CHESSBOARD COMPLEXES AND MATCHING COMPLEXES

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0. Introduction

The objective of this paper is to study the geometry of admissible rook configurations on a general $m \times n$ chessboard. An admissible configuration is any non-taking placement of rooks, i.e., a placement which does not allow any two of them to be in the same row or in the same column. The collection of all these placements forms a simplicial complex, called the chessboard complex, which will be denoted by $\Delta_{m,n}$.

Chessboard complexes appeared for the first time in the thesis of P. Garst [7], where they were obtained as special coset complexes of the symmetric group S_n. Recall (see e.g. [3], §2), that for every finite group G and collection $\mathcal{H} = \{H_1, \dots, H_m\}$ of subgroups, one defines the corresponding coset complex $\Delta(G, \mathcal{H})$ as the nerve of the family of cosets $\mathcal{F} = \{gH_i : g \in G, 1 \leq i \leq m\}$. Many important and natural complexes arise in this way, notable examples are Coxeter complexes and Tits buildings. Garst [7] obtained $\Delta_{m,n}, m \leq n$, as the coset complex $\Delta(S_n, \mathcal{H}_m)$, where S_n is the symmetric group and $\mathcal{H}_m = \{Stab(i) : 1 \leq i \leq m\}$. He showed that $\Delta_{m,n}$ is Cohen-Macaulay if and only if $2m \leq n + 1$.

The chessboard complex ${}_{m,n}$ seems to us a very natural object, and it is amusing to see how it arises in situations and constructions apparently unrelated to its definition or group-theoretical characterization.

A graph theorist might prefer to view $\Delta_{m,n}$ as the complex of all partial matchings in the complete bipartite graph $K_{m,n}$. In Section 4 we adopt this point of view and study also the matching complexes of the complete hypergraphs.

The complex $\Delta_{m,n}$ was obtained in [11] as the deleted join $[n]^{(m)}_{\Delta}$ and described as the complex $\mathcal{P}_{m,n}$ of all injective, partial, nonempty functions $f \subseteq [m] \times [n]$, $[m] = \{1, 2, \dots, m\}$. The complex $\mathcal{P}_{m,n}$ appeared in [11] in connection with some combinatorial geometric problems, namely the so called "Colored Tverberg's problem" and the problem of estimating the number of halving hyperplanes of a finite set of points in ^{d}. Further applications of the Colored Tverberg's theorem can be found in [1]. A key technical fact was, see [11, Theorem 3], that $\mathcal{P}_{m,2m-k} \cong \Delta_{m,2m-k}$ is $(m-k-1)$-connected for all $1 \leq k < m$, and that $\mathcal{P}_{m,n}$ is $(m-2)$-connected for $n \geq 2m-1$. The last result can be seen as a corollary of the result of Garst mentioned above and the fact that $\pi_1(\Delta_{m,n}) = 0$ if $m \geq 3, n \geq 5$.

These examples show that aside from being interesting objects on their own, chessboard complexes, and in particular their connectivity properties, turn out to be important from the point of view of certain applications. They, together with their higher-dimensional analogues and the closely related matching complexes of graphs and hypergraphs, are possible candidates for configuration spaces in various Combinatorial Geometric problems.

Here is a brief description of the content of this paper. Section 1 is devoted to the proof that every chessboard complex $\Delta_{m,n}$ is at least $(\nu-2)$-connected, where $\nu=\min\{m,n,[\frac{m+n+1}{3}]\}$. In Section 2 a detailed analysis of some small chessboard complexes is given, showing in particular that the bound found in the previous section cannot in general be improved. The section ends with a brief discussion of manifolds which arise in connection with "almost square" chessboard complexes $\Delta_{n,n+1}$. The results established in the first two sections permit us to make some observations concerning the depth of the associated Stanley-Reisner rings. For example, it is shown in Section 2 that the depth of the Stanley-Reisner ring $\mathbf{k}[\Delta_{m,n}]$ in general depends on the characteristic of the field $\mathbf{k}$. Higher-dimensional chessboard complexes $\Delta_{n_1,\dots,n_k}$, and the class of matching complexes of complete k-hypergraphs, are introduced and studied in Sections 3 and 4. The main objective of study are their connectivity properties, and nontrivial lower bounds are established, containing the result about chessboard complexes ${}_{m,n}$ as a special case.

Convention: We use the "matrix" notation for denoting $(m\times n)$-chessboards and the corresponding complexes $\Delta_{m,n}$, see Figure 1. So, we tacitly assume throughout the paper, that if (i,j) are coordinates of an elementary square $a_{i,j}$ in a chessboard $[m]\times[n]$, then i denotes the row and j the column of $a_{i,j}$.

Figure 1.

1. Connectivity and depth of chessboard complexes

The following theorem gives a lower bound for the connectivity of a chessboard complex.

Theorem 1.1. *The chessboard complex $\Delta_{m,n}$ is $(\nu-2)$-connected, where*

$$\nu = min\left\{m,n,\left[\frac{m+n+1}{3}\right]\right\}.$$

The proof of Theorem 1.1 relies on the following lemma, which is a slightly sharper version of [5, Theorem 4.10]. We will give an elementary proof, patterned on that used in [5]. The lemma can also be proved by a simple spectral sequence argument combined with the usual use of the Seifert-Van Kampen and Hurewicz theorems.

Lemma 1.2 (Nerve lemma). *Let Δ be a simplicial complex and $\{L_i\}_{i=1}^n$ a family of subcomplexes such that $\Delta = \cup_{i=1}^n L_i$. Suppose that every nonempty intersection $L_{i_1} \cap L_{i_2} \cap \ldots \cap L_{i_t}$ is $(k-t+1)$-connected for $t \geq 1$. Then Δ is k-connected if and only if $\mathcal{N}(\{L_i\}_{i=1}^n)$, the nerve of the covering $\{L_i\}_{i=1}^n$, is k-connected.*

Proof. Our argument is based on the following two facts (here Avert ()):

(i) *If $\cap 2^A$ is $(k-1)$-connected and is k-connected, then $\cup 2^A$ is k-connected.*

(ii) *If $\cap 2^A$ and $\cup 2^A$ are k-connected, then is k-connected.*

These facts are easily deduced from the Mayer-Vietoris exact sequence and the Seifert-Van Kampen and Hurewicz theorems. For a completely elementary proof, see Lemmas 4.8 and 4.9 of [5].

