Math. Program., Ser. A
DOI 10.1007/s10107-008-0211-3

FULL LENGTH PAPER

Weighted restricted 2-matching

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Received: 2 June 2004 / Accepted: 23 January 2008

Abstract A perfect 2-matching of a graph is a vector assigning values 0, 1, or 2 to the edges such that the sum of values of edges incident with any node is equal to 2. For restricted perfect 2-matchings, we are also given a collection of "allowed" odd cycles, and restrict ourselves to those perfect 2-matchings the support of which contains no odd cycle not in this collection. Given a graph and a collection of allowed odd cycles, we provide a TDI description of the convex hull of restricted perfect 2-matchings. The description has large coefficients and is given implicitly, thus polynomial time separation or optimization is not straightforward. In order to have such an algorithm, one has to specify the collection of allowed odd cycles. For any fixed number k, we solve optimization in strongly polynomial time for the special cases when the collection consists of odd cycles of length less than k, or odd cycles of length more than k. These solved special cases include minimum weight perfect matching, minimum weight triangle-free and/or pentagon-free perfect perfect 2-matching, and a bunch of other relaxations of the travelling salesman problem. Our algorithm is based on a primal–dual approach, and the unweighted algorithm of Cornuéjols and Hartvigsen, which is used as a subroutine. The TDI description also may be regarded as a generalization of their unweighted min–max formula.

Keywords Matching polytope · 2-matching · TDI description · TSP

Mathematics Subject Classification (2000) 05C70 · 90C27

G. Pap is a member of the MTA-ELTE Egerváry Research Group (EGRES). Supported by European MCRTN Adonet, Contract Grant No. 504438.
Research supported by the Hungarian National Foundation for Scientific Research Grant, OTKA T037547, and the Hungarian Academy of Sciences.

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1 Introduction

Edmonds' description of the perfect matching polytope [7] is a nice display of the polyhedral approach, and is essential to solve the weighted problem. This approach has been generalized to extensions of matching, one of which is a nice description of the triangle-free perfect 2-matching polytope by Cornuéjols and Pulleyblank [2], and a strongly polynomial time algorithm to find a minimum weight triangle-free perfect 2-matching. Thus a natural question is whether one can also find a minimum weight pentagon-free perfect 2-matching, which was proposed in Schrijver [11, p. 544], with the remark that the approach to the triangle-free problem does not extend to the pentagon-free problem. In this paper we apply a more involved technique to solve the pentagon-free problem, and more generally, a technique to solve a whole bunch of weighted restricted 2-matching problems. The description of the polytopes in this paper turned out to be more complex than the two special cases mentioned above. While most known TDI descriptions only have 0, 1, 2 or 0, ± 1 coefficients, here we need inequalities with arbitrarily large coefficients (see [4]).

For the background of the extensions of matching related to the topic of the paper, please refer to Sect. 1.2 on polyhedral results and Sect. 1.3 on unweighted results.

The main results of the paper are the polyhedral description, and a strongly polynomial time algorithm for the weighted problem. Pointers to these results are given at the end of the Introduction, with a summary of the methods used in the proof. Before that, we need some notation.

1.1 Notation

Let G be an undirected graph, with possible loops or parallel edges. Fix an arbitrary collection $\mathcal{H}$ of some (maybe none) odd length cycles in G (A *cycle* is the edge set of a 2-regular, connected subgraph, the length of it is given by the number of edges. A loop is regarded as an odd cycle). An *$\mathcal{H}$-matching* is the edge set of a subgraph the components of which are single edges and cycles in $\mathcal{H}$. An $\mathcal{H}$-matching is *perfect* if it covers all nodes. The *$\mathbb{R}^E$-incidence vector x_M* of an $\mathcal{H}$-matching M is given by $x_M(e) := 1$ if e is in a cycle component of M, $x_M(e) := 2$ if $\{e\}$ is a single edge component of M, all other entries of x_M are zero. In what follows, we will not make a difference between the $\mathcal{H}$-matching and its $\mathbb{R}^E$-incidence vector. Let $\mathcal{P}$ denote the convex hull of perfect $\mathcal{H}$-matchings.

Given a graph $G = (V, E)$, and a node $v \in V$, the set of edges incident with v is denoted by $d(v)$. Given a subset $U \subseteq V$, an edge is said to be induced in U if both of its ends are in U. The set of edges induced in U is denoted by $E[U]$. Thus, the subgraph induced by U is defined by $G[U] := (U, E[U])$. Moreover, the set of nodes not in U having at least one neighbor in U is denoted by $\Gamma_G(U)$.

1.2 Known weighted results

There are some well-known classes of polytopes arising as $\mathcal{P}$ for a choice of $\mathcal{H}$. Let us first consider the choice $\mathcal{H} = \emptyset$. In this case, $\mathcal{P} = 2\mathcal{M}$, where $\mathcal{M}$ is the so-called perfect matching polytope. Edmonds [7] gave a description of the perfect matching polytope, here we mention a stronger theorem due to Pulleyblank and Edmonds [10], Cunningham and Marsh III [5]. A system of linear inequalities $Ax \le b$ is called *TDI* or *totally dual integral* if for any integer vector c, the dual program of maximizing cx subject to $Ax \le b$ has an integer optimum solution y, whenever the optimum value is finite (a definition due to Edmonds and Giles [8]).

Theorem 1 (Pulleyblank and Edmonds [10], Cunningham and Marsh [5]) *For a graph $G = (V, E)$, the perfect matching polytope is determined by the TDI system*

$$x \ge 0 \tag{1}$$
$$x(d(v)) = 1 \quad \text{if } v \in V \tag{2}$$
$$x(E[U]) \le (|U| - 1)/2 \quad \text{if } U \subseteq V \text{ for which } G[U] \text{ is factor-critical.} \tag{3}$$

Moreover, for any integer vector c there is an integer optimum dual solution y_v, y_U such that $\{E[U] : y_U > 0\}$ is a nice family (defined in Sect. 2).

Consider choosing for $\mathcal{H}$ all odd cycles, except for the cycles of length three: in this case a perfect $\mathcal{H}$-matching is called a *triangle-free perfect 2-matching*. Cornuéjols and Pulleyblank [2] gave the following description of the triangle-free perfect 2-matching polytope.

Theorem 2 (Cornuéjols and Pulleyblank [2]) *For an undirected graph $G = (V, E)$, the triangle-free perfect 2-matching polytope is determined by the TDI system*

$$x \ge 0 \tag{4}$$
$$\frac{1}{2}x(d(v)) = 1 \quad \text{if } v \in V \tag{5}$$
$$x(E[a, b, c]) \le 2 \quad \text{if } \{a, b, c\} \subseteq V \tag{6}$$

For another example, consider the set $\mathcal{H}$ of all odd cycles of length at least 5, and possibly some additional cycles of length 3. In this case we only have a theorem describing $\mathcal{P}$ if G is simple:

Theorem 3 (Cornuéjols and Pulleyblank [2]) *If G is a simple graph, and $\mathcal{H}$ contains all odd cycles of length at least 5, then $\mathcal{P}$ is determined by the following TDI system*

$$x \ge 0 \tag{7}$$
$$\frac{1}{2}x(d(v)) = 1 \quad \text{if } v \in V \tag{8}$$
$$x(ab) + x(bc) + x(ca) \le 2 \quad \text{if } \{ab, bc, ca\} \subseteq E, \{ab, bc, ca\} \notin \mathcal{H} \tag{9}$$

In fact, it was an open problem whether there is a good description of this polyhedron if G is not simple. The paper gives a positive answer to a more general problem, but the description given here is slightly more complicated.

The last two systems do not have integer coefficients, but we get integer TDI systems if we replace (5), (8) by $x(d(v)) = 2$ and add $x(E[U]) \le |U|$ for all sets $U \subseteq V$.

1.3 Known unweighted results

For a basic introduction to matching theory, see [6]. In the sequel, the most important notion from matching theory is factor-criticality: an undirected graph is defined to be *factor-critical* if the deletion of any node leaves a graph with a perfect matching.

Let $\mathcal{H}'$ be a set of some odd cycles in $G = (V, E)$ and some nodes in V. If a node is in $\mathcal{H}'$ let us call it a *pseudo-node*. An $\mathcal{H}'$-matching is a node-disjoint collection of edges, pseudo-nodes and cycles in $\mathcal{H}'$, which is perfect if it covers V. The size of an $\mathcal{H}'$-matching is the number of covered nodes. Let $\nu^{\mathcal{H}'}(G)$ denote the maximum size of an $\mathcal{H}'$-matching. We define the graph G to be $\mathcal{H}'$*-critical* if it is factor-critical and there is no perfect $\mathcal{H}'$-matching in G. An induced subgraph $G[V']$ is $\mathcal{H}'$*-critical* if it is factor-critical and there is no perfect $\mathcal{H}'$-matching in $G[V']$. For a subset $X \subseteq V$, we define $c_G^{\mathcal{H}'}(X)$ to be the number of $\mathcal{H}'$-critical components in $G - X$.

We will make use of the following theorem, which is a special case of a theorem in [1] by Cornuéjols and Hartvigsen.

Theorem 4 (Cornuéjols and Hartvigsen [1]) *Let $G = (V, E)$ be a graph, and let $\mathcal{H}'$ be a collection of odd cycles and pseudo-nodes. Then there is a perfect $\mathcal{H}'$-matching in G if and only if there is no set $X \subseteq V$ such that $c_G^{\mathcal{H}'}(X) > |X|$.*

A stronger version is the following Edmonds–Gallai type structural characterization, for which we define the following sets, which are the analogues of the sets A, C, and D for the matching problem. Here $D = D_{\mathcal{H}'}(G)$ is defined to be the set of nodes $v \in V$ for which there is an $\mathcal{H}'$-matching of size $\nu^{\mathcal{H}'}(G)$ which exposes v. Let $A = A_{\mathcal{H}'}(G) = \Gamma_G(D)$.

Theorem 5 (Cornuéjols and Hartvigsen [1]) *Let $G = (V, E)$ be a graph, and let $\mathcal{H}'$ be a collection of odd cycles and pseudo-nodes. Then $\nu^{\mathcal{H}'}(G) = \min_{X \subseteq V} |V| + |X| - c_G^{\mathcal{H}'}(X)$ and this minimum is attained by set $X = A$. Furthermore,*

1. *The components of $G[D]$ are $\mathcal{H}'$-critical and the other components of $G - A$ have a perfect $\mathcal{H}'$-matching, and*
2. *If a set $X \subseteq V$ is minimizing the minimum above, and Z is the union of $\mathcal{H}'$-critical components in $G - X$, then $D \subseteq Z$.*
3. *There is a maximum $\mathcal{H}'$-matching which covers each node a in A by an edge ad with $d \in D$.*

Cornuéjols and Pulleyblank [3] considered the case when $\mathcal{H}'$ is the set of odd cycles of length at least k, as a relaxation of the Hamiltonian cycle problem. They showed that finding a perfect $\mathcal{H}'$-matching is polynomially solvable for any k. Complexity issues depend on recognizing $\mathcal{H}'$-critical graphs. If a factor-critical graph has a perfect $\mathcal{H}'$-matching then there is a perfect $\mathcal{H}'$-matching using exactly one member of $\mathcal{H}'$. The state of the art is that a polynomial-time recognition algorithm could be given in the following cases, where k is a fixed non-negative integer (not part of the input):

1. $\mathcal{H}'$ contains all odd cycles longer than k, and maybe some short cycles, too.
2. The members of $\mathcal{H}'$ can be listed in polynomial time. For example each member of $\mathcal{H}'$ has length less than k.

