ON THE STRONG LAW OF LARGE NUMBERS FOR
LOGARITHMICALLY WEIGHTED SUMS

By

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1. Introduction

Recently much effort has been devoted to the study of the so-called almost sure central limit theory. Probabilists are publishing several papers on extensions of classical weak limit theorems in a generalized formulation involving logarithmic average and logarithmic density. As a starting point of these studies, in 1988 BROŚAMLER and SCHATTE independently proved the following version of the a.s. central limit theorem.

Suppose \( X_1, X_2, \ldots \) are i.i.d. random variables with mean 0 and variance 1 (in fact, stronger moment conditions were first required, but later they proved to be superfluous). As usual, let \( S_n = X_1 + X_2 + \ldots + X_n \). Considering that

\[
\lim_{n \to \infty} P(S_n/\sqrt{n} < x) = \Phi(x) = \int_{-\infty}^{x} (2\pi)^{-1/2} \exp\left\{-y^2/2\right\} \, dy
\]

for every \( x \), one can naturally ask whether the (random) sequence of integers \( n \) for which \( S_n/\sqrt{n} < x \) holds has density a.s. In other words, does \( \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} I(S_i/\sqrt{i} < x) \) exist with probability 1 (where \( I(.) \) denotes the indicator of the event in parentheses)? The answer is negative, but a weaker statement can still be proved, namely, the integer sequence in question has logarithmic density \( \Phi(x) \) with probability 1:

\[
\lim_{n \to \infty} \frac{1}{\log n} \sum_{i=1}^{n} \frac{1}{i} I(S_i/\sqrt{i} < x) = \Phi(x) \quad \text{a.s}
\]

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This result has been extended and generalized by several authors. A brand new paper of Berkés and Dehling [1] provides a systematic study of logarithmic analogues of classical limit theorems. They also present a general method which can be applied to all similar problems. It is based on the fact that under very mild growth conditions on the partial sums $S_n$ of an independent (not necessarily identically distributed) sequence $\{X_n\}$ the a.s limit behaviour of the sequences

$$\frac{1}{\log n} \sum_{i=1}^{n} \frac{1}{i} I\left(\frac{S_i - b_i}{a_i} < x\right) \quad \text{and} \quad \frac{1}{\log n} \sum_{i=1}^{n} \frac{1}{i} P\left(\frac{S_i - b_i}{a_i} < x\right)$$

coincide. More precisely, defining $\xi_i = I\left(\frac{S_i - b_i}{a_i} < x\right) - P\left(\frac{S_i - b_i}{a_i} < x\right)$ one can write

$$\lim_{n \to \infty} \frac{1}{\log n} \sum_{i=1}^{n} \frac{1}{i} \xi_i = 0 \quad \text{a.s.}$$

(cf. Remark 1 in Section 4).

This limit result shows that the almost sure central limit theorems are not stronger than their classical counterparts, as it was first thought to be the case. On the contrary, counterexamples have been found even in the "nice" case of i.i.d. summands where the sums had a limit distribution in the a.s. sense, but not in the ordinary sense [2]. Similar phenomenon was observed in connection with random sequences of other types.

The present paper is motivated by the increasing need of results of type (1) in the study of a.s. weak limits. We try and find conditions on the sequence $\{\xi_n, n \geq 1\}$ that are sufficient to guarantee (1). Although in the above mentioned applications the $\xi_n$ are always bounded, it was quite natural to extend our attention to the unbounded case which proved to be even more interesting.

2. Strong laws of large numbers

Define $\ell(x) = \log x$ for $x \geq e$ and $\ell(x) = 1$ for $x < e$. Let $\ell_1(x) = \ell(x)$ and $\ell_k(x) = \ell(\ell_{k-1}(x))$ for $k \geq 2$.

Our main result is essentially an improved version of Serfling's strong law of large numbers [6].

