Packing non-returning A-paths

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Outline

- Background:
  - Packing A-paths – Gallai's Theorem (1961)
  - Packing $\mathcal{A}$-paths – Mader's Theorem (1978)
  - Packing non-zero A-paths in group-labeled graphs –
    – Chudnovsky et al. (2004, manuscript)
  - Special cases
- A new interpretation of these results
- A min-max formula for a slightly wider class of problems
- A delta-matroid structure of these packings
- Some remarks on the short proof

A-paths, Gallai's Theorem

$G = (V, E)$ an undirected graph. $A \subseteq V$ a set of **terminals**.

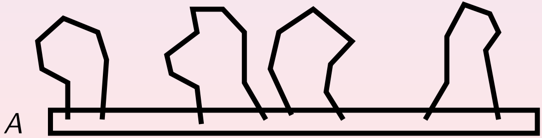
Definition

A-path:
- an s–t path P in G for some $s \neq t, s, t \in A$,
- $V(P) \cap A = \{s, t\}$.

Packing of A-paths: a family of fully node-disjoint A-paths.

Problem

Given G, A as above, find a maximum packing of A-paths.



A-paths, Gallai's Theorem

Theorem (Tibor Gallai, 1961)

Given an undirected graph $G = (V, E)$, and a set $A \subseteq V$, then the maximum cardinality $\widehat{\nu}(G, A)$ of a packing of A-paths is equal to

$$\min_{U \subseteq V} \; |U| + \sum_K \left\lfloor \frac{1}{2} |K \cap A| \right\rfloor \tag{1}$$

where the sum ranges over the components K of $G - U$.

Proof: A reduction to maximum matching, Berge-Tutte formula:

Let $G' := (V', E')$ with

$V' := V \cup \{v' : v \in V - A\}$, and $E' := \{uv' : uv \in E\} \cup \{uu' : u \in V\}$.

Then $$\widehat{\nu}(G, A) = \nu(G') - |V - A|. \quad \blacksquare$$

How to define the set of **"allowed A-paths"**?

Given an undirected graph $G = (V, E)$, and a set $A \subseteq V$.
Consider a set $\mathcal{M}$ of A-paths – these will be the "allowed A-paths".

Problem (the general scheme)

Find a maximum packing of allowed A-paths.

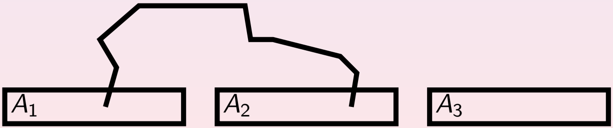
- Put constraints on $\mathcal{M}$ to get a solvable problem?
- How is $\mathcal{M}$ encoded?

The definition of $\mathcal{A}$-paths

$G = (V, E)$ an undirected graph. $A \subseteq V$ a fixed set of **terminals**.
$\mathcal{A} = \{A_1, \cdots, A_k\}$ a partition of A.

Definition

$\mathcal{A}$-**path**:
- a path P in G starting in a node $s \in A_i$ and ending in a node $t \in A_j$ with $i \neq j$,
- $V(P) \cap A = \{s, t\}$.

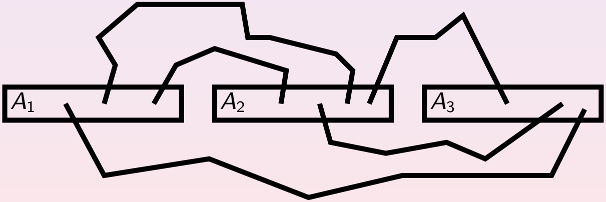


Definition

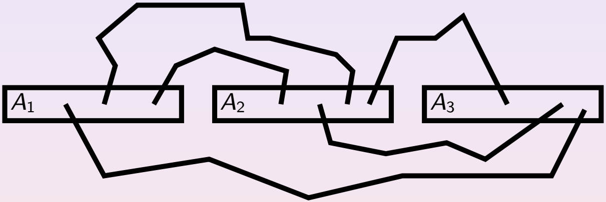
Packing of $\mathcal{A}$-paths: a family of fully node-disjoint $\mathcal{A}$-paths.