The conclusion of this paper is that in both cases the weighted problem is also tractable. This settles an unpublished conjecture of Pulleyblank and Loebl [9]. They conjectured that the maximum weight $\mathcal{H}$-matching problem is polynomially solvable if $\mathcal{H}$ consists of triangles.

1.4 Summary of the main results and methods

The first main result is the complete polyhedral description of restricted 2-matchings given in Theorem 7, and the total dual integrality of a modified description given in Theorem 8 (The polyhedral description is quite technical, so we postpone the precise formulation). These results are proved by using a primal–dual technique, which is based on two concepts. Firstly, we plug in the unweighted algorithm of Cornuéjols, Hartvigsen. Secondly, in the unweighted algorithm, we need an oracle to test critical subgraphs.

The second main result concerns the oracle used in the primal–dual algorithm, which is not granted for all special cases. Note that TSP may be formulated as a minimum weight restricted 2-matching problem! On the other hand, we show that there is a bunch of special cases which can be solved in strongly polynomial time. In particular for the two special cases above, for which the unweighted problem is known to be solvable in polynomial time, we show that the weighted problem is solvable in strongly polynomial time. We prove the following theorem, which is the main algorithmic result in the paper:

Theorem 6 *Let k be a fixed non-negative integer. Suppose we are given a graph $G = (V, E)$ with edge-weights and a collection $\mathcal{H}$ of odd cycles such that either*

1. *$\mathcal{H}$ contains all odd cycles longer than k and maybe some shorter cycles, or*
2. *$\mathcal{H}$ can be enumerated in polynomial time (for example every member of $\mathcal{H}$ has length less than k).*

Then there is a strongly polynomial time algorithm (achieving running time bounded by a polynomial in the size of the graph) to find a a minimum weight perfect $\mathcal{H}$-matching.

This result is proved by a primal–dual method, exploiting some more technical results on the polyhedral description. Note that the running time of the algorithm depends on k, which is not part of the input; otherwise it would imply an algorithm for TSP. Note that a solution of the case of unit weights (i.e. in the unweighted case) follows from results cited above.

2 The polyhedral description

In this section, we provide the crucial definitions in order to formulate the polyhedral description, and propose the main polyhedral results of the paper.

A family of sets is called *laminar* if any two of the sets are disjoint or one of them is a subset of the other. A laminar family of edge sets $\mathcal{L}$ is called a *nice family* if for all sets $F \in \mathcal{L}$ the subgraph $G(F) = (V(F), F)$ is factor-critical, the family $\{V(F) : F \in \mathcal{L}\}$ is laminar, and in addition it has one of the following two equivalent properties:

1. For any node $v \in V$ there is a matching M that exposes node v, and for every $F \in \mathcal{F}$ we have $|M \cap F| = (|V(F)| - 1)/2$.
2. For an arbitrary member $F \in \mathcal{F}$, let $F_1, F_2, \ldots, F_k$ denote the maximal members of $\mathcal{F}$ which are proper subsets of F. Then the graph we get from $G(F)$ by contracting the edges in $\cup F_i$ is factor-critical.

By this definition, we may have a pair of subsets in $\mathcal{F}$ with $F_1 \subseteq F_2$ and $V(F_1) = V(F_2)$. For a set $F \in \mathcal{L}$, let $\mathcal{L}_F$ denote the truncation of $\mathcal{L}$ to subsets of F, including F itself. The truncations to the maximal sets in $\mathcal{L}$ are called *components*.

A pair $\mathcal{F} = (\mathcal{L}, m)$ is called a *nice system* if $\mathcal{L}$ is a nice family, and $m : \mathcal{L} \to \mathbb{R}$ (with $m(F) \geq 0$ for any set $F \in \mathcal{L}$) is a non-negative weight- or "multiplicity"-function (Note that we also consider fractional multiplicities). Let $||\mathcal{F}|| := \sum_{F \in \mathcal{L}} m(F)(|V(F)| - 1)$ and for a vector $x \in \mathbb{R}^E$ let $x(\mathcal{F}) := \sum_{F \in \mathcal{L}} m(F)x(F)$. Let $\chi_{\mathcal{F}}(e) := \sum_{F \in \mathcal{L}} m(F)|\{e\} \cap F|$ be the $\mathbb{R}^E$-characteristic vector of $\mathcal{F}$. Then $x(\mathcal{F}) = x \cdot \chi_{\mathcal{F}}$ follows. For a set $F \in \mathcal{L}$ let us denote by $\mathcal{F}_F$ the truncation $(\mathcal{L}_F, m|_{\mathcal{L}_F})$. The components of $\mathcal{F}$ are those nice systems obtained by restricting the function m to a component of $\mathcal{L}$.

Definition 1 Let $G(\mathcal{F}) = (V(\mathcal{F}), E(\mathcal{F}))$ where $E(\mathcal{F}) := \cup\{F : F \in \mathcal{F}\}$ and $V(\mathcal{F}) := \cup\{V(F) : F \in \mathcal{F}\}$.

Lemma 1 *If $\mathcal{F}$ has one component, then for any node $v \in V(\mathcal{F})$ there is a 2-matching x_v with no cycle, $x_v(d(v)) = 0$, $x_v(d(u)) = 2$ for $u \in V(\mathcal{F}) - v$ (i.e. x_v is twice a matching in $G(\mathcal{F}) - v$), and $x_v(\mathcal{F}) = ||\mathcal{F}||$.*

Proof The straightforward proof is left to the reader. □

Definition 2 Let $\mathcal{F}$ be a nice system. The inequality $x(\mathcal{F}) \leq ||\mathcal{F}||$ is called *valid* if it holds for any $\mathcal{H}$-matching x; in this case the system $\mathcal{F}$ will also be called valid.

It is easy to see that $\mathcal{F}$ is valid if and only if its components are valid. Keep in mind that in the definition of validity we considered $\mathcal{H}$-matchings, while the polytope $\mathcal{P}$ is the convex hull of perfect $\mathcal{H}$-matchings. However, it is easy to see the following.

Proposition 1 *If a nice system $\mathcal{F}$ with one component is not valid, then there is a perfect $\mathcal{H}$-matching x in $G(\mathcal{F}) = (V(\mathcal{F}), E(\mathcal{F}))$ for which $x(\mathcal{F}) > ||\mathcal{F}||$.*

The main theorem of the paper is the following polyhedral description.

Theorem 7 *Let $G = (V, E)$ be a graph, and $\mathcal{H}$ be a collection of odd cycles. Then $\mathcal{P}$ is determined by*

$$x \geq 0 \tag{10}$$

$$\frac{1}{2}x(d(v)) = 1 \quad \text{if } v \in V \tag{11}$$

$$x(\mathcal{F}) \leq ||\mathcal{F}|| \quad \text{if } \mathcal{F} \text{ is a valid system.} \tag{12}$$

There is an infinite number of inequalities for valid systems $\mathcal{F}$. To get a traditional polyhedral description, consider a fixed nice family $\mathcal{L}$. We define $C_{\mathcal{L}} = \{m : \mathcal{L} \to \mathbb{R}^+, (\mathcal{L}, m)$ is valid$\}$, then

$$C_{\mathcal{L}} = \Bigg\{ m \geq 0\,, \tag{13}$$

$$\sum_{F \in \mathcal{L}} m(F)\,(|V(F)| - 1 - x(F)) \geq 0 \;\; \text{for } \mathcal{H}\text{-matchings } x \text{ of } G \Bigg\} \tag{14}$$

thus $C_{\mathcal{L}}$ is a polyhedral cone, let $\mathcal{G}_{\mathcal{L}}$ denote a finite generator. The set of solutions of (10)–(12) is the same as of

$$x \geq 0 \tag{15}$$

$$\frac{1}{2} x(d(v)) = 1 \quad \text{if } v \in V \tag{16}$$

$$x(\mathcal{F}) \leq ||\mathcal{F}|| \quad \text{if } \mathcal{F} = (\mathcal{L}, m) \text{ for some } m \in \mathcal{G}_{\mathcal{L}} \tag{17}$$

Notice that we have two complications in comparison with the Theorems 1 and 2. The general class of polytopes $\mathcal{P}$ cannot be described by inequalities with $0, \pm 1$ coefficients, and we also need edge sets of subgraphs which are not induced by a node set. However, we will show in Sect. 5 that there is a TDI description with integer coefficients:

Theorem 8 *Let $G = (V, E)$ be a graph, and $\mathcal{H}$ be a collection of odd cycles. The following is a TDI description of $\mathcal{P}$ which has integer coefficients*:

$$x \geq 0 \tag{18}$$

$$x(d(v)) = 2 \quad \textit{if } v \in V \tag{19}$$

$$x(E[U]) \leq |U| \quad \textit{if } U \subseteq V \tag{20}$$

$$x(\mathcal{F}) \leq ||\mathcal{F}|| \quad \textit{if } \mathcal{F} \textit{ is a valid system with integer multiplicity.} \tag{21}$$

We get a finite TDI description with integer coefficients if we only put the valid inequalities in (21) for $\mathcal{F}$ corresponding to a Hilbert base of some $C_{\mathcal{L}}$.

3 Decompositions of valid inequalities

In this section we prove a number of technical lemmas, which will then be used in the description of the primal–dual algorithm.

Definition 3 A valid system $\mathcal{F}$ is of type (α) if it has one component, m is equal to 1 on the maximal set and zero elsewhere.

Definition 4 A valid system $\mathcal{F}$ is of type (β) if it has one component, and there is a perfect $\mathcal{H}$-matching x^* in $G(\mathcal{F})$ such that $x^*(\mathcal{F}) = ||\mathcal{F}||$.

To proceed with the proof we will need the following technical lemmas, Lemma 5 will be the most important in the next section.