We may without loss of generality assume that the subcomplexes L_i are induced, i.e., that $L_i = \cap 2^{A_i}$ where $A_i = \text{vert}\ (L_i)$. This can always be achieved by passing to the barycentric subdivision. Then satisifes the following two conditions:

(iii) $2^{A_1} \cup \ldots \cup 2^{A_n}$,

(iv) *if $A_{i_1} \cap \ldots \cap A_{i_t} \neq \emptyset$ then $\cap 2^{A_{i_1} \cap \ldots \cap A_{i_t}}$ is $(k-t+1)$-connected, for $t \geq 1$.*

It follows from (*i*) and (*ii*) that is k-connected if and only if $^* = \cup 2^{A_1}$ is k-connected. Furthermore, * also satisfies conditions (*iii*) and (*iv*). Let us check (*iv*), and for this assume that $B = A_{i_1} \cap \ldots \cap A_{i_t} \neq \emptyset$. We have

(v) $^* \cap 2^B = \cap 2^B \cup 2^{A_1 \cap B}$.

If $A_1 \cap B = \emptyset$, then $^* \cap 2^B = \cap 2^B$ is $(k-t+1)$-connected by (*iv*). If $A_1 \cap B \neq \emptyset$, then since $\cap 2^B$ is $(k-t+1)$-connected and $\cap 2^B \cap 2^{A_1 \cap B} = \cap 2^{A_1 \cap B}$ is $(k-t)$-connected by (*iv*), we conclude from (*i*) and (*v*) that $^* \cap 2^B$ is $(k-t+1)$-connected.

The same argument can now be repeated, showing that * is k-connected if and only if $^{**} =^* \cup 2^{A_2}$ is k-connected, and that ** satisfies conditions (*iii*) and (*iv*), and so on. In the end this leads to the conclusion that is k-connected if and only if the complex $' = 2^{A_1} \cup 2^{A_2} \cup \ldots \cup 2^{A_n}$ is k-connected. But the ordinary nerve lemma (see [4,(10.6)] or [5, Theorem 4.3]) shows that $'$ is homotopy equivalent to the nerve $\{A_i\}_{i=1}^n$, which is the same as the nerve $\{L_i\}_{i=1}^n$.

Proof of Theorem 1.1. Let us assume that $m \leq n$. The proof will be carried out by induction on m, starting at $m = 2$. The chessboard complex $\Delta_{2,n}$ is clearly connected if $n > 2$ and otherwise non-empty.

We will cover the complex $\Delta_{m,n}$ by the following family of subcomplexes. Let L_i, $i = 1, ..., n$, be the collection of all rook placements on a $m \times n$ chessboard which either contain the square $(1, i)$ or else can be legally extended to contain that square.

Every proper subfamily of $\{L_i\}_{i=1}^n$ has nonempty intersection, so the nerve of the covering $\{L_i\}_{i=1}^n$ is the boundary of an $(n-1)$-simplex, i.e. topologically an $(n-2)$-sphere. Hence the nerve is $(n-3)$-connected, and certainly $n - 3 \geq \nu - 2$.

For the induction step we must verify that $L_{i_1} \cap \ldots \cap L_{i_t}$ is $(\nu - t - 1)$-connected. For $t = 1$ this follows from the observation that each L_i is a cone, and hence contractible. For $t \geq 2$, any intersection $L_{i_1} \cap \ldots \cap L_{i_t}$ is again a chessboard complex $\Delta_{m-1,n-t}$ which is, by the induction hypothesis, $(\mu - 2)$-connected, where $\mu = \min\{m-1, n-t, \left[\frac{m+n-t}{3}\right]\}$. It is now easy to check that $\mu - 2 \geq \nu - t - 1$.

A simplicial complex Δ is called *homotopy-Cohen-Macaulay* if $\operatorname{link}_\Delta(\sigma) = \{\tau \in \Delta \mid \tau \cup \sigma \in \Delta, \tau \cap \sigma = \emptyset\}$ is $(\dim \operatorname{link}_\Delta(\sigma) - 1)$-connected for all $\sigma \in \Delta \cup \{\emptyset\}$. For background and references concerning this and related notions of topological Cohen-Macaulayness, see [4] and [10]. The following is implied by Theorem 1.1.

Corollary 1.3. *The $(\nu-1)$-skeleton of the chessboard complex $\Delta_{m,n}$ is homotopy-Cohen-Macaulay.*

Proof. If $\sigma \in \Delta_{m,n}^{(\nu-1)}$ and $|\sigma| = s$, then $\operatorname{link}_{\Delta_{m,n}^{(\nu-1)}}(\sigma) \cong \Delta_{m-s,n-s}^{(\nu-s-1)}$. Now $\Delta_{m-s,n-s}$ is $(\mu - 2)$-connected with $\mu = \min\{m-s, n-s, \left[\frac{m+n+1-2s}{3}\right]\} \geq \nu - s$, so its $(\nu - s - 1)$-skeleton is certainly $(\nu - s - 2)$-connected.

The Stanley-Reisner ring $\mathbf{k}[\Delta_{m,n}]$ of $\Delta_{m,n}$, $\mathbf{k}$ a field, can be described as follows. Take the polynomial ring $\mathbf{k}[x_{ij}]$ over an $m \times n$ matrix of indeterminates x_{ij}, $1 \leq i \leq m$, $1 \leq j \leq n$, and let I be the ideal generated by relations $x_{ab}x_{cd} = 0$ whenever exactly one of $a = c$ or $b = d$ hold. Then, $\mathbf{k}[\Delta_{m,n}] = \mathbf{k}[x_{ij}]/I$. See [10] for the basic theory of Stanley-Reisner rings $\mathbf{k}[\Delta]$.

Corollary 1.4. *depth*$(\mathbf{k}[\Delta_{m,n}]) \geq \nu$, *for every field* $\mathbf{k}$.

Proof. A result of Smith [9] (also proved in [6]) characterizes the depth of a Stanley-Reisner ring $\mathbf{k}[\Delta]$ in terms of the skeleta $\Delta^{(j)}$ as follows:

$$\operatorname{depth}(\mathbf{k}[\Delta]) = 1 + \max\{j \mid \Delta^{(j)} \text{ is Cohen-Macaulay over } \mathbf{k}\}.$$

Hence, the result follows from Corollary 1.3.

We will show in the next section that $\mathrm{depth}(\mathbf{k}[\Delta_{m,n}])$ is sensitive to the characteristic of $\mathbf{k}$ and hence not a combinatorial invariant. However, we believe that the results of this section are sharp in the following sense.

