{C} - \mathcal{C}' + \{Z \cup \bigcup\mathcal{C}'\}$ is a blossom family.

Case 3. Suppose that $x||_{S-\cup\mathcal{C}} + 2\chi_u \in P(f^{\mathcal{C}})$. In this case, by applying Lemma 5 for members of $\mathcal{C}$, we construct either a matching larger than x, or a ntcdc. This is done as follows. By assertion (5), there is a (not necessarily even) vector $z \in \mathbb{N}^{\cup\mathcal{C}}$ such that $x' := [z, x||_{S-\cup\mathcal{C}} + 2\chi_u] \in P(f)$, and $z(B_i) = f(B_i) - 1$ for $i = 1, \cdots, k$. Thus, for an arbitrary index $i \in \{1, \cdots, k\}$ we get that $x'||_{S-B_i} \in P(f^{B_i})$. By applying Lemma 5 for B_i, we either construct a ntcdc, or we may replace entries of x' in B_i with even numbers, and retain the above properties. By repeating this procedure for $i = 1, \cdots, k$ we retain a matching x' that is larger than x.

4 Applications

4.1 A Parity Constrained Orientation Theorem

Frank, Jordán and Szigeti [4] proved that the existence of a k-rooted-connected orientation with prescribed parity of in-degrees can be characterized by a partition type condition. Recently, Király and Szabó [7] proved that the connectivity requirement in this parity constrained orientation problem can be given by a more general non-negative intersecting supermodular function. It is well-known that all these problems can be formalized as polymatroid parity problems. In this section we show that it is possible to formalize the problem of Király and Szabó in such a way that the arising polymatroid function has no non-trivial double circuits. So Theorem 1 can be applied to yield the result in [7].

$H = (V, \mathcal{E})$ is called a *hypergraph* if V is a finite set and $\emptyset \notin \mathcal{E}$ is a collection of multisets of V, the set of *hyperedges* of H. If in every hyperedge $h \in \mathcal{E}$ we designate a vertex $v \in h$ as the *head vertex* then we get a *directed hypergraph* $D = (V, \mathcal{A})$, called an *orientation* of H. For a set $X \subseteq V$, let $\delta_D(X)$ denote the set of directed hyperedges *entering* X, that is the set of hyperedges with head in X and at least one vertex in $V - X$.

Let $p : 2^V \to \mathbb{N}$ be a function with $p(\emptyset) = p(V) = 0$. An orientation D of a hypergraph $H = (V, \mathcal{E})$ *covers* p if $|\delta_D(X)| \geq p(X)$ for every $X \subseteq V$. In a *connectivity orientation problem* the question is the existence of an orientation covering p. When we are talking about *parity constrained orientations*, we are looking for connectivity orientations such that the out-degree at each vertex is of prescribed parity. Now define $b : 2^V \to \mathbb{Z}$ by

$$b(X) = \sum_{h \in \mathcal{E}} h(X) - |\mathcal{E}[X]| - p(X) \quad \text{for} \quad X \subseteq V, \tag{7}$$

where $\mathcal{E}[X]$ denotes the set of hyperedges $h \in \mathcal{E}$ with $h \cap (V - X) = \emptyset$, and h equivalently stands for the hyperedge and its multiplicity function. It is clear that if $x : V \to \mathbb{N}$ is the out-degree vector of an orientation covering p then $x \in B(b)$. The contrary is also easy to prove, see e.g. in [14]:

Lemma 6. *Let $H = (V, \mathcal{E})$ be a hypergraph, $p : 2^V \to \mathbb{N}$ be a function with $p(\emptyset) = p(V) = 0$, and $x : V \to \mathbb{N}$. Then H has an orientation covering p such that the out-degree of each vertex $v \in V$ is $x(v)$ if and only if $x \in B(b)$.*

The function $b : 2^V \to \mathbb{Z}$ is said to be *intersecting submodular* if (1) holds whenever $X \cap Y \neq \emptyset$. Similarly, $p : 2^V \to \mathbb{Z}$ is *intersecting supermodular* if $-p$ is intersecting submodular. If $b : 2^V \to \mathbb{N}$ is a non-negative, non-decreasing intersecting submodular function then we can define a polymatroid function $\widehat{b} : 2^V \to \mathbb{N}$ by $\widehat{b}(X) = \min\left\{\sum_{i=1}^{t} b(X_i) \ : \ X_1 \dot\cup X_2 \dot\cup \ldots \dot\cup X_t = X\right\}$ for $X \subseteq V$, which is called the *Dilworth truncation* of b. It is also well-known that, if $p : 2^V \to \mathbb{N}$ is intersecting supermodular with $p(V) = 0$, then p is non-increasing.

