Lattices in graphs with polynomial growth

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Abstract

We present results on the structure of graphs with polynomial growth. For certain classes of these graphs we prove that they contain subgraphs which are similar to the lattice graph $\mathbf{L}^d$. These results are then applied to investigate isoperimetric properties of certain classes of graphs with polynomial growth. In addition these structural results can also be applied to percolation problems.

1 Introduction

In [Tr2] Trofimov characterizes the automorphism groups of graphs with polynomial growth. In [Tr1] he gives an even stronger characterization of those graphs with polynomial growth on which a group of bounded automorphisms acts transitively. He shows that these graphs are quite similar to Cayley graphs of $\mathbf{Z}^d$. In this paper we investigate this relationship in detail.

Our results are then applied to the investigation the isoperimetric properties of infinite graphs, a topic which already obtained considerable attention in the literature (see e.g. [BSz], [Va]). We apply structural properties of certain classes of graphs with polynomial growth to obtain new bounds for their isoperimetric numbers. These structural properties are furthermore also applied to obtain results about percolation problems in these graphs.

2 Terminology and premilinary results

We call a graph $X = (V, E)$ *vertex-transitive* (sometimes briefly *transitive*) if for every pair x, $y \in V(X)$ there is an automorphism g such that $y = g(x)$. By $\mathrm{Aut}(X)$ we denote the group of all automorphisms of the graph X.

The concept of growth plays an important role throughout this paper. By

$$f_X(v, n) = |\{w \in V(X) | d(v, w) \leq n\}|\,, n \geq 1,$$

we denote the growth function of the graph X with respect to some $v \in V(X)$. If X is transitive this function clearly does not depend on a particular vertex, therefore we denote it by $f_X(n)$. If there are constants c_1, c_2, and an integer $d \geq 1$ such that $c_1 n^d \leq f_X(n) \leq c_2 n^d$ holds, then we say that X has *polynomial growth* and d is called the *growth degree of* X.

If a group $G \leq \mathrm{Aut}(X)$ acts transitively on $V(X)$, then an *imprimitivity system of* G *on* X is a partition τ of $V(X)$ into subsets called *blocks*, such that every element of G induces a permutation on the blocks of τ. The corresponding quotient graph of X is denoted by X_τ. If M_1 and M_2 are two blocks of τ then (M_1, M_2) is an edge of X_τ if and only if some vertex in M_1 is adjacent to some vertex in M_2. Denote by G_τ the homomorphic image of G. For simplicity we denote this homomorphism also by $\tau : G \to G_\tau$. We mention that the orbits of a normal subgroup $N \triangleleft G$, where G acts transitively on X, always give rise to an imprimitivity system of G on X.

An element $g \in \mathrm{Aut}(X)$ is called *bounded* if there exists a constant k, depending upon g, such that $d(x, g(x)) \leq k$ for every $x \in V(X)$. By $B(X) \subseteq \mathrm{Aut}(X)$ we denote the group of all bounded automorphisms of X, by $B_0(X) \subseteq B(X)$ the set of all bounded automorphisms of finite order.

In [Tr1] the following characterization of those graphs which admit a transitive group of bounded automorphisms was given:

Theorem 2.1. *Let X be a connected, locally finite graph. Then $B(X)$ acts transitively on X if and only if there is an imprimitivity system τ with finite blocks of $B(X)$ on X such that $B(X)_\tau$ is a finitely generated free Abelian group.*

We remark that the blocks of τ may be chosen as the orbits of $B_0(X)$ on X. Refining Trofimov's result it was shown in [GISW^2] that $B_0(X)$ is a normal locally finite periodic subgroup of $\mathrm{Aut}(X)$ which has finite orbits on $V(X)$.

For convenience we call a two-way infinite path a *2-path*. An automorphism which acts with infinite orbits on a graph X is called a *type 2* automorphism.