**Theorem 1.** Let $\xi_1, \xi_2, \ldots$ be arbitrary random variables with finite variances. Suppose there exist a positive non-increasing function $h(.)$ on
the positive numbers and a positive integer \( m \) such that

\[
\int_1^{\infty} h(z) \frac{\ell_m(z)}{z \ell(z)} \, dz < \infty
\]

and

\[
|E(\xi_i \xi_j)| \leq h(j/i) \quad \text{for all } 1 \leq i \leq j.
\]

Then

\[
\lim_{n \to \infty} \frac{1}{\ell(n)} \sum_{i=1}^{n} \frac{1}{i} \xi_i = 0 \quad \text{a.s.}
\]

If, in addition, the random variables \( \{\xi_n, n \geq 1\} \) are uniformly bounded, (2) can be weakened to require that

\[
\int_1^{\infty} h(z) \frac{1}{z \ell(z)} \, dz < \infty
\]

PROOF. For arbitrary positive number \( t > 1 \) let us introduce \( \xi(t) = \sum_{1 \leq i < t} \frac{1}{i} \xi_i \). We are going to apply the standard method of subsequences (see e.g. the proof of Theorem 3.7.3. of [7]). Choosing a sufficiently sparse subsequence \( \{N_k, k \geq 1\} \) we first show that \( E(\sum_{k} \eta(N_k)^2 / \ell(N_k)^2) < \infty \), from which \( \eta(N_k)/\ell(N_k) \to 0 \) follows with probability 1. Then we only have to check if

\[
\lim_{k \to \infty} \left( \max_{N_k < n < N_{k+1}} |\eta(n) - \eta(N_k)| / \ell(N_k) \right) = 0 \quad \text{a.s.}
\]

Let \( a = (a_k, k \geq 1) \) be an increasing sequence of positive numbers. For estimating the expectation of \( \mu^2(s, t \mid a) = \max_{t \leq a_k < s} (\eta(a_k) - \eta(s))^2, 1 \leq s < t, \) one can use Serfling's maximal inequality [5]. Let us define \( g(s, t) = 2 \sum_{s \leq i \leq j < t} \frac{1}{ij} h\left(\frac{i}{j}\right) \). Then we clearly have

\[
E(\eta(t) - \eta(s))^2 = \sum_{s \leq i < t} \sum_{s \leq j < t} \frac{1}{ij} E(\xi_i \xi_j) \leq g(s, t),
\]

and for every \( 1 \leq s < t < u \)

\[
g(s, t) + g(t, u) \leq g(s, u).
\]

Hence it follows that

\[
E(\mu^2(s, t \mid a)) \leq 6(\nu(s, t \mid a))^2 g(s, t),
\]
where \( \nu(s,t \mid a) \) stands for the number of \( a_k \) satisfying \( s \leq a_k < t \).

Let us estimate \( g(s,t) \). For \( 1 \leq i < j \) let \( D_{ij} \) denote the parallelogram with vertices \((i,j)\), \((i,j+1)\), \((i+1,j+1)\) and \((i+1,j+2)\), i.e.,

\[
D_{ij} = \{(x,y): j \leq x < j + 1, j - i - 1 < y - x \leq j - i\}.
\]

It is easy to see that for all \((x,y) \in D_{ij}\)

\[
\frac{1}{xy} h \left( \frac{y}{x} \right) \geq \frac{1}{(i+2)(j+1)} h \left( \frac{j}{i} \right) \geq \frac{1}{6ij} h \left( \frac{j}{i} \right),
\]

hence

\[
g(s,t) \leq \sum_{s \leq i < j < t} 12 \int_{D_{ij}} \frac{1}{xy} h \left( \frac{y}{x} \right) dx dy + \frac{2}{s^2} h(1)
\]

\[
\leq 12 \int_{\{s \leq x < y < t+1\}} \frac{1}{xy} h \left( \frac{y}{x} \right) dx dy + \frac{4h(1)}{s}.
\]