Lemma 2 *If $\mathcal{F}$ is a valid system with one component, $m(F) > 0$ for all sets $F \in \mathcal{L}$, and if $F_1 \in \mathcal{L}$ is not the maximal set, then the truncation $\mathcal{F}_{F_1}$ is not of type (β).*

Proof Suppose for contradiction, that x is a perfect $\mathcal{H}$-matching in $(V(\mathcal{F}_{F_1}), E(\mathcal{F}_{F_1}))$ and $x(\mathcal{F}_{F_1}) = ||\mathcal{F}_{F_1}||$. Then we construct x' by adding edges with weight 2 so that for all sets $F \in \mathcal{L}$, $F \not\subseteq F_1$ we have $x'(F) = |V(F)| - 1$ or $x'(F) = |V(F)|$, and for sets with $F \supseteq F_1$ only the second alternative holds. This x' can be constructed using property 1 for a $v \in V(F_1)$. Then the following calculation gives a contradiction with $\mathcal{F}$ being valid:

$$x'(\mathcal{F}) = x(\mathcal{F}_{F_1}) + \sum_{\mathcal{L}-\mathcal{L}_{F_1}} m(F)x'(F) > ||\mathcal{F}_{F_1}|| + \sum_{\mathcal{L}-\mathcal{L}_{F_1}} m(F)(|V(F)| - 1) = ||\mathcal{F}||$$

(The inequality holds since m is positive on the maximal set, which is in $\mathcal{L} - \mathcal{L}_{F_1}$). □

Lemma 3 *Suppose $\mathcal{F}$ is a valid system with one component and with $m(F) > 0$ for each set $F \in \mathcal{L}$. Then for all sets $F \in \mathcal{L}$ which is not the maximal set in $\mathcal{L}$,*

(a) *the system $\mathcal{F}_F$ is valid, and*
(b) *for any $\mathcal{H}$-matching x, the equality $x(\mathcal{F}_F) = ||\mathcal{F}_F||$ implies that $x(F') = |V(F')| - 1$ for all sets $F' \in \mathcal{L}_F$.*

Proof We prove the lemma by induction "from inside to outside". Suppose we have a set F_1, such that (a) and (b) holds for all sets F in $\mathcal{L}_{F_1} - F_1$.

Suppose F_1 is not the maximal set, and x is an $\mathcal{H}$-matching such that $x(\mathcal{F}_{F_1}) \geq ||\mathcal{F}_{F_1}||$. Then $x(F_1) = |V(F_1)|$ leads to a contradiction, since an $\mathcal{H}$-matching x' could be constructed as in Lemma 2 for which $x'(\mathcal{F}) > ||\mathcal{F}||$. Thus $x(F_1) \leq |V(F_1)| - 1$ and then we get statement (a) for F_1 by

$$x(\mathcal{F}_{F_1}) = x(F_1)m(F_1) + \sum x(\mathcal{F}_{D_i}) \leq (|V(F_1)| - 1)m(F_1) + \sum ||\mathcal{F}_{D_i}|| = ||\mathcal{F}_{F_1}||$$

where D_i are the maximal sets in $\mathcal{L}_{F_1} - F_1$.

If for x we have $x(\mathcal{F}_{F_1}) = ||\mathcal{F}_{F_1}||$, then all equalities $x(\mathcal{F}_{D_i}) = ||\mathcal{F}_{D_i}||$ must hold. Then by induction we get that for each set $F' \in \mathcal{L}_{F_1} - F_1$ the equality $x(F') = |V(F')| - 1$ holds. This also implies $x(F_1) = |V(F_1)| - 1$ and then b) holds for F_1. □

Notice, that (a) also holds for the maximal set, but (b) does not hold necessarily.

Lemma 4 *Suppose $\mathcal{F}$ is a valid system with one component, with $m(F) > 0$ for each set $F \in \mathcal{L}$, and $\mathcal{F}$ is not of type (β). Then for each set $F \in \mathcal{L}$ there is a multiplicity m_F on $\mathcal{L}_F$ so that for the systems $\widehat{\mathcal{F}}_F := (\mathcal{L}_F, m_F)$*

1. $\widehat{\mathcal{F}}_F \sim (\alpha)$, *or*
2. $\widehat{\mathcal{F}}_F \sim (\beta)$ *and $m_F(F') > 0$ for all sets $F' \in \mathcal{L}_F$,*

and there are coefficients $\lambda_F > 0$ *for which*

$$\chi_{\mathcal{F}} = \sum_{F \in \mathcal{L}} \lambda_F \cdot \chi_{\widehat{\mathcal{F}}_F}. \tag{22}$$

Proof The proof goes by induction on $|\mathcal{L}|$. For $|\mathcal{L}| = 1$ the statement holds, since $\mathcal{F}$ is a positive multiple of a system of type (α).

Let F_0 be the maximal set in $\mathcal{L}$. By Lemma 3, all truncations of $\mathcal{F}$ are valid, and by Lemma 2 no proper truncation of $\mathcal{F}$ is of type (β). Thus by induction one can give for each set $F \in \mathcal{L} - F_0$ a system $\widehat{\mathcal{F}}_F = (\mathcal{L}_F, m_F)$ as in 1. or 2. and coefficients $\lambda_F > 0$ for which

$$\chi_{\mathcal{F}} - m(F_0) \cdot \chi_{F_0} = \sum_{F \in \mathcal{L} - F_0} \lambda_F \cdot \chi_{\widehat{\mathcal{F}}_F} \tag{23}$$

holds. There are two cases. First, if there is no perfect $\mathcal{H}$-matching in $G(\mathcal{F})$ then $m_{F_0}(F_0) := 1$ and $m_{F_0}(F) := 0$ (for $F \neq F_0$) and $\lambda_{F_0} := m_{\mathcal{F}}(F_0) > 0$ give equality in (22).

Second, if there is at least one perfect $\mathcal{H}$-matching in $G(\mathcal{F})$. Since $\mathcal{F}$ is not of type (β), for each perfect $\mathcal{H}$-matching x in $(V(F_0), F_0)$ we have $x(\mathcal{F}) < ||\mathcal{F}||$. For a number $t > 0$ we let $\mathcal{F}^t := (\mathcal{L}, m^t)$ where $m^t(F) = m(F)$ if $F \neq F_0$, and $m^t(F_0) = t$. Let $t_0 := m(F_0)$, then $\mathcal{F}^{t_0} = \mathcal{F}$. There is a uniquely defined number T for which $\mathcal{F}^T$ is of type (β). Then $T > t_0$ holds, let $\widehat{\mathcal{F}}_{F_0} := \mathcal{F}^T$ and $\lambda'_{F_0} := t_0/T$, $\lambda'_F := (1 - t_0/T)\lambda_F$. This gives the desired decomposition 22. □

Lemma 5 *Suppose* $\mathcal{F}$ *is a valid system with one component, with* $m(F) > 0$ *for each set* $F \in \mathcal{L}$*, and* $\mathcal{F}$ *is not of type* (β)*. If* x *is an* $\mathcal{H}$*-matching for which* $x(\mathcal{F}) = ||\mathcal{F}||$*, then* $x(F') = |V(F')| - 1$ *for all sets* $F' \in \mathcal{L}$*.*

Proof Take the decomposition $\chi_{\mathcal{F}} = \sum_{F \in \mathcal{L}} \lambda_F \cdot \chi_{\widehat{\mathcal{F}}_F}$ in (22), then $||\mathcal{F}|| = \sum_{F \in \mathcal{L}} \lambda_F \cdot ||\widehat{\mathcal{F}}_F||$ holds, too. Since $\lambda_F > 0$, this implies $x(\widehat{\mathcal{F}}_F) = ||\widehat{\mathcal{F}}_F||$.

Suppose for $F' \in \mathcal{L}$ we have $x(F'') = |V(F'')| - 1$ for all sets $F'' \in \mathcal{L}$, $F'' \subsetneq F'$. Then $x(F') = |V(F')| - 1$ follows from $x(\widehat{\mathcal{F}}_F) = ||\widehat{\mathcal{F}}_F||$. □

Lemma 4 implies that the generator $\mathcal{G}_{\mathcal{L}}$ of the cone $\mathcal{C}$ (see 13) can be chosen so that each member is a nice system of type (α) or (β). The valid systems of type (α) are determined by an edge set. The valid systems $\mathcal{F}$ of type (β) can be described as follows: consider a fixed nice system $\mathcal{L}$, and a perfect $\mathcal{H}$-matching x^* in $(V(\mathcal{L}), \mathcal{L})$. We define $C_{\mathcal{L},x^*} = \{m : \mathcal{L} \to \mathbb{R}^+, (\mathcal{L}, m) \text{ is valid, and } x^*(\mathcal{L}, m) = ||(\mathcal{L}, m)||\}$. Then

$$C_{\mathcal{L},x^*} = \Bigg\{ m \geq 0, \tag{24}$$

$$\sum_{F \in \mathcal{L}} m(F) \big(|V(F)| - 1 - x^*(F)\big) = 0, \tag{25}$$

$$\sum_{F \in \mathcal{L}} m(F) \left(|V(F)| - 1 - x(F)\right) \geq 0 \quad \text{for } \mathcal{H}\text{-matchings } x \text{ of } G \Bigg\} \tag{26}$$

thus $C_{\mathcal{L},x^*}$ is a polyhedral cone. Let $\mathcal{G}(\alpha)$ and $\mathcal{G}(\beta)$ be the set of valid systems we get from the finite generators of these cones. Then Theorem 7 implies that $\mathcal{P}$ is determined by

$$x \geq 0 \tag{27}$$
$$\frac{1}{2}x(d(v)) = 1 \quad \text{if } v \in V \tag{28}$$
$$x(\mathcal{F}) \leq ||\mathcal{F}|| \quad \text{if } \mathcal{F} \in \mathcal{G}(\alpha) \cup \mathcal{G}(\beta). \tag{29}$$

4 The primal–dual method

Theorem 7 is proved in this section by a primal–dual method, which is sketched as follows. We apply the following principle iteratively: We maintain a dual solution, and we either find a primal solution that satisfies the complementary slackness conditions (which is equivalent with an instance of the unweighted problem), or we improve on the dual solution (which is done by using the unweighted min–max formula). Note that for a polynomial time algorithm we need some oracles to deal with critical subgraphs, which will be discussed in the following section.

Instead of taking the dual program of (10)–(12), we consider a bunch of linear programs each of which corresponds to a nice family. There is a large, but finite number of nice families in any graph G. For a nice family $\mathcal{L}$, consider the following linear program in $\mathbb{R}^{V \cup \mathcal{L}}$:

$$\min \sum_{v \in V} y_v + \sum_{F \in \mathcal{L}} m_F \cdot (|V(F)| - 1) \tag{30}$$
$$\frac{1}{2}y_u + \frac{1}{2}y_v + \sum_{uv \in F \in \mathcal{L}} m(F) \geq c_{uv} \quad \text{for } uv \in E - E(\mathcal{L}) \tag{31}$$
$$\frac{1}{2}y_u + \frac{1}{2}y_v + \sum_{uv \in F \in \mathcal{L}} m(F) = c_{uv} \quad \text{for } uv \in E(\mathcal{L}) \tag{32}$$
$$m_F \geq 0 \quad \text{for } F \in \mathcal{L} \tag{33}$$
$$\sum_{F \in \mathcal{L}} m(F)\,(|V(F)| - 1 - x(F)) \geq 0 \quad \text{for } \mathcal{H}\text{-matchings } x \text{ of } G. \tag{34}$$

Notice that the part (33)–(34) is equivalent to $(\mathcal{L}, m)$ being valid, and the objective in (30) is equal to $\sum_{v \in V} y_v + ||(\mathcal{L}, m)||$. Thus, for some solution (y, m) of (31)–(34) one can easily construct a dual solution of (10)–(12) of the same objective value. We abbreviate the objective in (30) by $(y, m) \cdot b$.