We call the Cayley graph $\mathbf{L}^d$, $d \geq 1$, of the free Abelian group $\mathbf{Z}^d$ with respect to its free generators the *d-dimensional lattice*. Let Y be a graph homeomorphic to some $\mathbf{L}^d, d \geq 1$, where $\{a_1, \ldots, a_d\}$ denotes a free generating set of $\mathbf{Z}^d$ (the edges of $\mathbf{L}^d$). If Y in addition has the property that it is obtained from $\mathbf{L}^d$ by replacing each edge corresponding to some a_i, $1 \leq i \leq d$, by a path of fixed length l_i, respectively, then Y is called $(l_1, \ldots, l_d)$ - *homeomorphic* to $\mathbf{L}^d$. If $\mathbf{L}^d$ is a Cayley graph of $\mathbf{Z}^d$ with respect to the generators $\{a_1, \ldots, a_d\}$ then any subgraph of $\mathbf{L}^d$ corresponding to generators $\{a_{i_1}, \ldots, a_{i_{d-1}}\}$ is called a *hyperplane* H of $\mathbf{L}^d$. We also say that H is *orthogonal* to the *direction* a_{i_d}.

3 Subgraphs

We recall a Lemma from [Se]. It can be seen as a first look at the subgraph structure. For the sake of completeness we include its proof.

Lemma 3.1. *Let X be a graph and let T_1 and T_2, $|T_1| = |T_2| \geq 1$, denote two finite orbits of a group $G \leq \mathrm{Aut}(X)$ on X. By Y we denote the bipartite subgraph of X spanned by T_1 and T_2. ($V(Y) = T_1 \cup T_2$, $E(Y) = \{(v, w) \in E(X) | v \in T_1,\ w \in T_2\}$.) If $E(Y) \neq \emptyset$ then there exists a complete matching of Y.*

Proof. For each pair $v_0, v \in T_1$ there is an automorphism $\alpha \in G$ such that $\alpha(v_0) = v$. Denote by $N(v_0)$ the vertices of T_2 which are adjacent to v_0. Since $\alpha(N(v_0)) \subset T_2$ every $v \in T_1$ has the same number of neighbours in T_2. Analogously this holds for the vertices in T_2. Since $|T_1| = |T_2|$ we obtain that Y is a regular graph. □

Under additional assumptions on the automorphisms of the graphs in consideration we can show much more:

Theorem 3.2. *Let X be a connected, locally finite, vertex-transitive graph of quadratic growth, such that the group of bounded automorphisms $B(X)$ acts transitively on $V(X)$. Then X is spanned by a finite number of pairwise disjoint, isomorphic subgraphs which are $(l_1, 1)$ - homeomorphic to $\mathbf{L}^2$.*

To fix notation we mention that a denotes a preimage of a_τ under τ^{-1} if a_τ is an automorphism of X_τ. We need the following two technical lemmas to prove this theorem.

Lemma 3.3. *Let τ be an imprimitivity system with finite blocks of cardinality m of a graph X and let $a_\tau \in \mathrm{Aut}(X_\tau)$ be a type 2 automorphism which leaves invariant a 2-path P_τ in X_τ. Then there exists an integer p, $1 \leq p \leq m!$, such that some $a^p \in \tau^{-1}(a_\tau^p)$ leaves invariant m disjoint 2-paths in X.*

Proof. By Lemma 3.1, the 2-path P_τ can be lifted to m disjoint 2-paths $P_1^1, \ldots, P_m^1$. Let $P_\tau = (\ldots, p_{-1}^\tau, p_0^\tau, p_1^\tau, \ldots)$. Each of the vertices

p_i^τ, $i \in \mathbf{Z}$, represents a finite set $T_i \subset V(X)$, $|T_i| = m$, and every 2-path $P_1^1, \ldots, P_m^1$ meets each of these sets in exactly one vertex. We denote $P_j^1 = (\ldots, t_{-1}^j, t_0^j, t_1^j, \ldots)$, $1 \leq j \leq m$.

Without loss of generality we can also assume that $a_\tau(p_i^\tau) = p_{i+1}^\tau$ holds. Then $b(T_i) = T_{i+1}$, $i \in \mathbf{Z}$, holds for each $b \in \tau^{-1}(a_\tau)$. Let a now denote one particular preimage of a_τ under τ^{-1}. Since a acts with m orbits on $S = \bigcup_{i \in \mathbf{Z}} T_i$, such that each orbit of a on S contains exactly one vertex of each T_i, we can find an integer q such that a^q maps a vertex of a 2-path P_k^1, $k \in \{1, \ldots, m\}$, onto a vertex of the same path. Without loss of generality we can assume that $a^q(t_0^1) = t_q^1$ holds. Then $P_1^2 = \bigcup_{i \in \mathbf{Z}} a^{iq}(P_1^1(t_0^1, t_q^1))$ is a 2-path which is left invariant by a^q. Also the 2-paths $P_j^2 = \bigcup_{i \in \mathbf{Z}} a^{iq}(P_j^1(t_0^j, t_q^j))$, $1 \leq j \leq m$, are pairwise disjoint, since the finite paths $P_j^1(t_0^j, t_q^j)$ are pairwise disjoint.