Conjecture 1.5. *For all m, n :*

(i) The chessboard complex $\Delta_{m,n}$ is not $(\nu - 1)$-connected,

(ii) depth$(\mathbf{k}[\Delta_{m,n}]) = \nu$, for some field $\mathbf{k}$.

The second part of the conjecture can be shown to be equivalent to the statement:

(iii) $\Delta_{m,n}^{(\nu)}$ is not homotopy-Cohen-Macaulay.

For this one must use Smith's characterization of depth [9] together with the 1-connec-tivity of links that can be derived from Theorem 1.1. Hence, part (i) of the conjecture implies part (ii).

We have verified Conjecture 1.5 for all $m \le n$ with $m \le 5$, with three exceptions: $_{4,6}, \Delta_{5,7}$ and $\Delta_{5,8}$. See the next section for the details. Also, both parts of the conjecture are true when $n \ge 2m - 1$. In that case $\nu = m$, $\Delta_{m,n}$ is $(m-2)$-connected, and it is easy to see that its Euler characteristic is not equal to 1, so $\Delta_{m,n}$ cannot be $(m-1)$-connected. Also, since $\mathrm{depth}(\mathbf{k}[\Delta_{m,n}]) \le \dim(\mathbf{k}[\Delta_{m,n}]) = \dim(\Delta_{m,n}) + 1 = m$, Corollary 1.4 shows that $\mathrm{depth}(\mathbf{k}[\Delta_{m,n}]) = m$ for all fields $\mathbf{k}$. This statement about depth is equivalent to the Cohen-Macaulayness of $_{m,n}$ when $n \ge 2m - 1$, which was proved by Garst [7].

A smaller chessboard can be embedded in a larger one in many ways. The simplest are the embeddings in one of the corners of the larger chessboard. Each of these embeddings defines in a natural way an embedding of the corresponding chessboard complexes. If $A \subset [m]$ and $B \subset [n]$, where $[m] = \{1, \dots, m\}$, then the complex $\Delta_{A,B}(m,n) = \Delta_{A,B}$ is defined as the subcomplex of $\Delta_{m,n}$ which consists of all simplices contained in $A \times B$. Obviously, if $A = k$ and $B = \ell$ then $\Delta_{A,B} \cong \Delta_{k,\ell}$. Since $\Delta_{A,B} * \Delta_{A^c,B^c} \subset \Delta_{m,n}$ we observe that also joins of chessboard complexes can be naturally embedded in bigger chessboard complexes. Since $\Delta_{1,2}, \Delta_{2,1}, \Delta_{2,3}, \Delta_{3,2}$ are spheres, we obtain by this construction many useful embeddings of spheres in chessboard complexes. The spheres obtained by this construction together with the complexes of the form $\Delta_{m,m+1}$, seem to be natural candidates for subcomplexes which support nontrivial homology classes in $\Delta_{m,n}$. In light of Theorem 1.1, where the mod 3 congruence class of $m+n+1$ played an important role, we select the following two special cases. Let $\Sigma_k^1 = \Delta_{2,3} * (\Delta_{2,1} * \Delta_{1,2})^{*(k)}$ be a $(2k+1)$-dimensional sphere embedded in $\Delta_{3k+2,3k+3}$, and $\Sigma_k^2 = \Delta_{2,3} * (\Delta_{2,1} * \Delta_{1,2})^{*(k-1)} * \Delta_{2,1}$ a $(2k)$-dimensional sphere embedded in $\Delta_{3k+1,3k+1}$. Since $\Delta_{3k+2,3k+3}$ is $(2k)$-connected and $\Delta_{3k+1,3k+1}$ is $(2k-1)$-connected (Theorem 1.1), one is led to the following conjecture, which also suggests an approach to Conjecture 1.5.

Conjecture 1.6. *The spheres Σ_k^1 and Σ_k^2 represent nontrivial homology classes in the chessboard complexes $\Delta_{3k+2,3k+3}$ and $\Delta_{3k+1,3k+1}$.*

2. Some special chessboard complexes

Here we will make a detailed study of some small chessboard complexes $\Delta_{m,n}$. Since the topology of $\Delta_{m,n}$ is relatively well understood when $n \geq 2m-1$ (the Cohen-Macaulay case) we will focus on chessboards which are square $\Delta_{m,m}$, or nearly-square $\Delta_{m,m+1}$.

The Euler-Poincaré relation for $\Delta_{m,n}$ $(m \leq n)$ has the form

$$\sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} \binom{n}{k} k! = \beta_0^{m,n} - \beta_1^{m,n} + \beta_2^{m,n} - \ldots + (-1)^{m-1} \beta_{m-1}^{m,n} \,,$$

where $\beta_i^{m,n} = \operatorname{rank}(H_i(\Delta_{m,n};))$. We observe that $\beta_{m-1}^{m,m} = 0$, since every codimension one simplex of $\Delta_{m,m}$ is contained in exactly one top-dimensional simplex. Also, $\beta_{m-1}^{m,m+1} = 1$, since $\Delta_{m,m+1}$ is an orientable pseudomanifold, the orientation coming from the signs of the permutations of the set $\{1, \ldots, m+1\}$ determined by the maximal simplices. Finally, Theorem 1.1 shows that $\beta_i^{m,n} = 0$ for $1 \leq i \leq \nu - 2$, and $\beta_0^{m,n} = 1$ if $\nu \geq 2$. Putting this information together in some interesting cases, we obtain the following:

$$\begin{array}{lcr}
\chi(\Delta_{3,3}) = -3 & & {}^{3,3}_{1} \quad = 4 \\
\chi(\Delta_{3,4}) = 0 & & {}^{3,4}_{1} \quad = 2 \\
\chi(\Delta_{4,4}) = 16 & \Rightarrow & \beta_2^{4,4} = 15 \\
\chi(\Delta_{4,5}) = 20 & \Rightarrow & \beta_2^{4,5} = 20 \\
\chi(\Delta_{5,6}) = -150 & \Rightarrow & \beta_3^{5,6} = 152
\end{array}$$

This verifies Conjecture 1.5 in these cases. The verification for $\Delta_{5,5}$ will be a consequence of Proposition 2.3 below.

By deleting the last (or any other) row of the $(m \times n)$-chessboard one obtains an $(m-1) \times n$-chessboard, so the complex $\Delta_{m-1,n}$ can be seen as a subcomplex of $\Delta_{m,n}$. Similarly, $\Delta_{m,n-1}$ can be realized as a subcomplex of $\Delta_{m,n}$ by deleting a column from the chessboard $[m] \times [n]$.

The following simple proposition gives a description of the quotient CW-complex of these two complexes. The proof is not difficult and will be left to the reader.