Here

\[
\int_{\{s \leq x < y < t+1\}} \frac{1}{xy} h \left( \frac{y}{x} \right) dx dy \leq \int_{\{s \leq x < y < t+1, 1 \leq \frac{y}{x} < \frac{t+1}{s}\}} \frac{1}{xy} h \left( \frac{y}{x} \right) dx dy.
\]

Substituting \( z = y/x \) we obtain for the right-hand side that

\[
= \int_s^{t+1} \frac{dx}{x} \int_1^{t+1} \frac{h(z)}{z} dz = \log \left( \frac{t+1}{s} \right) \int_1^{t+1} \frac{h(z)}{z} dz \leq \]

\[
\leq 2 \left( \log \left( \frac{t}{s} \right) + \int_1^{t/s} \frac{h(z)}{z} dz \right) \leq 2 \log \left( \frac{t}{s} \right) \int_1^{t/s} \frac{h(z)}{z} dz + \frac{2h(1)}{s}.
\]

Thus

\[
(7) \quad g(s,t) \leq 24 \log \left( \frac{t}{s} \right) \int_1^{t/s} \frac{h(z)}{z} dz + \frac{6h(1)}{S}.
\]

Particularly, from (5) and (7) it follows that

\[
\mathbb{E} \left( \sum_k \frac{\eta(N_k)^2}{\ell(N_k)^2} \right) \leq \sum_k \frac{24}{\ell(N_k)} \int_1^{N_k} \frac{h(z)}{z} dz + \sum_k \frac{6h(1)}{\ell(N_k)^2} =
\]
Similarly, if $N_{k+1}/N_k \to \infty$, then by (6) and (7) we get

$$
\mathbb{E} \left( \sum_k \frac{1}{\ell(N_k)^2} \mu^2(N_k, N_{k+1} | a) \right) \leq
$$

$$
\leq \text{const} \cdot \sum_k \frac{\ell(\nu(N_k, N_{k+1} | a))^2}{\ell(N_k)^2} \left[ \ell \left( \frac{N_{k+1}}{N_k} \right) \int_1^{N_{k+1}/N_k} \frac{h(z)}{z} \, dz + \frac{1}{N_k} \right] \leq
$$

$$
\leq \text{const} \cdot \int_1^{\infty} \frac{h(z)}{z} \left[ \sum_{k: N_{k+1}/N_k > z} \frac{\ell(\nu(N_k, N_{k+1} | a))^2}{\ell(N_k)^2} \ell \left( \frac{N_{k+1}}{N_k} \right) \right] \, dz + \text{const} \cdot \sum_k \frac{\ell(N_{k+1})^2}{N_k \ell(N_k)^2}.
$$

Let us first deal with the case of not necessarily bounded random variables. We first note that (2) can be replaced equivalently with the following condition:

$$
\int_1^{\infty} h(z) \frac{\ell_m(z)^2}{z \ell(z)} \, dz < \infty
$$

for some integer $m > 1$. Let us introduce

$$
N_k = \exp \left( \exp \left( \frac{k}{\ell_m(k)^2} \right) \right) \quad \text{and} \quad a_k = \exp \left( \exp \left( \frac{k}{\ell_m(k)^2} \right) \right).
$$

Then

$$
\nu(N_k, N_{k+1} | a) \sim \left( \frac{\ell_{m-1}(k)}{\ell_m(k)} \right)^2 \quad \text{and} \quad \ell \left( \frac{N_{k+1}}{N_k} \right) \sim \frac{\ell(N_k)}{\ell_m(k)^2}.
$$