We now assume that we have already found an integer s and 2-paths $P_1^{n+1}, \ldots, P_n^{n+1}$ such that $a^s(P_j^{n+1}) = P_j^{n+1}$, $1 \leq j \leq n$, holds for some n, $1 \leq n < m$. Renumbering the vertices, we again assume that $P_k^{n+1} = (\ldots, t_{-1}^k, t_0^k, t_1^k, \ldots)$, $1 \leq k \leq n$. Repeating the above procedure with a^s instead of a and the 2-paths P_k^{n+1} instead of P_k^1 we thus obtain an integer r (a multiple of s) and at least one additional 2-path P_{n+1}^{n+2} such that the 2-paths $P_1^{n+2}, \ldots, P_{n+1}^{n+2}$ are left invariant by a^r. (We emphasize that $P_1^{n+2} = P_1^{n+1}, \ldots, P_n^{n+2} = P_n^{n+1}$ hold.)

Since the T_i, $i \in \mathbf{Z}$, all have cardinality m we conclude that $q \leq m$. Also, if we have already determined the 2-paths $P_1^{n+1}, \ldots, P_n^{n+1}$ it is clear that $r \leq m - n$. Hence $p \leq m!$ holds for every integer p such that a^p leaves invariant our m disjoint 2-paths. □

Lemma 3.4. *Let τ be an imprimitivity system with finite blocks of cardinality m of a graph X and let $a_\tau \in \mathrm{Aut}(X_\tau)$ be a type 2 automorphism which leaves invariant the 2-paths $\ldots, P_{-1}^\tau, P_0^\tau, P_1^\tau, \ldots$ in X_τ. Then there exists an integer $1 \leq q \leq m!$ such that some $a^q \in \tau^{-1}(a_\tau^q)$ leaves invariant the 2-paths $\ldots, P_1^{-1}, \ldots, P_m^{-1}, P_1^0, \ldots, P_m^0, P_1^1, \ldots, P_m^1, \ldots$ in X, where the P_j^i, $i \in \mathbf{Z}$, $1 \leq j \leq m$, are pairwise disjoint preimages of the P_τ^i under τ^{-1}, respectively.*

Proof. By Lemma 3.3, we can find sets P_j^i, $i \in \mathbf{Z}$, $1 \leq j \leq m$, such that there is a $p_i \leq m!$ such that a^{p_i} leaves invariant every P_j^i, respectively. Hence, there are only finitely many different p_i and therefore the least common multiple q of all p_i has the required properties. Obviously $q \leq m!$ also holds. $\square$

Proof of Theorem 3.2. By Theorem 2.1 there exists an imprimitivity system τ of $B(X)$ on X with finite blocks of cardinality $m \geq 1$, such that X_τ is a Cayley graph of $\mathbf{Z}^2$. Hence there are two automorphisms $a_\tau, b_\tau \in B_\tau(X)$ such that $H = \langle a_\tau, b_\tau \rangle \cong \mathbf{Z}^2$ acts transitively on X_τ. Obviously there exist infinitely many pairwise disjoint 2-paths $\dots, P_\tau^{-1}, P_\tau^0, P_\tau^1, \dots$ in X_τ which are left invariant by a_τ and span X_τ. By Lemma 3.4 these paths can be lifted to 2-paths $\dots, P_1^{-1}, \dots, P_m^{-1}, P_1^0, \dots, P_m^0, P_1^1, \dots, P_m^1, \dots$ in X which are left invariant by some a^p, $a \in \tau^{-1}(a_\tau)$.

