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MATCHINGS AND NONRAINBOW COLORINGS*

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Abstract. We show that the maximum number of colors that can be used in a vertex coloring of a cubic 3-connected plane graph G that avoids a face with vertices of mutually distinct colors (a rainbow face) is equal to $\frac{n}{2} + \mu^* - 2$, where n is the number of vertices of G and μ^* is the size of the maximum matching of the dual graph G^*.

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1. Introduction.

Colorings of embedded graphs with face-constraints have recently drawn the attention of several groups of researchers. The very first question that comes to one's mind in this area is the following.

QUESTION 1. *What is the minimal number of colors needed to color an embedded graph in such a way that each of its faces is incident with vertices of at least two different colors; i.e., there is no monochromatic face?*

This problem can be found in the work of Zykov [23] who studied the notion of *planar hypergraphs* and was further explored by Kündgen and Ramamurthi [16] for hypergraphs arising from graphs embedded in surfaces of higher genera. As an example of results obtained in this area, let us mention that every graph embedded on a surface of genus ε has a coloring with $O(\sqrt[3]{\varepsilon})$ colors [5] that avoids a monochromatic face.

An opposite type of question, motivated by results of anti-Ramsey theory, is the following.

QUESTION 2. *What is the maximal number $\chi_f(G)$ of colors that can be used in a coloring of an embedded graph G with no rainbow face; i.e., a face with vertices of mutually distinct colors?*

In our further considerations, we call a vertex coloring of G with no rainbow face a *nonrainbow coloring* of G. Notice that, unlike in the case of ordinary colorings, the goal in this scenario is to *maximize* the number of used colors. Though it may take some time to digest the concept, the setting is so natural that it has recently appeared independently in papers of Ramamurthi and West [20] and of Negami [17] (see also [1, 2, 18] for some even earlier results of this favor). In fact, Negami addressed the following extremal-type question (equivalent to Question 2).

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QUESTION 3. *What is the smallest number $k(G)$ of colors such that every vertex-coloring of an embedded graph G with $k(G)$ colors contains a rainbow face?*

It is not hard to see that $\chi_f(G) = k(G) - 1$ and the results obtained in either of the scenarios translate smoothly to the other one.

We now briefly survey results obtained in the direction of Questions 2 and 3 for planar graphs. Ramamurthi and West [21] noticed that every plane graph G has a nonrainbow coloring with at least $\alpha(G) + 1$ colors where $\alpha(G)$ is the independence number (stability) of G. In particular, every plane graph G of order n has a coloring with at least $\lceil \frac{n}{4} \rceil + 1$ colors by the Four Color Theorem. Also, Grötzsch's theorem [9, 22] implies that every triangle-free plane graph has a nonrainbow coloring with $\lceil \frac{n}{3} \rceil + 1$ colors. It was conjectured in [21] that this bound can be improved to $\lceil \frac{n}{2} \rceil + 1$. Partial results on this conjecture were obtained in [14] and the conjecture has eventually been proven in [12]. More generally, Jungić, Kráľ, and Škrekovski [12] proved that every planar graph of order n with girth $g \geq 5$ has a nonrainbow coloring with at least $\lceil \frac{g-3}{g-2} n - \frac{g-7}{2(g-2)} \rceil$ colors if g is odd, and $\lceil \frac{g-3}{g-2} n - \frac{g-6}{2(g-2)} \rceil$ colors if g is even. All of these bounds are the best possible.

Complementary to the lower bounds on $\chi_f(G)$ presented in the previous paragraph, there are also results on upper bounds on $\chi_f(G)$. Negami [17] investigated nonrainbow colorings of plane triangulations G and showed that $\alpha(G) + 1 \leq \chi_f(G) \leq 2\alpha(G)$. In [6], it was shown that $\chi_f(G) \leq \lfloor \frac{7n-8}{9} \rfloor$ for n-vertex 3-connected plane graphs G, $\chi_f(G) \leq \lfloor \frac{5n-6}{8} \rfloor$ if $n \not\equiv 3 \pmod 8$, and $\chi_f(G) \leq \lfloor \frac{5n-6}{8} \rfloor - 1$ if $n \equiv 3 \pmod 8$ for 4-connected plane graphs G, and $\chi_f(G) \leq \lfloor \frac{43}{100} n - \frac{19}{25} \rfloor$ for 5-connected plane graphs G. The bounds for 3- and 4-connected graphs are the best possible.

Besides results on nonrainbow colorings of graphs with no short cycles and nontrivially connected plane graphs, there are also results on specific families on plane graphs, e.g., the numbers $\chi_f(G)$ were also determined for all semiregular polyhedra [11].