If there is at least one perfect $\mathcal{H}$-matching x in G, then $c \cdot x$ is a lower bound on $(y, m) \cdot b$ for a solution of some system (31)–(34). Choose $\mathcal{L}$ for which the minimum $(y, m) \cdot b$ in (30) is minimal, let (y, m) be a minimizing vector. The minimum is also attained by a pair $\mathcal{L}$, (y, m) for which $m(F) > 0$ holds for all sets $F \in \mathcal{L}$ (the other sets can be eliminated from the laminar family). To prove the existence of a perfect $\mathcal{H}$-matching x for which $c \cdot x = (y, m) \cdot b$, we need these complementary slackness conditions:

$$x_{uv} > 0 \;\text{ implies }\; \frac{1}{2}y_u + \frac{1}{2}y_v + \sum_{uv \in F \in \mathcal{L}} m(F) = c_{uv} \tag{35}$$

$$x(\mathcal{L}, m) = ||(\mathcal{L}, m)||. \tag{36}$$

Let $E^= = E^=(\mathcal{L}, y, m)$ denote the set of *tight edges*, the edges uv for which equality holds in (31) or (32). Let $G^= = G^=(\mathcal{L}, y, m) = (V, E^=)$, and let $\overline{G} = \overline{G}(\mathcal{L}, y, m) = (\overline{V}, \overline{E})$ be the graph we get from $G^=$ by contracting all edges in $E[\mathcal{L}]$. Notice that this way we could get a lot of loops or parallel edges, and it is of great importance to keep them. For a set $\overline{U} \subseteq \overline{V}$ let U denote set of the corresponding nodes in V.

Let $\mathcal{L}_{(\beta)}$ denote the union of those components of $\mathcal{L}$ which is of type (β). We call a node in $\overline{V}$ a pseudo-node if it corresponds to a component of $\mathcal{L}_{(\beta)}$.

$$\mathcal{M} = \mathcal{M}(\mathcal{L}, y, m) := \{x : x \text{ is an } \mathcal{H}\text{-matching with } x(F) = |V(F)| - 1 \text{ for all sets } F \in \mathcal{L}\} \tag{37}$$

$$\mathcal{H}^\star = \mathcal{H}^\star(\mathcal{L}, y, m) := \{\text{odd cycles, which appear in some } x \in \mathcal{M}\}. \tag{38}$$

Let $\overline{\mathcal{H}} = \overline{\mathcal{H}}(\mathcal{L}, y, m)$ denote the set of the pseudo-nodes, plus the odd cycles in $\overline{G}$ we get from $\mathcal{H}^\star$ by contracting the edges in $E(\mathcal{L})$. It follows from the definition of $\mathcal{M}$, that the contraction produces only cycles. For an $\mathcal{H}^\star$-matching x in $\mathcal{M}$ let $x/\mathcal{L}$ denote the resulting $\overline{\mathcal{H}}$-matching in $\overline{G}$ after contracting the edges in $E(\mathcal{L})$. It is easy to check, that the cycles in $\overline{\mathcal{H}}$ can be described in another way:

Proposition 2 *Consider an odd cycle C in $\overline{G}$, let B be the set of nodes in V contracted to a node in $V(C)$, let $\mathcal{F}_B$ denote the truncation to the sets in $E[B]$. For a node $z \in V(C)$ let $\mathcal{F}_z$ denote the component of $\mathcal{F}$ contracted to z. Cycle C is in $\overline{\mathcal{H}}$ if and only if there is a perfect $\mathcal{H}$-factor x_C in $G[B]$ which has $x_C/\mathcal{L}_B = \chi(C)$ and $x_C(\mathcal{F}_B) = ||\mathcal{F}_B||$ (or equivalently, it has $x_C(\mathcal{F}_z) = ||\mathcal{F}_z||$ for each node $z \in V(C)$).*

Proposition 3 *If there is a perfect $\overline{\mathcal{H}}$-matching in $\overline{G}$, then there is a perfect $\mathcal{H}$-matching x in G for which $(y, m) \cdot b = c \cdot x$.*

Proof In a perfect $\overline{\mathcal{H}}$-matching, each node $\overline{v} \in \overline{V}$ is covered by a pseudo-node, an odd cycle from $\overline{\mathcal{H}}$, or an edge.

If a node $\overline{v}$ is covered by an edge $\overline{e} \in \overline{E}$, then there is an edge $uv \in E^=$ which is mapped onto $\overline{e}$ by the contraction. Let $\mathcal{F}_{\overline{v}}$ denote the component of $\mathcal{F}$ contracted to $\overline{v}$, then $v \in V(\mathcal{F}_{\overline{v}})$. We define x on the edges in $E(\mathcal{F}_{\overline{v}})$ by the vector x_v for $\mathcal{F}_{\overline{v}}$ from Lemma 1.

If a node $\overline{v}$ is covered by a pseudo-node, then we define x on the edges in $E(\mathcal{F}_{\overline{v}})$ by the vector x^* from the definition of (β).

If a node $\overline{v}$ is covered by an odd cycle $C \in \overline{\mathcal{H}}$, let B denote the set of nodes in V contracted to a node in $V(C)$. We define x on the edges in $E(\mathcal{F}_{\overline{v}})$ by the vector x_C from Proposition 2.

The proof is completed by checking the complementary slackness conditions (35)–(36). Here (35) is due to using only the edges in $E^=$, (36) we get from the definitions of x^* and x_C. □

If there is no perfect $\overline{\mathcal{H}}$-matching in $\overline{G}$, then by Theorem 4 there is a set $\overline{X} \subseteq \overline{V}$ for which the number of $\overline{\mathcal{H}}$-critical components in $\overline{G} - \overline{X}$ is strictly greater than $|\overline{X}|$.

Let $\mathcal{F} := \mathcal{F}(\mathcal{L}, y, m)$ be the system with nice family $\mathcal{L}$ and multiplicity m. For $\overline{K} \subseteq \overline{V}$ we define $\mathcal{F}_{E[K]}$ to be the truncation of $\mathcal{F}$ to the subsets of $E[K]$.

Lemma 6 *If $\overline{G}[\overline{K}]$ is an $\overline{\mathcal{H}}$-critical component in $\overline{G} - \overline{X}$ for some $\overline{K} \subseteq \overline{V}$, then either*

(a) *$G^=[K]$ is $\mathcal{H}$-critical, or*
(b) *for each perfect $\mathcal{H}$-matching x in $G^=[K]$ we have $x(\mathcal{F}_{E[K]}) < ||\mathcal{F}_{E[K]}||$.*

Proof Since $\overline{G}[\overline{K}]$ is $\overline{\mathcal{H}}$-critical, all components of $\mathcal{L}_{(\beta)}$ are node-disjoint from K. For each $\mathcal{H}$-matching x in $G^=[K]$ we have $x(\mathcal{F}_{E[K]}) \leq ||\mathcal{F}_{E[K]}||$, since $\mathcal{F}$ is valid. Suppose for contradiction, that x is a perfect $\mathcal{H}$-matching in $G^=[K]$ for which $x(\mathcal{F}_{E[K]}) = ||\mathcal{F}_{E[K]}||$. By Lemma 5 we get that for each set $F \in \mathcal{F}_{E[K]}$ the equation $x(F) = |V(F)| - 1$ holds. Then it is easy to see, that $x/\mathcal{L}$ is a perfect $\overline{\mathcal{H}}$-matching in $\overline{G}[\overline{K}]$, a contradiction. □

Now, we have a set $\overline{X}$ at hand, which enables us to construct a solution of (31)–(34) for some other laminar family $\mathcal{L}'$ as follows. Let $\overline{K_i}$ ($i = 1, \dots, k$) be the set of $\overline{\mathcal{H}}$-critical components in $\overline{G} - \overline{X}$, here $k > |\overline{X}|$; furthermore we define $F_i := E^=[K_i]$. Let $X_1, \dots, X_l$ denote the maximal sets (of edges) in $\mathcal{L}$ which correspond to a node in $\overline{X}$. Let $\mathcal{L}' := \mathcal{L} \cup \{F_1, \dots, F_k\}$, and

$$\begin{aligned} m'(F_i) &:= m(F_i) + \varepsilon \quad \text{if } F_i \in \mathcal{L} \\ m'(F_i) &:= \varepsilon \quad \text{if } F_i \notin \mathcal{L} \\ m'(X_i) &:= m(X_i) - \varepsilon \quad \text{for } i = 1, \dots, l \\ m'(F) &:= m(F) \quad \text{otherwise} \\ y'(v) &:= y(v) \quad \text{if } v \in V - X - \cup K_i \\ y'(v) &:= y(v) + \varepsilon \quad \text{if } v \in X \\ y'(v) &:= y(v) - \varepsilon \quad \text{if } v \in \cup K_i \end{aligned} \tag{39}$$

Now we check for which value of ε we get a solution of (31)–(34). The dual change does not change $\frac{1}{2}y_u + \frac{1}{2}y_v + \sum_{uv \in F \in \mathcal{L}} m(F)$ for $uv \in E(\mathcal{L}')$, and does not decrease $\frac{1}{2}y_u + \frac{1}{2}y_v + \sum_{uv \in F \in \mathcal{L}} m(F)$ for $uv \in E^=$, thus for any $\varepsilon \geq 0$ we get a solution of (32), and (31) for edges in $E^=$. To get a solution of (31) for edges not in $E^=$, notice that the left hand side decreases by ε if $u, v \in \cup K_i$, and decreases by $\varepsilon/2$ if $u \in \cup K_i$ and $v \in V - X - \cup K_i$. Thus we get the upper bound

$$\begin{aligned} \min\{2c'(uv) \quad & \text{for } uv \notin E^=, u \in \cup K_i, v \in V - X - \cup K_i \\ c'(uv) \quad & \text{for } uv \notin E^=, u, v \in \cup K_i\}, \end{aligned}$$

where $c'(uv) := c(uv) - \frac{1}{2}y_u - \frac{1}{2}y_v - \sum_{uv\in F\in\mathcal{L}} m(F)$ is the "reduced" cost function. From (33) we get the upper bound

$$\min\{m(X_i) : i = 1, \ldots, l\},$$

which is positive since $m > 0$.

We need to choose $\varepsilon > 0$ to get a solution of (34) for $\mathcal{L}', m', y'$. It is enough to check that if $\mathcal{L}''$ is a component of $\mathcal{L}'$, then $\sum_{F\in\mathcal{L}''} m'(F)\,(|V(F)| - 1 - x(F)) \geq 0$ holds for any $\mathcal{H}$-matching x of G. The components of $\mathcal{F}'$ are $\mathcal{F}'_{F_i}$ and $\mathcal{F}'_{X_i}$, or are identical with a component of $\mathcal{F}$, we only have to check the first two cases. Thus the set of inequalities (34) for $\mathcal{L}', m', y'$ is equivalent with $\mathcal{F}'_{F_i}$ and $\mathcal{F}'_{X_i}$ being valid.