Proposition 2.1. *The quotient space $\Delta_{m,n}/\Delta_{m-1,n}$ is homeomorphic to the wedge of n copies of the suspension $S(\Delta_{m-1,n-1})$, i.e.*

$$\Delta_{m,n}/\Delta_{m-1,n} \approx \bigvee^{n} S(\Delta_{m-1,n-1}).$$

Let us now make a detailed analysis of the complex $\Delta_{3,4}$. The links of its vertices and edges are isomorphic to $\Delta_{2,3}$ and $\Delta_{1,2}$ respectively, which shows that $\Delta_{3,4}$ is a triangulation of a two-dimensional manifold. This manifold is orientable, as is any $\Delta_{n,n+1}$ (see above). The Euler characteristic of $\Delta_{3,4}$ is 0, so we conclude that $\Delta_{3,4}$ is a triangulation of a torus. A more informative way to reach this conclusion is via Figure 2, which shows that the universal covering of $\Delta_{3,4}$ is a triangulated honeycomb tesselation of the plane. So, we have the isomorphism $\Delta_{3,4} \cong^2 /\Gamma$ where Γ is the two dimensional lattice in ^{2} generated by vectors $x = \overrightarrow{AB}$ and $y = \overrightarrow{AC}$. Occasionally, it will be convenient to identify the group Γ, via the obvious isomorphism, with the group $\pi_1(\Delta_{3,4}) \cong H_1(\Delta_{3,4})$. Let $a = \overrightarrow{AX}$ and $b = \overrightarrow{AY}$.

Figure 2.

Figure 2 makes it possible to give a complete analysis of which subcomplexes of the form $\Delta_{2,3}$ and $\Delta_{3,2}$ contribute nontrivial classes to $H_1(\Delta_{3,4})$. Also, it gives a transparent picture of how the symmetric groups S_4 and S_3 act on $\Delta_{3,4}$ and $H_1(\Delta_{3,4})$. This and other related facts that will be needed later are recorded in the following lemma.

Lemma 2.2. *Let $e : \Delta_{3,3} \to \Delta_{3,4}$ be the natural inclusion map and $\bar{e} : H_1(\Delta_{3,3}) \to H_1(\Delta_{3,4})$ the induced homomorphism. Then the "horizontal" classes in $H_1(\Delta_{3,3})$, i.e. those classes determined by subchessboards $([3] \setminus \{i\}) \times [3]$, $1 \leq i \leq 3$, generate the kernel of $\bar{e}$. The "vertical" classes in $H_1(\Delta_{3,3})$, i.e. those classes determined by subchessboards of the form $[3] \times ([3] \setminus \{i\})$, $1 \leq i \leq 3$, are mapped to nonzero elements in $H_1(\Delta_{3,4})$ and in fact span a subgroup F of index 3 in $H_1(\Delta_{3,4})$. More precisely, having in mind the identification of $H_1(\Delta_{3,4})$ and Γ above, F is generated by the vectors $6a$ and $6b$. The group F is invariant under the obvious action of the symmetric group S_4 on $H_1(\Delta_{3,4})$, which implies that F is independent of the choice of embedding $e : \Delta_{3,3} \to \Delta_{3,4}$.*

Proof. Since $\Delta_{3,4}$ contains cones over every "horizontal" circle, the corresponding classes in $H_1(_{3,4})$ must vanish. Using Proposition 2.1, it is easy to see from the exact sequence of the pair $(\Delta_{3,4}, \Delta_{3,3})$

$$^3 \cong H_2(S(\Delta_{2,3}))^{\oplus 3} \to H_1(\Delta_{3,3})^{\bar{e}} H_1(\Delta_{3,4}) \to H_1(S(\Delta_{2,3}))^{\oplus 3} \cong 0$$

that $\bar{e}$ is an epimorphism. Its kernel is generated by the images of the three generators of $H_2(S(\Delta_{2,3}))^{\oplus 3}$, i.e. by the three horizontal classes. One can check by careful inspection of the triangulation of $\Delta_{3,4}$, see Figure 2, that the images of the vertical classes indeed generate the group F spanned by $6a$ and $6b$. Knowing that Γ is the group generated by $x = 4a + 2b$ and $y = 4b + 2a$, it is not difficult to deduce that F is a subgroup of index 3 in . The action of the group S_4 on $\Delta_{3,4}$ and Γ is described as follows. Let H be the reflection group generated by reflections in the sides of the triangle with vertices $(1,1), (3,4), (2,3)$. Then Γ is a normal subgroup of H and the quotient group $H/\Gamma \cong S_4$ acts on both $\Delta_{3,4}$ and Γ, which are exactly the actions we are interested in. This can be deduced directly from Figure 2. For example, the transposition of the second and the third column of the complex $\Delta_{3,4}$, can be identified as the reflection in the line determined by points $(1,1)$ and $(3,4)$. Indeed, the points $(1,2), (2,3), (3,3)$ are mapped to $(1,3), (2,2), (3,2)$ respectively, whereas all other vertices are not moved. From here it is easily seen that F is an invariant subgroup of Γ. The rest of the lemma follows from these observations.

Proposition 2.3. $\quad H_2(\Delta_{5,5}) \cong_3 \oplus_3 \oplus_3 \oplus_3.$

Proof. Using Proposition 2.1 one sees that the long exact sequence of the pair $(\Delta_{5,5}, \Delta_{4,5})$ has the following form:

$$\to \oplus_{i=1}^{5} H_2(\Delta_{4,4}^{i}) H_2(\Delta_{4,5}) \to H_2(\Delta_{5,5}) \to \oplus_{i=1}^{5} H_1(\Delta_{4,4}^{i})$$