Problem

Given $G, \mathcal{A}$ as above, find a maximum packing of $\mathcal{A}$-paths.



Special cases:
Berge-Tutte formula, Gallai's theorem, Menger's theorem



- Polynomial-time algorithm via matroid matching.

  (**L. Lovász**, Matroid matching and some applications, 1980)

- A direct reduction to linear matroid matching.

  (**A. Schrijver**, his book, page 1284.)

Theorem (Wolfgang Mader, 1978)

Given an undirected graph $G = (V, E)$, and a set $A \subseteq V$, and a partition $\mathcal{A} = \{A_1, \cdots, A_k\}$ of A. Then the maximum cardinality of a packing of $\mathcal{A}$-paths is equal to

$$\min \; |X| + \sum_K \left\lfloor \frac{1}{2} \left| K \cap \bigcup X_i \right| \right\rfloor \tag{2}$$

where the minimum ranges over sub-partitions $\{X, X_1, \cdots, X_k\}$ such that $A_i \subseteq X \cup X_i$, and the sum is taken over components K of $G - X - \bigcup E[X_i]$.

A structural description given in:
A. Sebő and L. Szegő: The path-packing structure of graphs (2004)

also, independently in:

Chudnovsky, et al.: Packing non-zero A-paths in group-labeled graphs (2004)

Mader 's Theorem, an interpretation

Theorem (Wolfgang Mader, 1978 – an interpretation)

Given an undirected graph $G = (V, E)$, and a set $A \subseteq V$, and a partition $\mathcal{A} = \{A_1, \cdots, A_k\}$ of A. Then the maximum cardinality of a packing of $\mathcal{A}$-paths is equal to

$$\min \ \widehat{\nu}(G, A \cup V(F)) \tag{3}$$

where the minimum ranges over sets $F \subseteq E$ such that there is no $\mathcal{A}$-path inside F.

Recall: $\widehat{\nu}(G, A)$ denotes the maximum cardinality of a packing of A-paths – determined by Gallai's Theorem.

Easy to check that "there is no $\mathcal{A}$-path inside F".

The easy part of the proof

Observation

If P is an $\mathcal{A}$-path and there is no $\mathcal{A}$-path in F, then some section of P is an $A \cup V(F)$-path.

Hence, if $\mathcal{P}$ is a packing of $\mathcal{A}$-paths and there is no $\mathcal{A}$-path in F, then $|\mathcal{A}| \leq \widehat{\nu}(G, A \cup V(F))$.

Motivation for this interpretation of Mader's Theorem

Motivation for this interpretation of Mader's Theorem:

- Parity in Mader's Theorem hidden inside an oracle for Gallai's formula for $\widehat{\nu}$.
- A similar interpretation can be given for results of
  Chudnovsky, et al.: Packing non-zero A-paths in group-labeled graphs (2004)
- Also extends to the general setting of non-returning A-paths. The method of proof is related to
  A. Schrijver: A short proof of Mader's $\mathcal{S}$-paths Theorem (2001)

Non-zero A-paths in group-labeled graphs – – Chudnovsky et al.

Definition

$G = (V, E)$ an undirected graph, s. t. a **reference orientation** is assigned to each edge. $A \subseteq V$ a fixed set of **terminals**.

Γ a fixed **group** (not necessarily commutative).
A **group-label** $\gamma(e) \in \Gamma$ is assigned to each edge $e \in E$.

For $ab = e \in E$ we define $\gamma(e, a) := \gamma(e)$ and $\gamma(e, b) := -\gamma(e)$.

Non-zero A-paths in group-labeled graphs – – Chudnovsky et al.