Let us mention that there are also results on mixed types of colorings in which we require that there is neither a monochromatic nor a rainbow face, e.g., [4, 13, 15]. For instance, it is known that each plane graph with at least five vertices has a coloring with two colors as well as a coloring with three colors that avoid both monochromatic and rainbow faces [3, 19].

The quantity $\chi_f(G)$ is also related to several parameters of the dual graph of G. In particular, $\frac{n}{2} + \mu^* - 2 \leq \chi_f(G) \leq n - \alpha^*$ for connected cubic plane graphs G [10], where α^* is the independence number of the dual graph G^* of G and μ^* is the size of the largest matching of G^*. In fact, it was conjectured that the first inequality is always an equality if G is 3-connected.

CONJECTURE 1. *The maximum number of colors used in a nonrainbow coloring of a cubic* 3*-connected plane graph G is related to the size of a maximum matching of its dual as follows:*

$$\chi_f(G) = \frac{n}{2} + \mu^* - 2.$$

We prove this conjecture. In our view, the fact that $\chi_f(G)$ only depends on the size of the largest matching of G^* in this specific case is quite surprising and deserves further investigation in a more general setting.

At the end of this paper, we briefly discuss generalizations and extensions of our results to cubic plane graph that need not be 3-connected. In particular, we show that the assumption that G is 3-connected cannot be relaxed.

2. Proof. If G is a plane graph, then $G^* = (V^*, E^*)$ denotes its plane dual and ϱ^* denotes the minimum size of an edge cover in G^*, i.e., the minimum size of a set of edges such that each vertex is incident with an edge in the set. Gallai's theorem relates the size of a maximum matching and the minimum edge-cover.

THEOREM 1 (see Gallai [7, 8]). *Let H be a graph without isolated vertices, μ the size of the maximum matching of H, and ρ the size of the minimum edge cover. The sum $\mu + \rho$ is equal to the number of vertices of H.*

By Euler's formula and Theorem 1, it holds that $\frac{n}{2} + \mu^* - 2 = n - \varrho^*$ for a 3-regular planar graph. Thus we prove the following theorem equivalent to the statement asserted in Conjecture 1.

THEOREM 2. *The maximum number of colors in a nonrainbow coloring of a cubic 3-connected planar graph $G = (V, E)$ is equal to $n - \varrho^*$.*

Proof. The easy part is to see that there is a nonrainbow coloring of that many colors. Let $E_C \subseteq E$ be the set of edges that corresponds to a minimum edge cover in G^*. The coloring is defined such that two vertices in V receive the same color if and only if they are in the same component of (V, E_c).

To prove the converse, we will rely on the min-max formula for edge cover, saying that the minimum size of an edge cover in a graph $G' = (V', E')$ without isolated vertices is equal to the maximum of $\sum_i \left\lceil \frac{1}{2}|K_i| \right\rceil$, where the maximum is taken over a vertex set $K \subseteq V'$, and K_i denotes the vertex sets of the components of $G'[K]$. This, for G^*, implies that

$$\varrho^* = \sum_i \left\lceil \frac{1}{2}|F_i| \right\rceil, \tag{1}$$

where, for some $F \subseteq V^*$, F_i are the components of $G^*[F]$. Let $V(F_i)$ denote the union of the boundaries of the faces in F_i, which is a subset of V. The sets $V(F_i)$ are disjoint. Hence, it suffices to prove for every nonrainbow coloring and every i that the number of colors appearing in $V(F_i)$ is no more than $|V(F_i)| - \left\lceil \frac{1}{2}|F_i| \right\rceil$. Fix an index i. Let $A_1, A_2, \ldots, A_k \subseteq V(F_i)$ denote the color-classes appearing in $V(F_i)$. We will prove that $k \le |V(F_i)| - \left\lceil \frac{1}{2}|F_i| \right\rceil$, which thus concludes our proof.

We say that a color-class A_j claims a face, if the boundary of that face contains at least two vertices in A_j. Let $Z_j \subseteq F_i$ denote the set of faces in F_i that are claimed by A_j. Now consider the graph $H = (A_j, Q_j)$, where $Q_j = \{a_f b_f : f \in Z_j\}$, where a_f, b_f are two distinct vertices in $A_j \cap f$ (for every face $f \in Z_j$ choose one such pair of vertices). Hence, G is cubic, implying that H is subcubic. Thus

$$|Q_j| \le \left\lfloor \frac{3}{2}|A_j| \right\rfloor. \tag{2}$$

Moreover, if $|A_j| = 1$, then $|Q_j| = 0$, and if $|A_j| = 2$, then $|Q_j| \le 2$ (since G is 3-connected). By considering these inequalities, and inequality (2) in case of $|A_j| \ge 3$, we get that $|Z_j| = |Q_j| \le 2(|A_j| - 1)$; i.e.,

$$|A_j| - 1 \ge \left\lceil \frac{1}{2}|Z_j| \right\rceil. \tag{3}$$

Thus

$$k = |V(F_i)| - \sum(|A_j| - 1) \le |V(F_i)| - \sum \left\lceil \frac{1}{2}|Z_j| \right\rceil \le |V(F_i)| - \left\lceil \frac{1}{2}|F_i| \right\rceil, \tag{4}$$