Here, $\Delta_{4,4}^{i} \cong \Delta_{4,4}$ denotes the chessboard complex obtained from the chessboard $[4] \times ([5] \setminus \{i\})$, and $\alpha = \oplus_{i=1}^{5} \alpha_i$ where $\alpha_i : H_2(\Delta_{4,4}^{i}) \to H_2(\Delta_{4,5})$ is the homomorphism induced by the inclusion map $e_i : \Delta_{4,4}^{i} \to \Delta_{4,5}$. Since $H_1(\Delta_{4,4}) = 0$, in order to determine the group $H_2(\Delta_{5,5})$ it will be necessary to determine the cokernel of α. In other words, we want to determine which classes in $H_2(\Delta_{4,5})$ can be represented as sums of classes coming from all of the groups $H_2(\Delta_{4,4}^{i}),\ i = 1, .., 5$. In fact, we will show that it suffices to take two of these groups, say $H_2(\Delta_{4,4}^{4})$ and $H_2(\Delta_{4,4}^{5})$. Let $\Delta_{3,4}^{i}$ be the complex based on the chessboard $([4] \setminus \{i\}) \times [4],\ 1 \leq i \leq 4$, and $\Delta_{3,3}^{i},\ 1 \leq i \leq 4$, the complex determined by the chessboard $([4] \setminus \{i\}) \times [3]$. Obviously $\Delta_{4,4}^{4} \cap \Delta_{3,4}^{i} = \Delta_{3,3}^{i}$. Let $\overline{\Delta}_{4,3} = \Delta_{4,4}^{4} \cap \Delta_{4,4}^{5} \cong \Delta_{4,3}$. Then the inclusion map $(\Delta_{4,4}^{4}, \overline{\Delta}_{4,3}) \to (\Delta_{4,5}, \Delta_{4,4}^{5})$ and the naturality of the exact sequence of the pair give rise to the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
H_2(\overline{\Delta}_{4,3}) & \to & H_2(\Delta_{4,4}^{4}) & \overset{\mu}{} & \oplus_{i=1}^{4} H_1(\Delta_{3,3}^{i}) & \overset{\gamma}{} & H_1(\overline{\Delta}_{4,3}) & \to & 0 \\
\downarrow & & \alpha_4 \downarrow & & \downarrow \beta & & & & \\
H_2(\Delta_{4,4}^{5}) & \underset{\alpha_5}{\to} & H_2(\Delta_{4,5}) & \underset{\nu}{} & \oplus_{i=1}^{4} H_1(\Delta_{3,4}^{i}) & \to & 0 & &
\end{array}$$

We want to find $H_2(\Delta_{4,5})/(\mathrm{im}(\alpha_4) + \mathrm{im}(\alpha_5))$. For this purpose it is enough to find the image of $\nu \circ \alpha_4 = \beta \circ \mu$ in $\oplus_{i=1}^{4} H_1(\Delta_{3,4}^{i})$. Since $\beta = \oplus_{i=1}^{4} \beta_i$, where β_i is induced by the corresponding inclusion maps, it is enough to find $\mathrm{im}(\beta_i \circ \mu)$ in $H_1(\Delta^i_{3,4})$. For notational convenience, we will assume $i = 4$, in the other cases the proof is similar. Let us first identify $\mathrm{im}(\mu) \cap H_1(\Delta^4_{3,3}) = \ker(\gamma) \cap H_1(\Delta^4_{3,3})$. By Lemma 2.2 we know that $\mathrm{im}(\mu) \cap H_1(\Delta^4_{3,3})$ consists of sums of classes determined by the complexes coming from chessboards $[3] \times ([3] \setminus \{i\}),\ 1 \leq i \leq 3$. On the other hand, this lemma implies that the images of these classes in $H_1(\Delta^4_{3,4})$ generate a subgroup F_i of index 3, hence there is an isomorphism $H_2(\Delta_{4,5})/(\mathrm{im}(\alpha_4) + \mathrm{im}({}_5)) \cong ({}_3)^{\oplus 4}$. A key observation in Lemma 2.2 was that the group F does not depend on the embedding $e : \Delta_{3,3} \to \Delta_{3,4}$. This implies that F_i does not change if $\Delta^4_{4,4}$ is replaced above by any of the complexes $\Delta^1_{4,4}, \Delta^2_{4,4}$ or $\Delta^3_{4,4}$. This leads immediately to the conclusion that $H_2(\Delta_{4,5})/\mathrm{im}(\alpha) \cong ({}_3)^{\oplus 4}$.

As a consequence of Proposition 2.3 one has $\beta^{5,5}_2 = 0$, and we know from before that $\beta^{5,5}_0 = 1$ and $\beta^{5,5}_1 = \beta^{5,5}_4 = 0$. Hence, from $\chi(\Delta_{5,5}) = -55$ we conclude that $\beta^{5,5}_3 = 56$. It follows that $\mathrm{depth}(\mathbf{k}[\Delta_{5,5}]) \leq 4$ for all fields $\mathbf{k}$. On the other hand, Corollary 1.4 shows that $\mathrm{depth}(\mathbf{k}[\Delta_{5,5}]) \geq 3$.

Proposition 2.4.

$$\mathit{depth}(\mathbf{k}[\Delta_{5,5}]) = \begin{cases} 3\,, & \mathit{if\ char}\ \mathbf{k} = 3 \\ 4\,, & \mathit{if\ char}\ \mathbf{k} \neq 3. \end{cases}$$

Proof. The characteristic 3 case follows directly from Smith's [9] characterization of depth and Proposition 2.3. In the other case we must show that $\Delta^{(3)}_{5,5}$ is Cohen-Macaulay over $\mathbf{k}$. The required vanishing of homology with coefficients in $\mathbf{k}$, char $\mathbf{k} \neq 3$, is implied by Proposition 2.3 and the Universal Coefficient Theorem for the 1-connected complex $\Delta^{(3)}_{5,5}$, and by Theorem 1.1 for all its proper links.

The only manifolds among chessboard complexes are $\Delta_{1,2}, \Delta_{2,3}, \Delta_{3,4}$ and their "twin" complexes obtained by interchanging the indices. However, it is interesting to note that $\Delta_{n,n+1},\ n \geq 4$, is almost a manifold, having singularities only at the points lying on its codimension three skeleton. For example, every point in $\Delta_{4,5}$ which is not a vertex possesses a neighborhood homeomorphic to a three dimensional ball. The star of each of the vertices in $\Delta_{4,5}$ is homeomorphic to a cone over a torus $\Delta_{3,4} \cong T^2$, so by removing a small open neighborhood around each of them one obtains a compact, orientable three-manifold M with boundary consisting of twenty copies of T^2. One can glue twenty solid torii along this boundary to obtain a closed three-dimensional manifold. This manifold can be used to obtain additional information about the homology of $\Delta_{4,5}$. One would hope that something similar can be done in higher dimensions, but that remains an interesting open problem.

3. Higher-dimensional chessboard complexes

In this section we will consider a higher-dimensional chessboard of shape $n_1 \times n_2 \times ... \times n_k$, where we suppose that $n_1 \leq n_2 \leq ... \leq n_k$. Here we assume that two rooks are in a non-taking position if and only if they do not belong to the same $(k-1)$-dimensional plane orthogonal to one of the axes of the chessboard. Again, we can in the same way assign to this chessboard its chessboard complex $\Delta_{n_1,n_2,...,n_k}$, where the vertices are the boxes of the chessboard and the simplices are the supports of non-taking rook placements. We determine an estimate for the connectivity of these complexes which generalizes Theorem 1.1. At the moment we are unable to provide either a proof or a counterexample for the claim that this estimate is sharp.