Thus if $p : 2^V \to \mathbb{N}$ is an intersecting supermodular function with $p(\emptyset) = p(V) = 0$ then $b : 2^V \to \mathbb{Z}$, as defined in (7), is a non-decreasing intersecting submodular function, but it is not necessarily non-negative. The following theorem can be proved using basic properties of polymatroid functions.

Theorem 3. *Let $H = (V, \mathcal{E})$ be a hypergraph and $p : 2^V \to \mathbb{N}$ be an intersecting supermodular function with $p(\emptyset) = p(V) = 0$. Define b as in (7). Then H has an orientation covering p if and only if $b(V) \leq \sum_{j=1}^{t} b(U_j)$ holds for every partition $U_1, U_2, \ldots, U_t$ of V.*

Let $H = (V, \mathcal{E})$ be a hypergraph and $T \subseteq V$. Our goal is to find an orientation of H covering p, where the set of odd out-degree vertices is as close as possible to T.

Theorem 4 (Király and Szabó [7]). *Let $H = (V, \mathcal{E})$ be a hypergraph, $T \subseteq V$, $p : 2^V \to \mathbb{N}$ be an intersecting supermodular function with $p(\emptyset) = p(V) = 0$, and assume that H has an orientation covering p. Define b as in (7). For an orientation D of H let $Y_D \subseteq V$ denote the set of odd out-degree vertices in D. Then*

$$\min\{|T \triangle Y_D| \ : \ D \text{ is an orientation of } H \text{ covering } p\} = \max\left\{b(V) - \textstyle\sum_{j=1}^{t} b(U_j) + |\{j : b(U_j) \not\equiv |T \cap U_j| \bmod 2\}|\right\}, \quad (8)$$

where the maximum is taken on partitions $U_1, U_2, \ldots, U_t$ of V.

An interesting corrollary is the following non-defect form, which is again a generalization of Theorem 3.

Theorem 5. *Let $H = (V, \mathcal{E})$ be a hypergraph, $T \subseteq V$, and let $p : 2^V \to \mathbb{N}$ be an intersecting supermodular function with $p(\emptyset) = p(V) = 0$. Then, H has an orientation covering p with odd out-degrees exactly in the vertices of T, if and only if*

$$b(V) \leq \textstyle\sum_{j=1}^{t} b(U_j) - |\{j : b(U_j) \not\equiv |T \cap U_j| \bmod 2\}| \qquad (9)$$

holds for every partition $U_1, U_2, \ldots, U_t$ of V.

Proof. For every $v \in T$ add a loop $2\chi_v$ to $\mathcal{E}$, resulting in the hypergraph $H' = (V, \mathcal{E}')$. Define b' as in (7), w.r.t. H'. As there is a straightforward bijection between the orientations of H and H', we have $\min\{|T \triangle Y_D| : D \text{ is an orientation of } H \text{ covering } p\} = \min\{|Y_{D'}| : D' \text{ is an orientation of } H' \text{ covering } p\}$, and $b(V) - \sum_{j=1}^{t} b(U_j) + |\{j : b(U_j) \not\equiv |T \cap U_j| \bmod 2\}| = b'(V) - \sum_{j=1}^{t} b'(U_j) + |\{j : b'(U_j) \text{ is odd }\}|$. Thus we can assume that $T = \emptyset$.

By Lemma 6, the integer vectors of $B(b)$ are exactly the out-degree vectors of the orientations of H covering p. Thus the $\geq$ direction is easy to check. Now we prove the other direction. As H has an orientation covering p, if $\emptyset \subsetneq U \subsetneq V$ then $b(U) + b(V - U) \geq b(V)$ by Theorem 3, implying that $b(U) \geq b(V) - b(V - U) \geq 0$.

Thus, b is non-decreasing, and we can define the polymatroid function $f = \widehat{b}$. We claim that it is enough to prove that $\nu(f) = \min \sum_{i=1}^{s} \left\lfloor \frac{1}{2} f(V_i) \right\rfloor$, where the minimum is taken over all partitions $V_1, V_2, \ldots, V_s$ of V. Indeed, using the definition of the Dilworth-truncation and that $b(V) = f(V)$ by Theorem 3, we get

$$\min\{|Y_D| : D \text{ is an ori. of } H \text{ covering } p\} = f(V) - 2\nu(f) =$$

$$= b(V) - \min\left\{\textstyle\sum_{i=1}^{s} f(V_i) - |\{i : f(V_i) \text{ is odd}\}| \ : \ V_1, \ldots, V_s \text{ partitions } V\right\} \leq$$

$$\leq b(V) - \min\left\{\textstyle\sum_{j=1}^{t} b(U_j) - |\{j : b(U_j) \text{ is odd}\}| \ : \ U_1, \ldots, U_t \text{ partitions } V\right\}.$$

Thus by Theorem 1 it is enough to prove that $\widehat{b}$ has no non-trivial compatible double circuits. The next lemma does the job.