Let X_i^j be the maximal sets in $\mathcal{L}_{X_i} - \{X_i\}$. Since

$$\chi_{\mathcal{F}'_{X_i}} = \left(1 - \frac{\varepsilon}{m(X_i)}\right) \cdot \chi_{\mathcal{F}_{X_i}} + \frac{\varepsilon}{m(X_i)} \cdot \sum_j \chi_{\mathcal{F}_{X_i^j}}$$

and since by Lemma 3 the systems $\mathcal{F}_{X_i}$ and $\mathcal{F}_{X_i^j}$ are valid, we get that the components $\mathcal{F}'_{X_i}$ are valid for each i and any $0 \leq \varepsilon \leq m(X_i)$. For a component $\mathcal{F}'_{F_i}$ and an $\mathcal{H}$-matching x we need:

$$\begin{aligned} 0 &\leq \sum_{F\in\mathcal{L}'_{F_i}} m'(F)\,(|V(F)| - 1 - x(F)) \\ &= \sum_{F\in\mathcal{L}'_{F_i}} m(F)\,(|V(F)| - 1 - x(F)) + \varepsilon \cdot (|V(F_i)| - 1 - x(F_i)) \\ &= ||\mathcal{F}_{F_i}|| - x(\mathcal{F}_{F_i}) + \varepsilon \cdot (|V(F_i)| - 1 - x(F_i))\,. \end{aligned} \tag{40}$$

The upper bound on ε is the minimum of the fraction $(||\mathcal{F}_{F_i}|| - x(\mathcal{F}_{F_i}))/(x(F_i) - |V(F_i)| + 1)$ on $\mathcal{H}$-matchings x for which $x(F_i) - |V(F_i)| + 1$ is positive. It is easy to see that $x(F_i) - |V(F_i)| + 1$ is positive if and only if x is a perfect $\mathcal{H}$-matching in $(V(F_i), F_i) = G^{=}[K_i]$. By Lemma 6, for these x's we have $||\mathcal{F}_{F_i}|| - x(\mathcal{F}_{F_i}) > 0$, the upper bound on ε will be $||\mathcal{F}_{F_i}|| - M(\mathcal{F}_{F_i})$, where the function M is defined by

$$M(\mathcal{F}) := \max\{x(\mathcal{F}) : x \text{ is a perfect } \mathcal{H}\text{-matching in } G(\mathcal{F})\}, \tag{41}$$

for a nice system $\mathcal{F}$.

We conclude that the exact bound which gives the maximum value of ε to get a dual change is given be the following formula.

Proposition 4 *Suppose we have a dual solution $\mathcal{L}$, y, m of* (31)–(34)*, and a set $\overline{X}$ as defined above. Then the maximum value of ε for which $\mathcal{L}'$, y', m' defined by* (39) *is a* *solution of* (31)–(34) *is*

$$\begin{aligned}&\min\{2c'(uv) \quad \textit{for } uv \notin E^{=}, u \in \cup K_i, v \in V - X - \cup K_i\\ &c'(uv) \quad \textit{for } uv \notin E^{=}, u, v \in \cup K_i\\ &m(X_i) \quad \textit{for } i = 1, \ldots, l\\ &||\mathcal{F}_{F_i}|| - M(\mathcal{F}_{F_i}) \quad \textit{for } i = 1, \ldots, k\}.\end{aligned} \tag{42}$$

If m is positive, then the formula gives a positive value for ε. The inequality $(y', m')b' = (y, m)b - (k - |\overline{X}|)\varepsilon < (y, m)b$ contradicts the choice of $\mathcal{L}$, y, m, this finishes the proof of the description (10)–(12) of $\mathcal{P}$. In fact, we have proved a little bit more, which is a strengthening of Theorem 1:

Theorem 9 *For each vector c, there is a nice family $\mathcal{L}$ for which the optimum value of* (30)–(34) *is equal to* $\max_{x \in \mathcal{P}} cx$.

5 Strongly polynomial running time and total dual integrality

Now let us discuss the oracles used in the primal–dual method. Clearly, to determine ε in the dual change (42), we need an oracle to determine values of $M(\cdot)$, and a vector x attaining maximum in (41). Another concern lies within the definition (37)–(38) of $\mathcal{H}^\star$, which is used in the definition of $\overline{\mathcal{H}}$. However, it is easy to see that these definitions can be evaluated by applying the oracle for $M(\cdot)$. An oracle for $M(\cdot)$ is not granted, but for those special cases mentioned earlier, we will show how to implement this oracle in strongly polynomial time in Sect. 6.

We are in position to prove the following statement, that is a strengthening of total dual integrality, and implies Theorem 8. We will prove this result by a strongly polynomial time algorithm that maintains a dual integral solution, until optimality is achieved.

Theorem 10 *For each integer vector c, there is a nice family $\mathcal{L}$ for which the optimum value of* (30)–(34) *is equal to* $\max_{x \in \mathcal{P}} cx$ *and is attained by an integer vector* (y, m).

The proof goes by carefully reading Sect. 4, and using the dual change for an algorithm maintaining dual integrality. This can be done by choosing the set $\overline{X}$ in a special way.

We start with $\mathcal{L}$ empty and (y, m) some integer solution of (30)–(34). Then we iteratively change $\mathcal{L}$ and (y, m) as written in Sect. 4: we find the sets X and K_i, change $\mathcal{L}$ and (y, m), and if an entry of m' is 0 then we eliminate the set from $\mathcal{L}'$. The value of ε is defined by the formula (42). If the minimum was taken over an emptyset, then it is easy to see that set X is as in Theorem 4 which verifies that there is no perfect $\mathcal{H}$-matching. In this case the algorithm stops by announcing the polytope is empty and exhibiting the set X.

From this point on we suppose there is a perfect $\mathcal{H}$-matching in G. If there is a perfect $\overline{\mathcal{H}}$-matching in $\overline{G}$, then Lemma 3 gives an optimum $\mathcal{H}$-matching, the algorithm terminates.

If there is no perfect $\overline{\mathcal{H}}$-matching, to maintain integrality of (y, m) let us choose X and K_i according to the following rule. Consider the sets $\overline{D} = D_{\overline{\mathcal{H}}}(\overline{G})$ and $\overline{A} =$

$A_{\overline{\mathcal{H}}}(\overline{G})$ from Theorem 5. Let $\overline{G}[\overline{A}_j \cup \overline{D}_j]$ be the components of $\overline{G}[\overline{D} \cup \overline{A}] - E[\overline{A}]$, where $\overline{A}_j \subseteq \overline{A}$ and $\overline{D}_j \subseteq \overline{D}$. Let $\overline{K_{ij}}$ $(i = 1, 2, \ldots, k_j)$ denote the components of $\overline{G}[\overline{D}]$ for which $\overline{K}_{ij} \subseteq \overline{D}_j$. Since $\overline{G}[\overline{D}]$ has $\overline{\mathcal{H}}$-critical components, each $\overline{G}[\overline{K}_{ij}]$ is a $\overline{\mathcal{H}}$-critical component in $\overline{G} - \overline{A}_j$.

Proposition 5 *For each j we have $k_j > |\overline{A}_j|$.*

Proof If $k_1 \leq |\overline{A}_1|$ holds, then $X = \overline{A} - \overline{A}_1$ must be a minimizing set for $|V| + |X| - c_{\overline{G}}^{\overline{H}}(X)$, thus $k_1 = |\overline{A}_1|$. This implies that $\overline{G} - (\overline{A} - \overline{A}_1)$ cannot have any other $\overline{\mathcal{H}}$-critical than $\overline{K}_{ij}$ for $j \neq 1, i = 1, 2, \ldots, k_j$. Thus the union of $\overline{\mathcal{H}}$-critical components of $\overline{G} - (\overline{A} - \overline{A}_1)$ is strictly smaller than $\overline{D}$, a contradiction with part 2 of Theorem 5. □

Let us choose $\overline{X} := \overline{A}_1$ and $\overline{K}_i := \overline{K}_{i1}$! If (y, m) is integer, then the parity of y is the same on each node of a component of $G^=$. By the choice of $\overline{X}$, y_v has the same parity for the nodes v in K_i and X. We get that ε is integer, since the values in (42) are integer, only $c'(uv)$ has the fractional expression $\frac{1}{2}y_u + \frac{1}{2}y_v$. By the above observation on parity, this must be integer for $uv \notin E^=$, $u, v \in \cup K_i$. Hence, after the dual change, (y', m') is an integer vector. To complete the proof, we need to show the finiteness of the algorithm. In fact, finiteness follows from integrality, since each dual change decreases the dual objective by at least 1. What we will show is that rule implies that the number of dual changes is bounded by a polynomial of $|V|, |E|$.

Proposition 6 *A dual change does not increase the deficiency* $\mathrm{def}(\mathcal{L}, y, m) := |\overline{V}| - \nu^{\overline{\mathcal{H}}}(\overline{G})$.

Proof To see this, notice that edges which "get untight" by the dual change, i.e. edges in $\overline{G}$ but not in $\overline{G}(\mathcal{L}', y', m')$ must be induced by $\overline{X}$ or must be in $d(\overline{X}, \overline{V} - \overline{X} - \overline{D}_1)$. Consider a maximum $\overline{\mathcal{H}}$-matching $\overline{x}$ as in part 3. of Theorem 5, then $\overline{x}$ does not use any of these edges. Thus $\overline{x}' := \overline{x}/\{E^=[K_i] : i = 1, 2, \ldots, k_1\}$ is a $\overline{\mathcal{H}}(\mathcal{L}', y', m')$-matching in $\overline{G}(\mathcal{L}', y', m')$ of the same deficiency. If some sets with $m(X_i) = 0$ for a node in $\overline{X}$ are eliminated, that $\overline{x}'$ can be expanded using Lemma 1.

Let $d(\mathcal{L}, y, m)$ denote the number of nodes of V corresponding to a node in $\overline{D}$.

Proposition 7 *If* $\mathrm{def}(y, m) = \mathrm{def}(y', m')$ *then* $d(\mathcal{L}', y', m') \geq d(\mathcal{L}, y, m)$.

Proof Let D' be the set of nodes in $\overline{G}(\mathcal{L}', y', m')$ which correspond to a node in $\overline{D} = \overline{D}(\mathcal{L}, y, m)$. We need to see that $D' \subseteq \overline{D}(\mathcal{L}', y', m')$, i.e. let $v' \in D'$.