Theorem 3.1. *The chessboard complex $\Delta_{n_1,n_2,...,n_k}$ is $(\nu-2)$-connected, where*

$$\nu = min\left\{ n_1, \left[\frac{n_1+n_2+1}{3}\right], ..., \left[\frac{n_1+n_2+...+n_k+k-1}{2k-1}\right]\right\}.$$

Proof. The chessboard complex is always non-empty, which verifies the $n_1 = 1$ and $n_1 = n_2 = 2$ cases. Furthermore, it is easy to see that the chessboard complex $\Delta_{n_1,n_2,...,n_k}$ is connected when $n_1 \geq 2$ and $n_2 \geq 3$. Therefore we may assume that $n_1 \geq 3$. We will use an induction argument. Let us suppose the statement of the theorem to be true for all chessboards whose sides are all smaller (and at least one strictly smaller) than for the given chessboard $[n_1] \times [n_2] \times ... \times [n_k]$, with $3 \leq n_1 \leq n_2 \leq ... \leq n_k$.

Let us think of the vertices of the chessboard complex as the boxes of a chessboard, and let us consider all the boxes in the first hyperplane (among n_1 of them) orthogonal to the first and the shortest of the axes. Let us assign to each of these boxes the subcomplex L_i of all rook placements which could be extended to a rook placement including the particular box i that we consider. Every such subcomplex is a cone with the vertex i as apex, and so all of them are contractible and therefore certainly $(\nu-2)$-connected. The subcomplexes L_i form a covering of $\Delta_{n_1,n_2,...,n_k}$.

The intersection of any two of these subcomplexes is the chessboard complex associated to a chessboard whose sides (not necessarily in increasing order) are at least $n_1-1, n_2-2, n_3-2, ..., n_k-2$. The intersection of any three elements of this covering is the chessboard complex associated to a chessboard whose sides are at least $n_1-1, n_2-3, n_3-3, ..., n_k-3$, and so on. The intersection of every n_2-1 elements of this covering is non-empty, and so its nerve has full (n_2-2)-skeleton and therefore is (n_2-3)-connected. It is easy to see that $n_2-3 \geq \nu-2$. So, it is enough to prove, by Lemma 1.2, that every intersection $L_{i_1} \cap L_{i_2} \cap ... \cap L_{i_t}$ is $(\nu-t-1)$-connected for $2 \leq t \leq \nu$.

Let us first consider the case $t \geq 3$. The intersection $L_{i_1} \cap ... \cap L_{i_t}$ is again a chessboard complex associated to a chessboard whose sides are at least n_1-1, n_2-

$t, ..., n_k - t$. By the induction hypothesis it is at least $(\mu - 2)$-connected, where

$$\mu = \min\left\{m_1, \left[\frac{m_1+m_2+1}{3}\right], ..., \left[\frac{m_1+m_2+...+m_k+k-1}{2k-1}\right]\right\}$$

and $m_1, m_2, ..., m_k$ are the numbers $n_1 - 1, n_2 - t, ..., n_k - t$ in increasing order. We want to show that $\mu \geq \nu - t + 1$. Since $t \geq 3$, it is easy to see that for $i \geq 2$ we have $\frac{ti}{2i-1} \leq t - 1$, and therefore

$$\frac{m_1+...+m_i+i-1}{2i-1} \geq \frac{n_1+...+n_i-ti+i-1}{2i-1} \geq \frac{n_1+...+n_i+i-1}{2i-1} - t+1 \geq \nu - t + 1.$$

So, $\left[\frac{m_1+m_2+...+m_i+i-1}{2i-1}\right] \geq \nu - t + 1$ for $i \geq 2$. For $i = 1$ we have two cases:

(i) if $m_1 = n_1 - 1$, then certainly $m_1 \geq n_1 - t + 1 \geq \nu - t + 1$;

(ii) if $m_1 = n_2 - t$ and if we suppose $m_1 < n_1 - t + 1$, then we have $n_1 = n_2$ and $\nu \leq \left[\frac{2n_2+1}{3}\right] \leq n_2 - 1$. Then again $m_1 \geq \nu - t + 1$.

All this means that $\mu \geq \nu - t + 1$.

Let us now assume $t = 2$. The intersection $L_{i_1} \cap L_{i_2}$ is a chessboard complex associated to a chessboard whose sides are at least $n_1 - 1, n_2 - 2, ..., n_k - 2$, and it is at least $(\mu - 2)$-connected where

$$\mu = \min\left\{m_1, \left[\frac{m_1+m_2+1}{3}\right], ..., \left[\frac{m_1+m_2+...+m_k+k-1}{2k-1}\right]\right\}$$

and $m_1, m_2, ..., m_k$ are the numbers $n_1 - 1, n_2 - 2, ..., n_k - 2$ in increasing order. We want to show that $\mu \geq \nu - 1$. We can see that $m_1 \geq \nu - 1$ as in the case $t \geq 3$.

Let us now prove that $\frac{m_1+m_2+...+m_i+i-1}{2i-1} \geq \nu - 1$ for $2 \leq i \leq k$. If one among the numbers $m_1, m_2, ..., m_i$ equals $n_1 - 1$, this is trivial, since in this case we have

$$\frac{m_1+...+m_i+i-1}{2i-1} \geq \frac{n_1+...+n_i-(2i-1)+i-1}{2i-1} = \frac{n_1+...+n_i+i-1}{2i-1} - 1 \geq \nu - 1.$$

So, let us suppose that $m_1 = n_2 - 2, m_2 = n_3 - 2, ..., m_i = n_{i+1} - 2 < n_1 - 1$. But, this is possible only if $i \leq k - 1$ and if $n_1 = n_2 = ... = n_{i+1}$. Let us denote this last number by n. We want to prove that $\left[\frac{m_1+...+m_i+i-1}{2i-1}\right] \geq \left[\frac{n_1+...+n_{i+1}+i}{2i+1}\right] - 1$, i.e. $\left[\frac{i(n-1)-1}{2i-1}\right] \geq \left[\frac{(i+1)(n-1)}{2i+1}\right]$, which would imply the required inequality. If $n \geq 2i+2$, then

$$\frac{i(n-1)-1}{2i-1} - \frac{(i+1)(n-1)}{2i+1} = \frac{n-1-(2i+1)}{(2i-1)(2i+1)} \geq 0,$$

and certainly $\left[\frac{i(n-1)-1}{2i-1}\right] \geq \left[\frac{(i+1)(n-1)}{2i+1}\right]$. So, we are left with the case $3 \leq n \leq 2i+1$.