Lemma 7. *Let $H = (V, \mathcal{E})$ be a hypergraph and let $p : 2^V \to \mathbb{N}$ an intersecting supermodular function with $p(\emptyset) = 0$. Suppose moreover that $b : 2^V \to \mathbb{Z}$ defined by (7) is non-negative and non-decreasing. Then the polymatroid function $f := \widehat{b}$ has no non-trivial compatible double circuits.*

Proof. Assume that $w : V \to \mathbb{N}$ is a non-trivial compatible double circuit of f with principal partition $W = W_1 \dot\cup W_2 \dot\cup \ldots \dot\cup W_d$. Clearly, $b(W) \geq w(W) - 2$. Let $1 \leq i < j \leq d$ and $Z = W - W_i$. As $w|_Z$ is a circuit, Lemma 2 yields that $w(Z) - 1 = f(Z) = \min \sum \{b(X_i) \ : \ X_1, \ldots, X_k \text{ partitions } Z\}$. However, if a non-trivial partition with $k \geq 2$ gave equality here, then we would have $f(Z) = \sum b(X_i) \geq \sum f(X_i) \geq \sum w(X_i) = w(Z) > f(Z)$, because $w|_{X_i} \in P(f)$. Thus $w(W - W_i) - 1 = b(W - W_i)$, and similarly, $x(W - W_j) - 1 = b(W - W_j)$. By applying intersecting submodularity to $W - W_i$ and $W - W_j$, and using that $w|_{W - W_i - W_j} \in P(f)$, we get $0 \geq b(W) - b(W - W_i) - b(W - W_j) + b(W - W_i - W_j) \geq (w(W) - 2) - (w(W - W_i) - 1) - (w(W - W_j) - 1) + w(W - W_i - W_j) = 0$, so equality holds throughout. As a corollary, each hyperedge $e \in \mathcal{E}[W]$ is spanned by one of the W_i's, and

$$\binom{d-1}{2}(b(W) + 2) = \binom{d-1}{2} w(W) =$$
$$= \sum_{1 \leq i < j \leq d} w(W - W_i - W_j) = \sum_{1 \leq i < j \leq d} b(W - W_i - W_j). \quad (10)$$

On the other hand,

$$\binom{d-1}{2} \sum_{h \in \mathcal{E}} h(W) = \sum_{1 \leq i < j \leq d} \sum_{h \in \mathcal{E}} h(W - W_i - W_j),$$

since $\sum_{h \in \mathcal{E}} h$ is modular, and

$$\binom{d-1}{2} p(W) \leq \sum_{1 \leq i < j \leq d} p(W - W_i - W_j),$$

since p is non-negative and non-increasing. Finally,

$$\binom{d-1}{2}|\mathcal{E}[W]| = \binom{d-1}{2}\sum_{i=1}^{d}|\mathcal{E}[W_i]| = \sum_{1\le i<j\le d}|\mathcal{E}[W-W_i-W_j]|.$$

By the definition of b, the last 3 equalities together contradict (10).

Let us give an example showing that polymatroids without non-trivial compatible double circuits are not closed under contractions. Let $V=\{v_1,v_2,v_3,v_4\}$, $\mathcal{E}=\{v_1v_i, v_iv_i : i\in\{2,3,4\}\}$, $p(\{v_1\})=1$ and $p(U)=0$ for the other sets. Then, by Lemma 7, $\widehat{b}$ has no non-trivial compatible double circuits, while the polymatroid obtained from $\widehat{b}$ by contracting an element in the prematroid from the preimage of v_1 has the non-trivial compatible double circuit $(1,2,2,2)$.