Futhermore, let $P_\tau^i = (\dots, v_{-1}^i, v_0^i, v_1^i, \dots)$, $i \in \mathbf{Z}$. Then X_τ contains a 2-path $Q_\tau = (\dots, v_0^{-1}, v_0^0, v_0^1, \dots)$ such that $b(v_0^i) = v_0^{i+1}$ for all $i \in \mathbf{Z}$. Clearly Q can be lifted to m disjoint 2-paths $Q_1, \dots, Q_m$ in X. Moreover each Q_j, $1 \leq j \leq m$, meets exactly one 2-path of each set $P_i = \{P_1^i, \dots, P_m^i\}$, $i \in \mathbf{Z}$. Since a^p leaves invariant every P_j^i, $i \in \mathbf{Z}, 1 \leq j \leq m$, each $a^p(Q_j)$ meets exactly the same 2-path P_j^i as Q_j. Let W^j denote the set of 2-paths intersecting with Q_j. Then each $Y_j = \bigcup_{i \in \mathbf{Z}} a^{ip}(Q_j) \cup W^j$ spans a subgraph homeomorphic to the 2-dimensional lattice. Also the Q_j, $1 \leq j \leq m$, are pairwise disjoint and they span X. In addition every vertex of the $a^{ip}(Q_j)$, $i \in \mathbf{Z}$, has valency 4, which means that only in the direction of a, the edges of the 2-dimensional lattice are substituted by paths of length $1 \leq l_1 \leq m!$. $\square$

A simple example of a graph X which satisfies all assumptions of Theorem 3.2 and contains a subgraph homeomorphic to the 2-dimensional lattice only with $l_1 = 3$ is given as follows:

X has vertex-set $V(X) = \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}_3 = \{(r, s, t) | r \in \mathbf{Z}, s \in \mathbf{Z}, t \in \{0, 1, 2\}\}$ and edge-set $E = E_1 \cup E_2$ where

$$E_1 = \{((r, s, t), (r+1, s, t) \text{ for all } (r, s) \in \mathbf{Z} \times \mathbf{Z} \text{ and } t \in \{0, 1, 2\}\}$$

and

$$E_2 = \{((r, s, t), (r, s+1, t+|r| \bmod 3)) \text{ for all } (r, s) \in \mathbf{Z} \times \mathbf{Z} \text{ and } t \in \{0, 1, 2\}\}.$$

To illustrate the proof of Theorem 3.2 we mention that in this example the sets $\{(r,s,t)|t=0,1,2\}$ give rise to the finite blocks of an imprimitivity system τ of $B(X)$ on X. The graph X_τ then is a Cayley graph of $\mathbf{Z}^2$ with respect to a free generating set $\{a_\tau, b_\tau\}$. Let the paths P_j^i be the 2-paths induced by the edges of E_1 and let Q_j be a 2-path induced by the edges $((r,s,t),(r,s+1,t)) \in E_2$. Then a^p for $p=3$ leaves invariant the 2-paths P_j^i if a_τ leaves invariant P_τ^i.

4 Applications

4.1 Expansion

Here we investigate the isoperimetric properties of vertex-transitive graphs with polynomial growth. Le X be a graph. Denote by ∂S the *vertex-boundary* of $S \subset V(X)$, that is, the set of those vertices which are adjacent to vertices of S, but are not contained in S. The *isoperimetric number* is defined as

$$i(X) = \inf_S \frac{|\partial S|}{|S|},$$

where S runs over all finite vertex sets of X. For graphs with non-exponential growth, and hence for graphs with polynomial growth, $i(X)=0$ holds (see e. g. [MW]). Hence, to study isoperimetric properties of these graphs, we need a different concept, the so called *d-dimensional isoperimetric number*

$$i^{(d)}(X) = \inf_S \frac{|\partial S|}{|S|^{\frac{(d-1)}{d}}},$$

where S is given as above. Varopoulous [Va] (see also [VSC]) showed that if X is a locally finite Cayley graph of an infinite group G with polynomial growth of degree d, then $i^{(d)}(X) \geq C$, where C depends on the generators which define the edges of X. This result was extended to locally finite, vertex-transitive graphs by Saloff-Coste [Sa].

A possible other approach is using the diameter diam(S) (instead of $|S|^{1/d}$) as a parameter in estimation of the *isoperimetric ratio* $|\partial S|/|S|$ of an arbitary vertex set S. The first results in this direction were obtained by Aldous [Al], Babai [Ba], Babai and Szegedy [BSz].