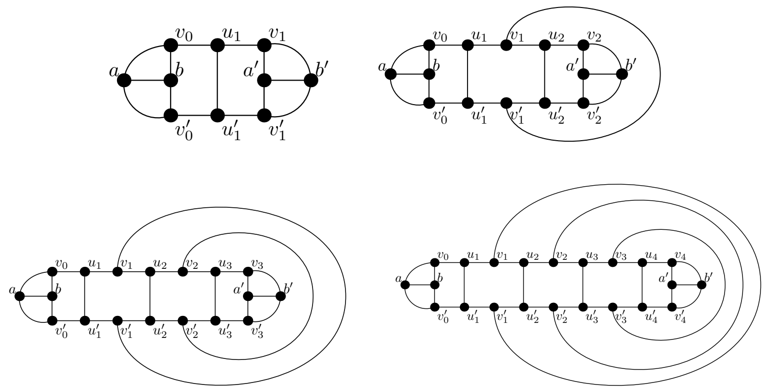

FIG. 1. *The graphs G_1, G_2, G_3, and G_4.*

where the last inequality follows from $|F_i| \leq \sum |Z_j|$ and the fact that the function $f(x) = \lceil \frac{x}{2} \rceil$ is superadditive over the natural numbers. ∎

Note that, as we discuss in section 4, the quantity $\chi_f(G)$ can be computed in polynomial time for 3-connected cubic plane graphs.

3. 2-connected cubic graphs. We first describe a construction of cubic 2-connected plane graphs G_ℓ, $\ell \geq 0$, which are our example graphs, given with a specific embedding into the plane. Start with two paths $v_0u_1v_1u_2v_2 \dots u_\ell v_\ell$ and $v_0'u_1'v_1'u_2'v_2' \dots u_\ell' v_\ell'$ and add the edges u_iu_i' for $i = 1, \dots, \ell$. Next, add an edge ab and join both a and b to both v_0 and v_0'. At the other ends of the paths, add an edge $a'b'$ and join both a' and b' to v_ℓ and v_ℓ'. Finally, add the edges v_iv_i', $i = 1, \dots, \ell-1$, in such a way that they are drawn in the outer face. The graphs G_1, G_2, G_3, and G_4 can be found in Figure 1. Observe that the graph G_ℓ is a 2-connected cubic graph with $n = 4\ell + 6$ vertices and $f = 2\ell + 5$ faces. Also observe that the maximum matching of the dual graph G_ℓ^* has size $\ell + 2$.

THEOREM 3. *For every integer m, there exists a 2-connected cubic plane graph with n vertices such that*

$$\chi_f(G) > \frac{n}{2} + \mu^* - 2 + m.$$

Proof. Consider a graph G_ℓ for $\ell = 3m+1$ and color the following pairs of vertices with the same color (distinct pairs with distinct colors): a and b; a' and b'; u_i and u_i' for $i = 1, 4, \dots, \ell$; and v_i and v_i' for $i = 2, 5, \dots, \ell - 2$. Each of the remaining vertices gets a unique color. This way we construct a nonrainbow coloring of G with $4\ell + 6 - (2m + 3) = 10m + 7$ colors. Hence, we have the following:

$$\chi_f(G) - \frac{n}{2} - \mu^* + 2 \geq 10m + 7 - (6m + 5) - (3m + 3) + 2 = m + 1.$$

The statement of the lemma now follows. ∎

4. Concluding remarks. The statement of Theorem 2 cannot be extended to all cubic plane graphs without any further assumptions. Since the size of the maximum matching in a graph can be computed in polynomial time, it is possible to determine $\chi_f(G)$ for 3-connected cubic plane graphs in polynomial time. For bridgeless cubic plane graphs which need not be 3-connected, it seems natural to consider a dynamic programming approach based on the structure of cuts of sizes one and two in the graph G. Such an algorithm can utilize the following generalization of Theorem 2.

THEOREM 4. *If G is a plane 3-connected cubic graph and F a subset of its faces, then*

$$\chi_f^F(G) = n + \mu^* - |F|,$$

where $\chi_f^F(G)$ is the maximum number of colors that can be used in a coloring such that no face of F is rainbow, and μ^* is the size of a maximum matching of $G^*[F]$.

Though we believed that this approach should have led to a polynomial-time algorithm for determining $\chi_f(G)$ of all cubic graphs, we were not able to obtain such an algorithm; we suspect the problem could be NP-complete.

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