4.2 A Planar Rigidity Problem

If $G=(V,E)$ is a graph and $p:V\to\mathbb{R}^2$ is an embedding into the Euclidean plane then (G,p) is said to be a *framework*. We think of the edges of G as rigid bars with flexible joins at the vertices. An *infinitesimal motion* means an assignment of velocities $x(v)\in\mathbb{R}^2$ to each vertex $v\in V$ such that the bar lengths are preserved, that is $(p(u)-p(v))\perp(x(u)-x(v))$. The framework (G,p) is called *rigid* if all infinitesimal motions of (G,p) correspond to isometries of $\mathbb{R}^2$. The question of pinning down a minimum vertex set resulting a rigid framework was solved by Lovász in his seminal paper [10] about matroid parity. We say that $G=(V,E)$ is *generic rigid* if all frameworks (G,p) with *algebraically independent* coordinates p are rigid. The problem of finding a vertex set $Z\subseteq V$ of minimum size such that $G+K_Z$ is generic rigid is left open by [10], and it was solved recently by Fekete [3]. For more on the 2-dimensional rigidity see Laman [8] and Lovász and Yemini [11].

The setup of [3] puts the problem into a bit more general setting. Let $G=(V,E)$ be a graph, and for $l\in\{2,3\}$ let $M_{2,l}$ be the matroid on ground set E such that $F\subseteq E$ is independent in $M_{2,l}$ if and only if $|F[X]|\le 2|X|-l$ for all $X\subseteq V$, $|X|\ge 2$. It can be proved that M is really a matroid. For clarity, $M_{2,2}$ is two times the cycle matroid of G, and so G has two edge-disjoint spanning trees if and only if $r_{2,2}(E)=2|V|-2$. As $M_{2,3}$ is the rigidity matroid of G, the graph G is generic rigid if and only if $r_{2,3}(E)=2|V|-3$. For $Z\subseteq V$ let $K_Z=(Z,E_Z)$ be the graph with vertex set Z having $4-l$ parallel edges between any two vertices of Z. Our goal is to find a set $Z\subseteq V$ of minimum size such that $E+E_Z$ has rank $2|V|-l$. For $l=2$, this is equivalent to shrinking a minimum vertex set Z such that G/Z has two edge-disjoint spanning trees.

We assume that E is independent in $M_{2,l}$, since if E is replaced by one of its bases then the solution set does not change. Fekete [3] proved the following lemma. For $X\subseteq V$ let $e(X)$ denote the number of edges having at least one end vertex in X.

Lemma 8 ([3]). *Let $l\in\{2,3\}$. Assume that E is independent in $M_{2,l}$ and that $r_{2,l}(E)<2|V|-l$. Let $Z\subseteq V$. Then $r(E+E_Z)=2|V|-l$ if and only if $e(Y)\ge 2|Y|$ for every $Y\subseteq V-Z$.*

Therefore, the goal is to find a set $Z \subseteq V$ of minimum size such that $e(Y) \geq 2|Y|$ for every $Y \subseteq V - Z$. Let $f : 2^V \to \mathbb{N}$ be the polymatroid function with $f(X) = \min_{Y \subseteq X} 2|Y| + e(X-Y)$, i.e. f is obtained from the polymatroid function $X \mapsto e(X)$ by deleting with the vector $(2, 2, \ldots, 2)$. Hence for $l = 2$ the value $|V| - \nu(f)$ means the minimum size of a set Z whose contraction results in a graph with two edge-disjoint spanning trees, and for $l = 3$ it is the minimum size of a set Z such that $G + K_Z$ is generic rigid. In [10] the computation of $\nu(f)$ is reduced to the matching problem of graphs, yielding a partition type characterization. This characterization follows from the previous results of this paper, too. First, by Lemma 7 with the choice $p = 0$, the polymatroid function $X \mapsto e(X)$ has no non-trivial compatible double circuits. As f is obtained from $X \mapsto e(X)$ by deletion, Claim 4 yields that nor f has. Thus, $\nu(f) = \min \sum_{j=1}^{t} \left\lfloor \frac{1}{2} f(U_j) \right\rfloor$, where the minimum is taken over all partitions $U_1, U_2, \ldots, U_t$ of V. By the definition of f, we get the following.

Theorem 6 (Fekete, [3]). *Let $l \in \{2, 3\}$. Assume that E is independent in $M_{2,l}$ and that $r_{2,l}(E) < 2|V| - l$. Then the minimum size of a set $Z \subseteq V$ such that $r(E + E_Z) = 2|V| - l$ is $|V| - \nu(f)$, where*

$$\nu(f) = \min \left| V - \bigcup_{j=1}^{t} U_j \right| + \sum_{j=1}^{t} \left\lfloor \frac{e(U_j)}{2} \right\rfloor,$$

where the minimum is taken over all subpartitions $U_1, U_2, \ldots, U_t$ of V.

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