Definition

An A-**path** in G is a sequence $P = (v_0, e_0, v_1, e_1, \dots, e_{k-1}, v_k)$ with $k \geq 1$ and $V(P) \cap A = \{v_0, v_k\}$.

Let us define the **weight of an** A-**path** P by $\gamma(P) := \gamma(e_0, v_0) + \gamma(e_1, v_1) + \cdots + \gamma(e_{k-1}, v_{k-1})$.

P is called **non-zero** if $\gamma(P) \neq 0$.

Problem

Given G, A, Γ, γ as above, find a maximum packing of non-zero A-paths.

Non-zero A-paths in group-labeled graphs – – Chudnovsky et al.

A min-max formula is for this problem is given in:

Chudnovsky, et al.: Packing non-zero A-paths in group-labeled graphs (2004)

Special cases:

- Berge-Tutte Formula
- Gallai's Theorem
- Mader's Theorem
- Packing odd-length A-paths
- Paths on surfaces with topological constraints (for example: non-contractible, or non-separating, or orientation-reversing)

Definition

$G = (V, E)$ an undirected graph, s. t. a **reference orientation** is assigned to each edge. $A \subseteq V$ a fixed set of **terminals**.

Ω an arbitrary **set of "potentials"**.
A **"potential of origin"** $\omega(a) \in \Omega$ for each terminal $a \in A$.

$\pi : E \rightarrow S(\Omega) = \{$ all permutations of Ω $\}$

For $ab = e \in E$ we define $\pi(e, a) := \pi(e)$ and $\pi(e, b) := \pi^{-1}(e)$
(the **"mapping of potential"** on edge ab.)

Definition: non-returning A-paths

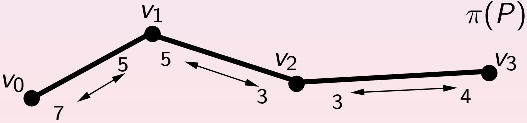
Definition

An A-**path** in G is a sequence $P = (v_0, e_0, v_1, e_1, \ldots, e_{k-1}, v_k)$ with $k \geq 1$ and $V(P) \cap A = \{v_0, v_k\}$.

Let $\pi(P) := \pi(e_0, v_0) \circ \pi(e_1, v_1) \circ \ldots \circ \pi(e_{k-1}, v_{k-1})$ define the **mapping on path** P.

P is called **non-returning** if $\pi(P)(\omega(v_0)) \neq \omega(v_k)$.

$\omega(v_0) = 7$



$\pi(P)(7) = 4$

Motivation for the definition of non-returning A-paths

Problem

Given G, A, ω, π as above, find a maximum packing of non-returning A-paths.

Special cases:

Non-zero A-paths in group-labeled graphs

(introduced by: Chudnovsky et al.)

Given $G = (V, E)$, a set $A \subseteq V$ of terminals, a colouring of A and a partial colouring of E.

Pack A-paths meeting at least two different colours!

Does not seem to be an instance for non-zero A-paths, but is an instance for non-returning A-paths.

(communicated by: L. Szegő, J.F. Geelen)

The Main Result

Theorem

Given G, A, ω, π as above, then the maximum cardinality of a packing of non-returning A-paths is equal to

$$\min \ \widehat{\nu}(G, A \cup V(F)) \tag{4}$$

where the miniumum ranges over sets $F \subseteq E$ such that there is no non-returning A-path inside F.

Recall: $\widehat{\nu}(G, A)$ denotes the maximum cardinality of a packing of A-paths – determined by Gallai's formula.

Easy to check that "there is no non-returning A-path inside F".

Three Equivalent Concepts

Problem

Pack non-returning A-paths.

Problem

Given a group-labeled graph with edge-labels of a group Γ. Let Γ' be a subgroup of Γ. Pack A-paths having weight not in Γ'.