The following tight estimation holds for infinite vertex-transitive graphs with polynomial growth [Lu]:

Theorem 4.1. *If X is an infinite, connected, locally finite graph with polynomial growth, then*

$$\frac{|\partial S|}{|S|} \geq \frac{2}{\operatorname{diam}(S)+1}$$

for every $S \subset V(X)$.

This bound is not sensitive to the growth degree of the graphs in question. Using Lemma 3.1 we prove a lower bound for the isoperimetric ratio, which increases with the growth degree of the graph.

Theorem 4.2. *Let X be a connected, locally finite graph, such that the group of bounded automorphisms $B(X)$ acts transitively on $V(X)$. If X has polynomial growth of degree d, then*

$$\frac{|\partial S|}{|S|} \geq \sum_{i=1}^{d} \frac{2}{(\operatorname{diam}(S)+1)^i}$$

for every finite set $S \subset V(X)$.

Proof. By Theorem 2.1 there exists an imprimitivity system τ of $B(X)$ on X with finite blocks of cardinality m such that X_τ is a Cayley graph of $\mathbf{Z}^d$. Consequently X_τ contains a d-dimensional lattice $\mathbf{L}^d$ as a subgraph. By Lemma 3.1 for two vertices connected in $\mathbf{L}^d$ (blocks of $B(X)$ on X) there exists a matching in X between the vertices of the blocks in question. Hence the lifting of a simple path of X_τ (and so of $\mathbf{L}^d$) contains m vertex disjoint paths in X. The union of these matchings forms a connected subgraph Y of X. Clearly $V(Y) = V(X)$. Without loss of generality we can assume that $X = Y$.

We prove our result by induction on d. If $d = 1$, then Y consists of m pairwise disjoint 2-paths and Theorem 4.1 completes the proof. Let our result hold for $d-1$ and let $\mathcal{H}$ be a family of the preimages of the $(d-1)$-dimensional hyperplanes of $\mathbf{L}^d$ which are orthogonal to an arbitrary fixed direction. By H we denote that preimage of a hyperplane of $\mathbf{L}^d$ which has maximal intersection with S, i.e. $S' = S \cap H$ has maximal cardinality among all $S \cap K, K \in \tau^{-1}\mathcal{H}$. Let $\partial_H S = \partial S \cap H$. Then $|\partial S| \geq |\partial_H S'| + 2\,|S'|$. This inequality holds since there exist $|S'|$ pairwise disjoint 2-paths each of which contains exactly one vertex of S' but no other vertex of H (these are 2-paths in the $d-$th direction). Hence on each of these $|S'|$ $2-$paths we find at least 2 vertices contained in ∂S. In addition ∂S clearly contains all vertices of $\partial_H S'$. Of course $|S| \leq |S'|\,(\text{diam}(S)+1)$ also holds. By our induction hypotheses for H we get

$$\frac{|\partial S|}{|S|} \geq \frac{1}{\text{diam}(S)+1}\left(2+\frac{|\partial_H S'|}{|S'|}\right) \geq \sum_{i=1}^{d} \frac{2}{(\text{diam}(S)+1)^i}.$$

□

We mention that this bound is tight for all $\mathbf{L}^d$, $d \geq 1$, if $|S| = 1$. Nevertheless we are convinced that it can be significantly improved. The next result indicates that we might not be wrong:

Proposition 4.3. *Denote by $S(m)$, $S(m) = \{w \in V(\mathbf{L}^d) | d(0,w) \leq m\}$ the ball of the d-dimensional lattice $\mathbf{L}^d$. Then for every m and d*

$$\frac{|\partial S(m)|}{|S(m)|} \geq \frac{d}{m+1}.$$

Proof. We define the generating function $f(z)$ of the number of lattice points having distance exactly m from the origin:

$$f(z) = \sum_{m=0}^{\infty} a_m z^m,\ a_m = |\{w \in V(\mathbf{L}^d) | d(0,w) = m\}|.$$