Let K be the component of $\overline{G}[\overline{D}]$ which corresponds to v', consider an $\overline{\mathcal{H}}$-matching $\overline{x}$ in $\overline{G}$ exposing v' for which each node a in $\overline{A}$ is covered by an edge ad with $d \in \overline{D}$, and $\overline{x}$ has deficiency $\mathrm{def}(y, m)$ (as in part 3 of Theorem 5). Then $\overline{x}/\{E^=[K_i] : i = 1, 2, \ldots, k_1\}$ is an $\overline{\mathcal{H}}(\mathcal{L}', y', m')$-matching exposing v', thus $v' \in \overline{D}(\mathcal{L}', y', m')$. □

Let $n(\mathcal{L}, y, m)$ components of $\overline{G}[\overline{D} \cup \overline{A}] - E[\overline{A}]$. Let $\mathcal{L}(\overline{A}(y, m))$ denote the union of truncations of $\mathcal{L}$ to nodes in $\overline{A}$.

Proposition 8 *If* $\mathrm{def}(\mathcal{L}, y, m) = \mathrm{def}(\mathcal{L}', y', m')$ *and* $d(\mathcal{L}', y', m') = d(\mathcal{L}, y, m)$ *hold, then* $n(\mathcal{L}', y', m') \leq n(\mathcal{L}, y, m)$ *and* $|\mathcal{L}(\overline{X}(y', m'))| \leq |\mathcal{L}(\overline{X}(y, m))|$.

Proof The first follows from the observation that each edge in $\overline{G}[\overline{D} \cup \overline{A}] - E[\overline{A}]$ stays tight. The second follows, since new members $\mathcal{L}'$ are in $\mathcal{D}(\mathcal{L}', y', m')$. □

Proposition 9 *If we have* $\operatorname{def}(\mathcal{L}, y, m) = \operatorname{def}(\mathcal{L}', y', m')$ *and* $d(\mathcal{L}', y', m') = d(\mathcal{L}, y, m)$ *and* $n(\mathcal{L}, y, m) = n(\mathcal{L}', y', m')$ *and* $|\mathcal{L}'(\overline{A}'(y', m'))| = |\mathcal{L}(\overline{A}(y, m))|$, *then the number of tight edges in* $G - A$ *increases (where* A *is the set of nodes in* V *corresponding to a node of* $\overline{A}$*).*

Proof The premises imply that an edge induced in a component $\overline{G}[\overline{K}_i]$ is determining the minimum (42), this edge is is getting tight for $\mathcal{L}', y', m'$. Only edges incident with a node of $\overline{X} \subseteq \overline{A}$ can get untight. □

We conclude, that the special choice of X assures that either $\operatorname{def}(\mathcal{L}, y, m)$ decreases or $d(\mathcal{L}, y, m)$ increases or $n(\mathcal{L}, y, m) + |\mathcal{L}(\overline{A}(y, m))|$ decreases or the number of tight edges in $G - A$ increases. Thus the number of dual changes needed is at most $|V|^2 \cdot |E|^2$. This provides an algorithm for finding a maximum weight perfect $\mathcal{H}$-matching together with an integer dual optimum, and proves that (15)–(17) is TDI.

proof of Theorem 8 For some fixed integer vector c, consider an integer optimum solution (y, m) of (30)–(34). Let $y'_v := \lfloor \frac{1}{2} y_v \rfloor$, let U_1 be the set of nodes v with y_v odd, and $z'_U := 1$ if $U = U_1$, otherwise $z'_U := 0$. Let ω' be a 0-1 vector with the only 1 in the entry for $\mathcal{F} = (\mathcal{L}, m)$. Then (y', z', ω') is an optimum dual solution of (18)–(21). □

6 Complexity considerations

Up to this point we have not discussed the complexity of the notions introduced. Recall the definition of $M(\mathcal{F})$, and let

$$M'(\mathcal{F}) := \max \{x(\mathcal{F}) : x \text{ is a perfect } \mathcal{H}\text{-matching in } G(\mathcal{F}) \text{ with exactly one odd cycle}\}. \tag{43}$$

The following lemmas are essential for proving results on complexity.

Lemma 7 (Cornuéjols and Pulleyblank [2]) *If G is factor-critical and there is a perfect $\mathcal{H}$-matching in G, then there is a perfect $\mathcal{H}$-matching in G which has exactly one odd cycle.*

Cornuéjols and Pulleyblank gave a short proof of Lemma 7, a short proof of the following weighted version could not have been given, we postpone the proof of Lemma 8 to Sect. 7.

Lemma 8 *Suppose* $\mathcal{F} = (\mathcal{L}, m)$ *is a valid system with one component and there is at least one perfect $\mathcal{H}$-matching in* $G(\mathcal{F})$*. Then* $M(\mathcal{F}) = M'(\mathcal{F})$*, i.e. the maximum in* $M(\mathcal{F})$ *is attained by a perfect $\mathcal{H}$-matching x which has exactly one odd cycle.*

We show how Lemma 8 can be used to construct an an algorithm to check whether a nice system with one component $\mathcal{F}$ is valid and if so, to find the maximum $M(\mathcal{F})$. This is of our great interest, since function M is necessary to determine the value of ε from Lemma 4.

Theorem 11 *Suppose we are given a graph $G = (V, E)$ and a collection $\mathcal{H}$ of odd cycles. Suppose that $\mathcal{H}$ satisfies one of the conditions in Theorem 6, that is, $\mathcal{H}$ can be enumerated in polynomial time, or there is a fixed non-negative integer k such that $\mathcal{H}$ consists of all odd cycles longer than k and some shorter odd cycles. Then there is a strongly polynomial time algorithm to determine $M(\mathcal{F})$ for any nice system $\mathcal{F}$ with one component.*

Proof We use a simple reduction to weighted perfect matching using a principle of Cornuéjols and Pulleyblank [3].

In the first case we try each cycle C in $\mathcal{H}$ which has $C \subseteq E(\mathcal{F})$. For each C we determine a maximum perfect matching in $G(\mathcal{F}) - V(C)$, and in the end we pick the maximum of all.

In the second case, to determine $M'(\mathcal{F})$, it is enough to find a maximum weight perfect 2-matching with only one cycle C and the length of C at least k, where k is a fixed odd number. This is equivalent with the following approach. We try each path $P \subseteq E(\mathcal{F})$ on k edges to find the maximum weight $M'(\mathcal{F}, P)$ of a perfect 2-matching with a cycle C for which $P \subseteq C$—in the end we pick the maximum over all possible P's. For a fixed path P connecting nodes a and b, it is easy to see that $M'(\mathcal{F}, P)$ is the sum of the maximum weight of a perfect matching M_a in $G(\mathcal{F}) - (V(P) - a)$ and the maximum weight of a perfect matching M_b in $G(\mathcal{F}) - (V(P) - b)$. We perform a weighted perfect matching algorithm to find these maximums, the sum of matchings M_a and M_b gives the optimum $M'(\mathcal{F}, P)$ (after eliminating even cycles). □

Lemma 9 *A nice system $\mathcal{F} = (\mathcal{L}, m)$ with one component is valid if and only if its proper truncations are valid and $M'(\mathcal{F}) \leq ||\mathcal{F}||$.*

Proof If $\mathcal{F}$ is valid, then all truncations of $\mathcal{F}$ are valid by Lemma 3 and $M'(\mathcal{F}) \leq ||\mathcal{F}||$ follows from the definition of validity.

Now, suppose the proper truncations of $\mathcal{F}$ are valid and $M'(\mathcal{F}) \leq ||\mathcal{F}||$. If $|\mathcal{L}| = 1$ then the validity of $\mathcal{F}$ follows from Lemma 7. If there is no perfect $\mathcal{H}$-matching in $G(\mathcal{F})$ then the validity of $\mathcal{F}$ follows merely from the validity of its truncations to the maximal proper truncations. Thus from now on we suppose $|\mathcal{L}| > 1$ and there is a perfect $\mathcal{H}$-matching in $G(\mathcal{F})$.

Let F_0 be the maximal set of $\mathcal{L}$ and let F_i $(i = 1, 2, \ldots, k)$ be the family of maximal sets in $\mathcal{L} - F_0$, we will use the notation $\mathcal{F}_i := \mathcal{F}_{F_i}$. Consider the nice system $\mathcal{F}^0 = (\mathcal{L}, m_0)$ with $m_0(F_0) := 0$ and $m_0(F) := F$ for $F \neq F_0$. Since

$$\chi_{\mathcal{F}^0} = \sum_{i \geq 1} \chi_{\mathcal{F}_i}$$

and $\mathcal{F}_i$ are valid, we get that $\mathcal{F}^0$ is valid. Hence by Lemma 8 $M(\mathcal{F}^0) = M'(\mathcal{F}^0)$,

$$M(\mathcal{F}) = M(\mathcal{F}^0) + m(F_0)|V(F_0)| = M'(\mathcal{F}^0) + m(F_0)|V(F_0)| \leq M'(\mathcal{F}),$$

thus $M(\mathcal{F}) = M'(\mathcal{F})$. For an $\mathcal{H}$-matching x we have

$$\begin{aligned} x(\mathcal{F}) &= m(F_0)x(F_0) + \sum_{i\geq 1} x(\mathcal{F}_i) \\ &\leq m(F_0)x(F_0) + \sum_{i\geq 1} ||\mathcal{F}_i|| \\ &= m(F_0)x(F_0) + ||\mathcal{F}|| - m(F_0)(|V(F_0)| - 1), \end{aligned} \tag{44}$$

If $\mathcal{F}$ would not be valid, then for some $\mathcal{H}$-matching x we would have $x(\mathcal{F}) > ||\mathcal{F}||$. Then by (44) x must be a perfect $\mathcal{H}$-matching in $G(\mathcal{F})$. This implies $M'(\mathcal{F}) = M(\mathcal{F}) > ||\mathcal{F}||$, a contradiction. □

The correctness of the following algorithm follows from these lemmas. By Proposition 1, this algorithm is in fact a separation algorithm for $\mathcal{C}_{\mathcal{L}}$.

Algorithm 1 *Checking validity and finding $M(\mathcal{F})$.*

1. *Determine $M'(\mathcal{F}_F)$ for each set $F \in \mathcal{F}$.*
2. *$\mathcal{F}$ is valid if and only if $M'(\mathcal{F}_F) \leq ||\mathcal{F}_F||$ holds for each $F \in \mathcal{F}$.*
3. *If $\mathcal{F}$ is valid, then $M(\mathcal{F}) = M'(\mathcal{F})$.*

7 Proof of Lemma 8

To prove 8 we will use the following observations on the matching polytope.

Proposition 10 *Suppose $G = (V, E)$ is a graph with a fixed node $t \in V$ and $|V|$ is even. Then the following system is a description of the perfect matching polytope for G.*

$$\begin{aligned} x &\geq 0 \\ x(d(v)) &= 1 \quad \text{for each node } v \in V \\ x(d(U)) &\geq 1 \quad \text{for each set } U \subseteq V \text{ with } G[U] \text{ factor-critical and } t \notin U \end{aligned} \tag{45}$$

Proof (A refinement of the proof in [11]). Suppose the polytope Q determined by (45) has a vertex x which is not in the convex hull of perfect matchings of G. We choose a counterexample with $|E| + |V|$ minimal. Hence $0 < x_e < 1$ for each $e \in E$, each degree in G is at least 2. If $|E| = |V|$, then G is an even cycle, thus no counterexample. So we must have $|E| > |V|$. As x is a vertex of Q, there are $|E|$ linearly independent constraints in (45) satisfied with equality. Since $|E| > |V|$, there is a set $U \subseteq V$, $|U| \geq 3$ with $G[U]$ factor-critical and $t \notin U$ for which equality $x(d(U)) = 1$ holds.