Problem

Given a family $\mathcal{W}$ of A-**walks** obeying the following **exchange property**:

> Consider walks W_1, W_2, W_3 such that each of $W_1 + W_2^{-1}$, $W_2 + W_3^{-1}$, $W_3 + W_1^{-1}$ is an A-walk. If $W_1 + W_2^{-1} \in \mathcal{W}$, then $W_2 + W_3^{-1} \in \mathcal{W}$ or $W_3 + W_1^{-1} \in \mathcal{W}$.

Pack A-paths which are in $\mathcal{W}$.

Remarks on the proof

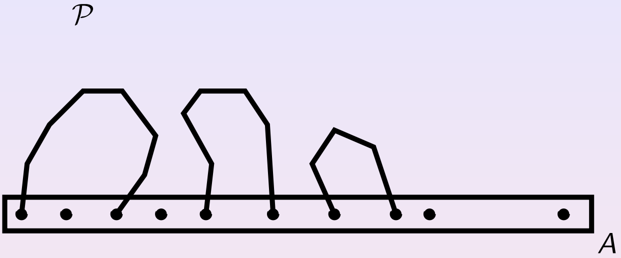
Definition

- A set $Z \subseteq A$ is called **exactly coverable** if there a a packing $\mathcal{P}$ of non-returning A-paths such that $Z = V(\mathcal{P}) \cap A$.
- A set $Z \subseteq A$ is called **coverable** if there a a packing $\mathcal{P}$ of non-returning A-paths such that $Z \subseteq V(\mathcal{P}) \cap A$.

Theorem

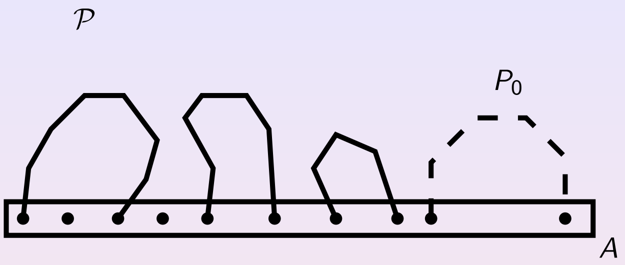
- *Exactly coverable sets form the family of independent sets of a delta-matroid.*
- *Coverable sets form the family of independent sets of a matroid.*

The key reduction principle in the proof of the Main Result



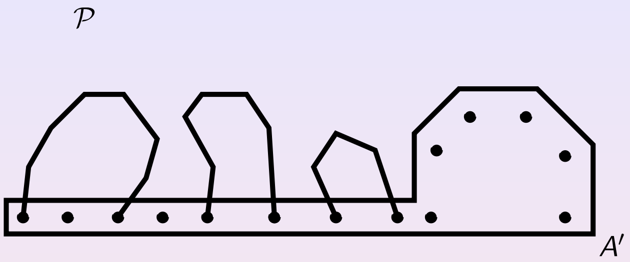
Consider a maximum packing $\mathcal{P}$ of non-returning A-paths.

The key reduction principle in the proof of the Main Result



Consider a maximum packing $\mathcal{P}$ of non-returning A-paths.
Suppose there is a (returning) A-path P_0 disjoint from $\mathcal{P}$.

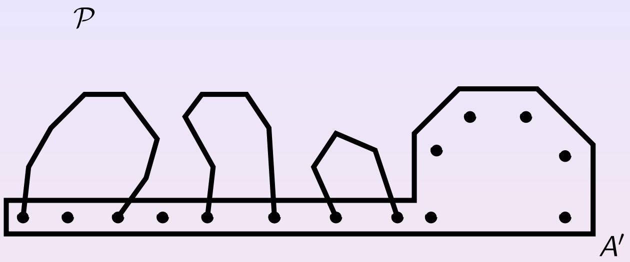
The key reduction principle in the proof of the Main Result



Define $G' := G - E(P_0)$, $A' := A \cup V(P)$, $\omega' := \ldots$, $\pi' := \ldots$.

Claim

$\mathcal{P}$ is maximum with respect to G, A, ω, π if and only if $\mathcal{P}$ is maximum with respect to G', A', ω', π'.



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