Since $\frac{(i+1)(n-1)}{2i+1} = (n-1)\frac{1}{2} + \frac{1}{2(2i+1)}$, the $2i - 1$ numbers $\left\{\frac{(i+1)(n-1)}{2i+1} \mid 3 \leq n \leq 2i+1\right\}$ are distributed equidistantly in i intervals $[1, 2], [2, 3], \ldots, [i, i+1]$, where the distance is $\frac{1}{2} + \frac{1}{2(2i+1)} > \frac{1}{2}$ and there are exactly two numbers in each interval except for the last one, which contains only one of these numbers. It is easy to see that the minimal value of the expression $(n-1)\frac{i+1}{2i+1}-[(n-1)\frac{i+1}{2i+1}]$, $3 \le n \le 2i+1$, is attained for $n=3$. So,

$$(n-1)\frac{i+1}{2i+1}-\left[(n-1)\frac{i+1}{2i+1}\right] \ge 1+\frac{1}{2i+1}-\left[1+\frac{1}{2i+1}\right]=\frac{1}{2i+1}.$$

Since $n \ge 3$, we have $(n-1)\frac{i+1}{2i+1}-\frac{i(n-1)-1}{2i-1}=\frac{2i+2-n}{(2i-1)(2i+1)} \le \frac{1}{2i+1}$, and therefore $\left[(n-1)\frac{i+1}{2i+1}\right] \le \left[\frac{i(n-1)-1}{2i-1}\right]$.

The following consequences are proved just like Corollaries 1.3 and 1.4, which they generalize.

Corollary 3.2. *The* $(\nu-1)$*-skeleton of* $\Delta_{n_1,n_2,\ldots,n_k}$ *is homotopy-Cohen-Macaulay.*

Corollary 3.3. *depth*$(\mathbf{k}[\Delta_{n_1,n_2,\ldots,n_k}]) \ge \nu$, *for every field* $\mathbf{k}$.

We believe that these results are sharp, in the sense that Conjecture 1.5 extends to the higher-dimensional chessboard complexes $\Delta_{n_1,n_2,\ldots,n_k}$.

¿From Theorem 3.1 it is clear that the connectivity of the complex $\Delta_{n_1,n_2,\ldots,n_k}$ is not greater than the connectivity of Δ_{n_1,n_2}. Actually, one can say something more.

Proposition 3.4. $H_*(\Delta_{n_1,n_2})$ *is a direct summand of* $H_*(\Delta_{n_1,n_2,\ldots,n_k})$.

Proof. We consider the projection mapping $\pi : \Delta_{n_1,n_2,\ldots,n_k} \longrightarrow \Delta_{n_1,n_2}$ and the inclusion mapping $i : \Delta_{n_1,n_2} \longrightarrow \Delta_{n_1,n_2,\ldots,n_k}$, where i is defined on vertices by $i(x_1,x_2)=(x_1,x_2,\ldots,x_2)$. It is easy to see that i can be extended over all simplices of Δ_{n_1,n_2}. Moreover, it is a trivial observation that $\pi \circ i = 1$. Therefore $\pi_* \circ i_* = 1$, which implies our statement.

4. Matching complexes

The matching complex $M(G)$ of a graph or hypergraph G is defined as the collection of all of its partial matchings (i.e. systems of pairwise disjoint edges). See [8] for general information about matchings.

As was mentioned in the introduction it is easy to see that the matching complex of the complete bipartite graph K_{n_1,n_2} is exactly the chessboard complex Δ_{n_1,n_2}. Namely, the edges of K_{n_1,n_2} are in bijective correspondence with the vertices of Δ_{n_1,n_2}, and every partial matching of K_{n_1,n_2} defines a non-taking rook placement on the $n_1 \times n_2$ chessboard, i.e. a simplex of Δ_{n_1,n_2}, and vice-versa. Moreover, chessboard complexes of higher-dimensional chessboards can be seen as the matching complexes of complete multipartite hypergraphs. The complete k-partite hypergraph $K_{n_1,n_2,\dots,n_k}$ is defined as the hypergraph whose set of vertices is the union of pairwise disjoint sets $V_1, ..., V_k$ of cardinalities $n_1, ..., n_k$, and whose k-edges are defined by sets of k vertices, one from each of the sets $V_1, V_2, ..., V_k$. Again, the simplices of the matching complex $M(K_{n_1,\dots,n_k})$ are the collections of pairwise disjoint k-edges, which correspond to the non-taking rook placements on the higher-dimensional chessboard $[n_1] \times \ldots \times [n_k]$, i.e. to the simplices of $_{n_1,\dots,n_k}$, and vice-versa.

In this section we shall treat the case of the complete k-graph on n vertices, which we denote by K_n^k, and especially the case of the complete graph K_n on n vertices which is the particular case obtained when $k = 2$. Let us remind the reader that the complete k-graph on n vertices is defined as the hypergraph with n vertices each k vertices of which span a k-multi-edge. Clearly, the matching complex of K_n^k could be described as the complex whose vertices are all the k-element subsets of an n-element set, and for which some vertices span a simplex if and only if the associated k-tuples are pairwise disjoint. Skeleta of the matching complex $M(K_n^k)$ are closely related to so called "Kneser hypergraphs", see [2].

Theorem 4.1. *The matching complex $M(K_n^k)$ is $(\nu - 2)$-connected, with $\nu = \left[\frac{n+2k-3}{2k-1}\right]$.*

Proof. The proof is carried out by induction on n, again using the nerve lemma. It is easy to check that the statement is true for $n \leq 4k-1$. Namely, all we have to show is that $M(K_n^k)$ is non-empty when $k \leq n \leq 2k$ and connected when $2k+1 \leq n \leq 4k-1$. But, having $n \geq 2k+1$, it is easy to connect any two vertices of $M(K_n^k)$ by a path of consecutive edges, exchanging in each two steps one element from the k-set associated to the first vertex with one element from the set associated with the second vertex.

So, let us take $n \geq 4k$ and assume that the statement of the theorem is true for all numbers smaller than n.