Consider the projections x' and x'' of x to the edges of the graphs G/U and $G/(V - U)$. It is easy to see, that these vectors satisfy (45) for $t' := t$ and $t'' := \{V - U\}$, respectively. By the minimality of $|E|+|V|$, x' and x'' must be in the perfect matching polytope for G/U and $G/(V - U)$. As in [11] one can show that x must also be in the perfect matching polytope of G, a contradiction. □

Using routine transformations and an uncrossing technique in [11] one can also prove the following extension of Proposition 10. A different proof of Proposition 11 can be given by observing that in a dual-changing algorithm for maximum weight perfect matching we can keep a dual solution which has no blossom containing t.

Proposition 11 *Suppose $G = (V, E)$ is a graph with a fixed node $t \in V$ and $|V|$ is even. Then*

$$\begin{aligned} x &\geq 0 \\ x(d(v)) &= 1 \quad \textit{for } v \in V \\ x(E[U]) &\leq (|U| - 1)/2 \quad \textit{for } U \subseteq V, G[U] \textit{ factor-critical and } t \notin U. \end{aligned} \tag{46}$$

is a description of the perfect matching polytope for G and for any vector c there is an optimum dual solution y_v, y_U of (46) for which $\{E[U] : y_U > 0\}$ is a nice family.

Let $G = (V, E)$ be an arbitrary graph with real weights c_e on the edges $e \in E$. Fix a node $s \in V$. Suppose x is twice the characteristic vector of a perfect matching M in $G - s$ of maximum weight $c \cdot x$. We call an odd cycle $Q \subseteq E$ *alternating* if $s \in V(Q)$ and $x(Q) = |V(Q)| - 1$.

Proposition 12 *Using the above notations, either*

(a) *there is an alternating odd cycle Q for which $c(Q) \geq x(Q)$, or*
(b) *there is a vector $y \in \mathbb{R}^V$ and a nice system $\mathcal{F}$ for which $s \notin V(\mathcal{F})$ and $y_s < 0$ and $c_{uv} \leq \frac{1}{2} y_u + \frac{1}{2} y_v + \chi_{\mathcal{F}}(uv)$ for each edge $uv \in E$ and $c \cdot x = \sum\{y_v : v \in V - s\} + ||\mathcal{F}||$.*

Proof Suppose c is integer. We construct an auxiliary graph $G' = (V + t, E')$ where $E' := E \cup \{tv$: for each edge $sv \in E\} + st$ with weight-function $c'_{uv} := c_{uv}$ and $c'_{tv} := c_{sv}$ if $u, v \in V - s$ and $c'_{st} := -1$. Let x' be twice the characteristic vector of the perfect matching $M + st$ in G', its weight is $c \cdot x - 2$. If the maximum weight of twice a perfect matching in G' is at least $c \cdot x$ then it is attained by some x' for which $x'(st) = 0$. Thus there is an alternating even cycle for x and x' which contains st, when we delete st from this cycle and identify t with s we get the alternating cycle Q as required in (a).

Suppose the maximum weight of twice a perfect matching in G' is $c \cdot x - 2$. Using Proposition 11 with the fixed node t we get a vector $y' \in \mathbb{R}^{V+t}$ and a nice system $\mathcal{F}' = (\mathcal{L}', m')$ for which $t \notin V(\mathcal{F}')$ and $c'_{uv} \leq \frac{1}{2} y'_u + \frac{1}{2} y'_v + \chi_{\mathcal{F}'}(uv)$ for each edge $uv \in E'$ and $c \cdot x - 2 = \sum\{y''_v : v \in V + t\} + ||\mathcal{F}'||$. Let $\mathcal{F}$ be the truncation of $\mathcal{F}'$ to the sets not containing s and let

$$\begin{aligned} y''_s = y''_t &:= \frac{1}{2}(y'_s + y'_t) \\ y''_v &:= y'_v + \sum\{m'(F) : F \in \mathcal{L}' \text{ and } s \in V(F')\} \quad \text{for } v \in V - s. \end{aligned}$$

For $y'', \mathcal{F}$ we also have $c'_{uv} \leq \frac{1}{2} y''_u + \frac{1}{2} y''_v + \chi_{\mathcal{F}}(uv)$ for $uv \in E'$ and $c \cdot x - 2 = \sum\{y''_v : v \in V + t\} + ||\mathcal{F}||$. Now $s, t \notin V(\mathcal{F})$ and since $M + st$ is optimal, by slackness condition we get that $\frac{1}{2}y''_s + \frac{1}{2}y''_t = -1$, thus $y''_s = y''_t = -1$. We have $c \cdot x = \sum\{y''_v : v \in V - s\} + ||\mathcal{F}||$, let us construct y as follows. $y_v := y''_v$ for each node v in $V - s$ and $y_s := -1$, then y and $\mathcal{F}$ is as required in (b). □

Proof of Lemma 8 Consider a valid system $\mathcal{F}$ with one component, suppose x is a perfect $\mathcal{H}$-matching in $G = (V, E) := (V(\mathcal{F}), E(\mathcal{F}))$, and $x(\mathcal{F}) = M(\mathcal{F})$, i.e. x has maximum weight for weight-function $c := \chi_{\mathcal{F}}$. We choose x which has a number k of odd cycles, namely $C_1, \ldots, C_k$. We suppose $\mathcal{H} = \{C_1, \ldots, C_k\}$, it will be easy to see that this does not affect any part of the proof, since in what follows we show that there is a maximum vector x using at most one of these cycles.

Suppose for contradiction that $k \geq 3$ and there is no maximizing x with less number of odd cycles. Consider $G^{C_i} = G[V(C_i)] = (V(C_i), E^{C_i})$ for some $i \leq k$, let x^{C_i}, c^{C_i} be the truncation to the edges $E^{C_i} = E[V(C_i)]$. Then x^{C_i} must be a maximizing perfect $\mathcal{H}$-matching in G_{C_i} with weight c^{C_i}. Thus, by applying the Theorem, there must be a vector $y^{C_i} \in \mathbb{R}^{V(C_i)}$ and a valid system $\mathcal{F}^{C_i} = (\mathcal{L}^{C_i}, m^{C_i})$ in G^{C_i} for which the following hold.

$$\frac{1}{2}y_u^{C_i} + \frac{1}{2}y_v^{C_i} + \sum_{uv \in F \in \mathcal{L}^{C_i}} m^{C_i}(F) \geq c_{uv} \quad \text{for } uv \in E^{C_i} \tag{47}$$

$$\frac{1}{2}y_u^{C_i} + \frac{1}{2}y_v^{C_i} + \sum_{uv \in F \in \mathcal{L}^{C_i}} m^{C_i}(F) = c_{uv} \quad \text{for } uv \in E(\mathcal{L}^{C_i}) \tag{48}$$

$$\sum_{v \in V(C_i)} y_v^{C_i} + ||\mathcal{F}^{C_i}|| = c^{C_i} \cdot x^{C_i}. \tag{49}$$

A sub-partition V^i of V will be called *special*, if there are vectors $y^i \in \mathbb{R}^{V^i}$ and valid systems $\mathcal{F}^i = (\mathcal{L}^i, m^i)$ in $G^i = G[V^i] = (V^i, E^i)$ for which $V(C_i) \subseteq V^i$ and the following hold.

$$\frac{1}{2}y_u^i + \frac{1}{2}y_v^i + \sum_{uv \in F \in \mathcal{L}^i} m(F) \geq c_{uv} \quad \text{for } uv \in E^i - E(\mathcal{L}^i) \tag{50}$$

$$\frac{1}{2}y_u^i + \frac{1}{2}y_v^i + \sum_{uv \in F \in \mathcal{L}^i} m^i(F) = c_{uv} \quad \text{for } uv \in E(\mathcal{L}^i) \tag{51}$$

$$\sum_{v \in V^i} y_v^i + ||\mathcal{F}^i|| = c^i \cdot x^i \tag{52}$$

(here c^i, x^i denote the truncation to E^i) and furthermore, $\mathcal{F}^i$ has one component with maximal set F^i for which $V(F^i) = V^i$, and x^i is a perfect $\mathcal{H}$-matching in V^i.

There is at least one special sub-partition, namely $V^i = V(C_i)$, consider a special sub-partition V^i of V for which $\cup V^i$ is maximal. Let $\mathcal{F}^*, y^*$ be the merging of $\mathcal{F}^i, y^i$. We construct an auxiliary graph $G' := (V', E')$ by $V' := V - \cup V^i + s$ as the new node set and let us construct the new edge set by identifying the nodes in $\cup V^i$ by s:

$$E' := E[V - \cup V^i] \cup \{sv: \text{ the copy of an edge } uv \text{ with } u \in \cup V^i, v \notin \cup V^i\}$$
$$\cup\{e: \text{ a loop on } s \text{ as the copy of } uv \text{ with } u \in V^i, v \in V^j \text{ for some } i \neq j\}.$$

The auxiliary weight function is defined as follows:

$$c'_{uv} := c_{uv} \quad \text{if } uv \in E[V - \cup V^i], u \in \cup V^i, v \notin \cup V^i$$
$$c'_{sv} := c_{uv} - \frac{1}{2}y^*_u \quad \text{if } sv \in E' \text{ is the copy of } uv \in E$$
$$c'_e := c_{uv} - \frac{1}{2}y^*_u - \frac{1}{2}y^*_u \quad \text{if } e \in E' \text{ is the copy of an edge } uv \text{ with}$$
$$u \in V^i, v \in V^j \text{ for some } i \neq j.$$

Let x' be the truncation of x to E', then x' is twice the characteristic vector of a perfect matching M in $G[V - \cup V^i] = G' - s$. Apply Proposition 12 to graph G' with s, x' and c'.