It is easy to see that

$$f(z) = (1 + 2\sum_{m=1}^{\infty} z^m)^d = \left(\frac{1+z}{1-z}\right)^d.$$

(This function is called the *nu function* in [So].) $f(z)$ satisfies the differential equation

$$(1+z)f'(z) = 2d\frac{f(z)}{1-z}$$

in a small neighbourhood of $z = 0$. Comparing the coefficents of z^m of both sides of this equation we obtain

$$(m+1)a_{m+1} + ma_m = 2d\sum_{i=0}^{m} a_i$$

Since

$$a_{m+1} \geq a_m$$

we have the lower bound

$$\frac{a_{m+1}}{\sum_{i=0}^{m} a_i} \geq \frac{d}{m+1}.$$

□

Continuing the remarks before Proposition 4.3 we conjecture here that the bound given in this proposition is the best possible bound for the isoperimetric ratio in lattices:

Conjecture 1. *For all finite sets $S \subset V(\mathbf{L}^d)$ with diameter $2m$, the balls $S(m)$ are those with the smallest isoperimetric ratio.*

Remark. We can alternatively define ∂S as the edge boundary of $S \subset V(X)$, the set of edges having one vertex in S. For a result like that in [Va] we obtain no essential difference. On the other hand, to prove a bound as in Theorem 4.2 we would have to choose an approach different from the above.

4.2 Percolation

Here we consider the *Bernoulli bond percolation* on some transitive infinite graphs. In bond percolation the edges of the graph X are independently *open* (or preserved) with probability p and *closed* (or deleted) with probability $1-p$. The corresponding product measure on the set of the edges is denoted by P_p. $C(x)$, named *open cluster*, is the component of the vertex x in the random subgraph; it is the set of vertices connected by open (preserved) edges. Denote by $\theta(p) = \theta(p, X, \text{bond})$ the *percolation probability*: $\theta(p) = P_p\{C(x) \text{ is infinite}\}$. It is clear that $\theta(p)$ does not depend on the vertex x. If $\theta(p) > 0$ it is known that there exists - with probability 1 - a unique infinite open cluster. In this case we also say that *percolation occurs*. It is proved that there exists a *critical probability* $p_c = \sup\{p : \theta(p) = 0\}$, for which $0 < p_c < 1$, $\theta(p) = 0$ if $p < p_c$ and $\theta(p) > 0$ if $p > p_c$.

As a general reference see e.g. [Gr] and for vertex-transitive graphs [M^{2}S].

By Theorem 3.2 we can obtain a bound for p_c. Using directly the fact that under the assumptions of Theorem 3.2 there exist $(l_1, 1)-$homeomorphs of $\mathbf{L}^d$ we can derive an upper bound on the critical probability.

For this purpose we consider the following bond percolation on $\mathbf{L}^2$: Each horizontal edge is open with probability p_h and each vertical edge is open with probability p_v. Kesten proved for the above anisotropic bond percolation process ([Ke] pp. 54) that if $p_h + p_v < 1$ then all open clusters are almost surely finite, and if $p_h + p_v > 1$ then there almost surely exists a unique infinite cluster. Using this result we get:

Theorem 4.4. *Let X be a connected, locally finite, vertex-transitive graph of quadratic growth, such that the group of bounded automorphisms $B(X)$ acts transitively on $V(X)$. If m is the size of the orbits of $B_0(X)$ on X then $p_c(p_c^{m!-1} + 1) \leq 1$ holds for the critical probability p_c of the bond percolation problem on X.* □

This bound may be rather poor, but it only depends on an algebraic constant of the graph in consideration. Of course improvements on the upper bound of l_1 in Theorem 3.2 would also improve this bound.

5 Concluding remarks

A result like Theorem 3.2 cannot be easily generalized to graphs X which satisfy the assumptions of Theorem 3.2 but have growth degree $d \geq 3$. This could only be done if we would know that the automorphism groups of such graphs always contain subgroups isomorphic to $\mathbf{Z}^d$ whenever $B(X)$ acts transitively on them. But this seems to be a property which is quite difficult to prove. Nevertheless we want to formulate it as a conjecture:

Conjecture 2. *Let X be a locally finite, connected graph of polynomial growth upon which $B(X)$ acts transitively. Then $Aut(X)$ always contains a subgroup isomorphic to $\mathbf{Z}^d$ where d is the growth degree of X.*

What can be easily shown for the lattice structure of those graphs is summarized in the following result:

Proposition 5.1. *Let X be as in Conjecture 2. Then X contains a subgraph contractible onto $\mathbf{L}^d$.*

Proof. Follows immediately from Theorem 2.1, Lemma 3.1 and the fact that all blocks of τ have the same finite diameter in X. □

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