Let $L_{i_2,\dots,i_k}$ (for $2 \leq i_2 < i_3 < \cdots < i_k \leq n$) be the collection of all simplices σ of $M(K_n^k)$ such that $\sigma \cup \{\{1, i_2, i_3, ..., i_k\}\} \in M(K_n^k)$. Then $\{L_{i_2,\dots,i_k} \mid 2 \leq i_2 < i_3 < \cdots < i_k \leq n\}$ is a covering of a certain subcomplex M' of $M(K_n^k)$. If k divides n then $M' = M(K_n^k)$, and if $n = dk + r$ with $0 < r < k$, then M' excludes all $(d-1)$-simplices $(A_1, ..., A_d)$, $A_i \cap A_j = \emptyset$, of $M(K_n^k)$ such that $1 \notin A_1 \cup ... \cup A_d$. In any case, the $\left(\left[\frac{n}{k}\right] - 2\right)$-skeleta of M' and of $M(K_n^k)$ are the same. Since $\left[\frac{n}{k}\right] - 3 \geq \nu - 2$ (this requires our assumption that $n \geq 4k$), it suffices to prove that M' is $(\nu-2)$-connected, which we will do by applying the nerve lemma to its covering by the subcomplexes $L_{i_2,i_3,\dots,i_k}$.

It is elementary to see that already $n \geq 4k-3$ implies

$$n - (\nu(k-1)+1) = n - \left(\left[\frac{n+2k-3}{2k-1}\right](k-1)+1\right) \geq k.$$

Since every ν elements of the covering determine at most $\nu(k-1)+1$ points, their intersection is a matching complex $M(K_t^k)$, where $t \geq k$, hence non-empty. So, the nerve of this covering has full $(\nu-1)$-skeleton, therefore it is $(\nu-2)$-connected. By the nerve lemma, it is therefore enough to prove that the intersection of every t elements of the covering is $(\nu-t-1)$-connected for $2 \leq t \leq \nu$.

Any t elements of the covering determine at most $t(k-1)+1$ points, so their intersection is a matching complex $M(K_r^k)$ where $r \geq n - t(k-1) - 1$. Therefore, by the induction hypothesis, this intersection is $\left(\left[\frac{n-t(k-1)-1+2k-3}{2k-1}\right]-2\right)$-connected. It is again elementary to see that

$$\left[\frac{n-t(k-1)-1+2k-3}{2k-1}\right]-2 = \left[\frac{n+2k-3}{2k-1}+\frac{(t-2)k}{2k-1}\right]-t-1 \geq \left[\frac{n+2k-3}{2k-1}\right]-t-1,$$

which proves the theorem.

Corollary 4.2. *The $(\nu-1)$-skeleton of $M(K_n^k)$ is homotopy-Cohen-Macaulay.*

Proof. Let $\sigma \in M(K_n^k)^{(\nu-1)}$, and $|\sigma| = s$. Then $\text{link}(\sigma) \cong M(K_{n-sk}^k)^{(\nu-s-1)}$. Since $M(K_{n-sk}^k)$ is $(\mu-2)$-connected with $\mu = \left[\frac{n-sk+2k-3}{2k-1}\right] \geq \nu - s$, it follows that $M(K_{n-sk}^k)^{(\nu-s-1)}$ is $(\nu-s-2)$-connected.

The Stanley-Reisner ring of the matching complex $M(K_n^k)$ has the following concrete description. Take the polynomial ring $\mathbf{k}[x_A, ...]$, with one indeterminate x_A for each k-element subset A of $\{1, 2, ..., n\}$. Then $\mathbf{k}[M(K_n^k)]$ is the quotient modulo the ideal generated by all products $x_A x_B$ for distinct subsets A and B such that $A \cap B \neq \emptyset$.

Corollary 4.3. *depth*$(\mathbf{k}[M(K_n^k)]) \geq \nu$, *for every field* $\mathbf{k}$.

We conjecture that the bounds obtained in this section are sharp, in the same sense as in Conjecture 1.5.

The following argument verifies this in case of the complex $M(K_8)$. The Euler characteristic of the complex $M(K_8)$ is :

$$\chi(M(K_8)) = \sum_{i=1}^{4} (-1)^{i-1} \binom{8}{2} \cdot \cdots \cdot \binom{8-2i+2}{2} \cdot \frac{1}{i!} = 133.$$

Since we have shown that $M(K_8)$ is 1-connected and since $H_3(M(K_8)) = 0$ (because each 2-dimensional simplex is contained in exactly one 3-dimensional simplex), this relation implies by the Euler-Poincaré formula that the second Betti number equals 132. Therefore, $M(K_8)$ is not 2-connected.

By similar reasoning one can check that the bound in Theorem 4.1 is sharp for $M(K_n)$, $n \leq 6$. We have not been able to settle the case $n = 7$, i.e. to determine whether $M(K_7)$ is 1-connected.

It is easy to see that the matching complex $M(K_n)$ contains, as a subcomplex, the chessboard complex Δ_{n_1,n_2} whenever $n_1 + n_2 = n$ (it is enough to partition the vertices of K_n in two groups of n_1 and n_2 elements respectively). Similarly, the matching complex of the complete k-graph on n vertices $M(K_n^k)$ contains, as a subcomplex, the chessboard complex $\Delta_{n_1,\dots,n_k}$ whenever $n_1 + \dots + n_k = n$.

It is interesting to observe that our estimates on the connectivity of the matching complex and the embedded chessboard complexes are closely related. More precisely, we have proved that the connectivity of the matching complex $M(K_n^k)$ is not smaller than the connectivity of the embedded chessboard complex obtained when $n_1 \leq n_2 \leq \dots \leq n_k \leq n_1 + 1$. Namely, by Theorem 3.1, the lower bound for the connectivity of the embedded chessboard complex is $\left[\frac{n+k-1}{2k-1}\right] - 2$, and Theorem 4.1 shows that the connectivity of the matching complex $M(K_n^k)$ is at least $\left[\frac{n+2k-3}{2k-1}\right] - 2 = \left[\frac{n+k-1}{2k-1} + \frac{k-2}{2k-1}\right] - 2$. This estimate is better, and in some cases strictly better. However, it should be noticed that for $k = 2$ these estimates coincide.

5. Final remark

In view of Corollary 1.3 it seems plausible to conjecture that the complex $\Delta_{m,n}^{(\nu-1)}$ is shellable. This is not known even for the case when $n \geq 2m - 1$, i.e. the Cohen-Macaulay chessboard complexes considered by Garst [7]. Similarly we suspect that the complexes described in Corollaries 3.2 and 4.2 are shellable. This suggests as a special instance the following "recreational" challenge: Arrange all non-taking placements of 5 rooks on the 8×8 chessboard in a shelling order.

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