First we will prove, that case (b) leads to a contradiction. Suppose there is a vector $y' \in \mathbb{R}^{V'}$ and a nice system $\mathcal{F}'$ for which $s \notin V(\mathcal{F}')$ and $y'_s < 0$ and $c'(uv) \leq \frac{1}{2}y'_u + \frac{1}{2}y'_v + \chi_{\mathcal{F}'}(uv)$ for each edge uv and $c' \cdot x' = \sum y'_v + ||\mathcal{F}'||$. Merging together y^*, y' and deleting the entry of s we get y^0. Merging together $\mathcal{F}^*$, $\mathcal{F}'$ we get $\mathcal{F}^0 = (\mathcal{L}^0, m^0)$ which is clearly valid. It is easy to see, that

$$\frac{1}{2}y^0_u + \frac{1}{2}y^0_v + \sum_{uv \in F \in \mathcal{L}^0} m^0(F) \geq c_{uv} \quad \text{for } uv \in E - E(\mathcal{L}^0) \tag{53}$$

$$\frac{1}{2}y^0_u + \frac{1}{2}y^0_v + \sum_{uv \in F \in \mathcal{L}^0} m^0(F) = c_{uv} \quad \text{for } uv \in E(\mathcal{L}^0) \tag{54}$$

$$\sum_{v \in V} y^0_v + ||\mathcal{F}^0|| = c \cdot x. \tag{55}$$

Furthermore, since the entry of s was negative we get that each edge uv with $u \in V^i$ and $v \notin V^i$ (for some i) must not be tight in (53). This means that if for each i and each edge $uv \in E$ with $u \in V^i$ and $v \notin V^i$, then $\frac{1}{2}y^0_u + \frac{1}{2}y^0_v + \sum_{uv \in F \in \mathcal{L}^0} m^0(F) > c_{uv}$ holds. Since $k \geq 3$, there must be an $ab \in E$ edge that is not in $E(\mathcal{L}^0)$. By $0 \leq c_{ab} \leq \frac{1}{2}y^0_a + \frac{1}{2}y^0_b$ we get that at least one of y^0_a and y^0_b must be non-negative, suppose $y^0_a \geq 0$. Consider a characteristic vector z of twice a maximum weight perfect matching in $G - a$, clearly it has weight $c \cdot z = ||\mathcal{F}|| \geq c \cdot x$. On the other hand, $c \cdot z \leq \sum_{v \in V - a} y^0_v + ||\mathcal{F}^0|| = c \cdot x - y^0_a \leq c \cdot x$. By slackness conditions z can only be positive on tight edges, thus z would have to leave at least one node exposed in each V^i. This is a contradiction.

Suppose we have case (a) for the instance G', s, x', c' of the Proposition 12. Let Q be the edge set in G corresponding the M-alternating cycle Q'. Then Q is either an M-alternating odd ear on some V^i, or Q is an M-alternating odd path between V^i and V^j for some $i \neq j$.

In the first case we prove that if we replace V_i by $V^Q := V_i \cup V(Q)$ we get a special sub-partition, which contradicts the choice of the sub-partition. We only have to show the existence of a vector y^Q and a valid system $\mathcal{F}^Q = (\mathcal{L}^Q, m^Q)$ in $G^Q := G[V^Q] = (V^Q, E^Q)$ which is a solution of

$$\frac{1}{2}y_u^Q + \frac{1}{2}y_v^Q + \sum_{uv \in F \in \mathcal{L}^Q} m^Q(F) \geq c_{uv} \quad \text{for } uv \in E^Q - E(\mathcal{L}^Q) \tag{56}$$

$$\frac{1}{2}y_u^Q + \frac{1}{2}y_v^Q + \sum_{uv \in F \in \mathcal{L}^Q} m^Q(F) = c_{uv} \quad \text{for } uv \in E(\mathcal{L}^Q) \tag{57}$$

$$\sum_{v \in V^Q} y_v^Q + ||\mathcal{F}^Q|| = c^Q \cdot x^Q \tag{58}$$

and furthermore, $\mathcal{F}^Q$ must have one component with maximal set F^Q for which $V(F^Q) = V^Q$. Since x^Q is a maximum perfect $\mathcal{H}$-matching in G^Q, by the Theorem there must be a vector y^Q and a valid system $\mathcal{F}^Q$ in $G^Q := G[V^Q] = (V^Q, E^Q)$ which is a solution of (56)–(58). Consider the graph $\overline{G^Q}$ which we get from the tight edges and the contraction of the maximal sets in $\mathcal{L}^Q$. In case $\overline{G^Q}$ is factor-critical, then we add the set E^Q to $\mathcal{F}^Q$ with multiplicity zero to get a valid system as required.

Suppose for contradiction that $\overline{G^Q}$ is not factor-critical. Let $\overline{D}, \overline{A}, \overline{C}$ be the Gallai–Edmonds decomposition of $\overline{G^Q}$, let D, A, C be the sets corresponding to them in V. Since $D \neq V$ and x^Q has only one odd cycle, there must be an edge uv with $x^Q(uv) = 2$, $uv \notin E(\mathcal{L}^Q)$ and $v \notin D$. Since x^Q is maximum, edge uv must be tight, that is

$$\frac{1}{2}y_u^Q + \frac{1}{2}y_v^Q = \frac{1}{2}y_u^Q + \frac{1}{2}y_v^Q + \sum_{uv \in F \in \mathcal{L}^Q} m^Q(F) = c_{uv}. \tag{59}$$

Consider the maximum weight "twice a perfect matching" problem in the graphs $G^Q - u$ and $G^Q - v$, with maximum values w_u, w_v. It is easy to see

$$w_u \leq \sum_{z \in V^Q - u} y_z^Q + ||\mathcal{F}^Q|| = c^Q \cdot x^Q - y_u^Q. \tag{60}$$

Since $v \notin D$ we get that

$$w_v < \sum_{z \in V^Q - v} y_z^Q + ||\mathcal{F}^Q|| = c^Q \cdot x^Q - y_v^Q. \tag{61}$$

By (59), (60) and (61) we get

$$w_u + w_v < 2c^Q \cdot x^Q - 2c_{uv}. \tag{62}$$

If $uv \in E^i$, then we give a bound on w_u, w_v using $y^i, \mathcal{F}^i$ which satisfies (50)–(52). Let w'_u, w'_v denote the maximum weight of "twice a perfect matching" in the graphs $G^i - u$ and $G^i - v$. Let $M \cap Q$ denote the edges of x_Q on the ear Q. Clearly $w'_u + 2c(M \cap Q) \le w_u$ and $w'_v + 2c(M \cap Q) \le w_v$.

$$w'_u = ||\mathcal{F}^i|| + \sum_{z \in V^i - u} y^i_z = c^i \cdot x^i - y^i_u = c^Q \cdot x^Q - 2c(M \cap Q) - y^i_u,$$
$$w'_v = ||\mathcal{F}^i|| + \sum_{z \in V^i - v} y^i_z = c^i \cdot x^i - y^i_v = c^Q \cdot x^Q - 2c(M \cap Q) - y^i_v.$$

Then by (62) we get

$$\frac{1}{2} y^i_u + \frac{1}{2} y^i_v > c_{uv}$$

which is a contradiction, since $m^i \ge 0$ and the edge uv must be tight for $y^i, \mathcal{F}^i$.

If $uv \notin E^i$, then $uv \in M \cap Q$ is an edge on the ear Q. We will use a similar argument as in the last paragraph to get to a contradiction. Suppose the ear

$$Q = \{q_l r_l : l = 0, \ldots, t\} \cup \{r_l q_{l+1} : l = 0, \ldots, t-1\}$$

is connecting $q = q_0 \in V^i$ and $r = r_t \in V^i$. Then $V(Q) = \{q_l : l = 0, \ldots, t\} \cup \{r_l : l = 0, \ldots, t\}$ and $M \cap Q = \{r_l q_{l+1} : l = 0, \ldots, t-1\}$.

Suppose $u = r_j$ and $v = q_{j+1}$. By part (a) in Proposition 12 we know that

$$\frac{1}{2} y^i_q + \frac{1}{2} y^i_r \le c(\{q_l r_l : l = 0, \ldots, t\}) - c(\{r_l q_{l+1} : l = 0, \ldots, t-1\}). \tag{63}$$

Since the edges in $E(\mathcal{L}^i)$ are tight, it is easy to see

$$\begin{aligned} w_u &\ge ||\mathcal{F}^i|| + \sum_{z \in V^i - r} y^i_z + 2c(\{r_l q_{l+1} : l = 0, \ldots, j-1\}) \\ &\quad + 2c(\{q_l r_l : l = j+1, \ldots, t\}) \\ &\ge c^i \cdot x^i - y^i_r + 2c(\{r_l q_{l+1} : l = 0, \ldots, j-1\}) \\ &\quad + 2c(\{q_l r_l : l = j+1, \ldots, t\}) \end{aligned} \tag{64}$$

and similarly we get

$$\begin{aligned} w_v &\ge ||\mathcal{F}^i|| + \sum_{z \in V^i - q} y^i_z + 2c(\{r_l q_{l+1} : l = j+1, \ldots, t\}) \\ &\quad + 2c(\{q_l r_l : l = 0, \ldots, j\}) \\ &\ge c^i \cdot x^i - y^i_q + 2c(\{r_l q_{l+1} : l = j+1, \ldots, t\}) \\ &\quad + 2c(\{q_l r_l : l = 0, \ldots, j\}) \end{aligned} \tag{65}$$

hence by (63), (64) and (65)

$$\begin{aligned} w_u + w_v &\geq 2c^i \cdot x^i - y_r^i - y_q^i + 2c(\{q_l r_l : l = 0, \ldots, t\}) \\ &\quad +2c(\{r_l q_{l+1} : l = 0, \ldots, t-1\}) - 2c(q_j r_{j+1}) \\ &= 2c^i \cdot x^i - y_r^i - y_q^i + 4c(M \cap Q) + 2c(\{q_l r_l : l = 0, \ldots, t\}) \\ &\quad -2c(\{r_l q_{l+1} : l = 0, \ldots, t-1\}) - 2c_{uv} \\ &\geq 2c^i \cdot x^i + 4c(M \cap Q) - 2c_{uv} = 2c^Q \cdot x^Q - 2c_{uv} \end{aligned} \tag{66}$$

which is a contradiction with (62).

Now we get to the case when Q is an M-alternating odd path between $r \in V^i$ and $q \in V^j$ for some $i \neq j$. By part (a) in Proposition 12

$$\frac{1}{2} y_q^i + \frac{1}{2} y_r^i \leq c(\{q_l r_l : l = 0, \ldots, t\}) - c(\{r_l q_{l+1} : l = 0, \ldots, t-1\}). \tag{67}$$

Then we construct x' by changing the entries in $E^i \cup E^j \cup Q$ as follows. We take twice a perfect matching in $G^i - q$ and $G^j - r$, and we add twice the matching $Q - M$. Since the edges in $E(\mathcal{L}^i) \cup E(\mathcal{L}^j)$ are tight and for the maximal sets $F^{i,j}$ we have $V^i = V(F^i)$, $V^j = V(F^j)$, we can choose the perfect matchings in $V_i - q$ and $V_j - r$ to have value $||\mathcal{F}^i|| + \sum\{y_z^i : z \in V_i - r\}$ and $||\mathcal{F}^j|| + \sum\{y_z^j : z \in V_j - q\}$. By (52) these values are equal to $c^i \cdot x^i - y_r^i$ and $c^j \cdot x^j - y_q^j$. Thus the change of value on the edges in $E^i \cup E^j$ is at least $-y_r^i - y_q^j$. By (67) the change in the value on the edges of Q is at least $y_r^i + y_q^j$. This means, x' is an $\mathcal{H}$-factor having less number of odd cycles and $c \cdot x \leq c \cdot x'$. □

Acknowledgments The author thanks Tamás Király and Márton Makai for helpful discussions on the topic. The author is grateful to a referee for a suggestion on the organization of the paper.

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