Hence

$$
\sum_{k: N_{k+1}/N_k > z} \frac{\ell(\nu(N_k, N_{k+1} | a))^2}{\ell(N_k)^2} \ell \left( \frac{N_{k+1}}{N_k} \right) =
$$

$$
= O \left( \sum_{\ell(N_k) > \ell_m(k)^2 \ell(z)} \frac{1}{\ell(N_k)} \right) = O \left( \frac{1}{\ell(z)} \right)
$$
as $z \to \infty$. In addition,
\[ \sum_k \frac{\ell(N_{k+1})^2}{N_k \ell(N_k)^2} < \infty. \]
Thus by (9)
\[ E \left( \sum_k \frac{1}{\ell(N_k)^2} \mu^2(N_k, N_{k+1} | a) \right) \leq \text{const} \cdot \int_1^\infty \frac{h(z)}{z \ell(z)} dz < \infty. \]
This implies that
\[ \frac{1}{\ell(N_k)} \mu(N_k, N_{k+1} | a) \to 0 \quad \text{a.s.}, \]
and since
\[ \max \left\{ \left| \frac{\eta(a_i)}{\ell(a_i)} \right| : N_k \leq a_i < N_{k+1} \right\} \leq \left| \frac{\eta(N_k)}{\ell(N_k)} \right| + \frac{1}{\ell(N_k)} \mu(N_k, N_{k+1} | a), \]
we obtain that
\[ \limsup_{k \to \infty} \frac{\eta(a_k)}{\ell(a_k)} \leq \limsup_{k \to \infty} \frac{\eta(N_k)}{\ell(N_k)} \quad \text{with probability 1}. \]
Iterating this procedure one can conclude that (11) holds even for $a_k = \exp \left( \exp \left( \frac{k}{l(k)^2} \right) \right)$.

Now, let us substitute $N_k = \exp \left( \exp \left( \frac{k}{l_m(k)^2} \right) \right)$ into the estimation obtained in (8) for $E \left( \sum_k \frac{\eta(N_k)^2}{\ell(N_k)^2} \right)$. Routine calculation shows that
\[ \sum_{N_k > z} \frac{1}{\ell(N_k)} = O \left( \frac{\ell_m(z)^2}{\ell(z)} \right), \]
\[ \sum_k \frac{\eta(N_k)^2}{\ell(N_k)^2} \]
\[ E \left( \sum_k \frac{\eta(N_k)^2}{\ell(N_k)^2} \right) \leq \text{const} \cdot \int_1^\infty \frac{h(z) \ell_m(z)^2}{z \ell(z)} dz < \infty. \]
Consequently, $\lim_{k \to \infty} \frac{\eta(N_k)}{\ell(N_k)} = 0$ and therefore $\lim_{k \to \infty} \frac{\eta(a_k)}{\ell(a_k)} = 0 \quad \text{a.s.}$ for $a_k = \exp \left( \exp \left( \frac{k}{l(k)^2} \right) \right)$.

We have to prove that the fluctuation of the sequence $(\eta(n)/\ell(n), n \geq 1)$ between $a_k$ and $a_{k+1}$ is getting negligible as $k \to \infty$. This will be carried out in three steps.
Firstly, let \( b_k = \exp(\exp(k^{1/3})) \), thus \( a_k \approx b_k^3 / \log^3 k \) and \( \nu(a_k, a_{k+1} | b) = O(k^3) \). In addition, \( \ell\left(\frac{a_{k+1}}{a_k}\right) = \exp\left(\frac{k+1}{\ell(k+1)}\right) - \exp\left(\frac{k}{\ell(k)}\right) \sim \frac{\ell(a_k)}{\ell(k)} \), thus

\[
\sum_{k: a_{k+1}/a_k > z} \frac{\ell(\nu(a_k, a_{k+1} | b))^2}{\ell(a_k)^2} \ell\left(\frac{a_{k+1}}{a_k}\right) = O\left(\sum_{k: \ell(a_k) > \ell(k)\ell(z)} \frac{1}{\ell(a_k)}\right) = O\left(\frac{1}{\ell(z)}\right)
\]
as \( z \to \infty \). Hence by (9) we have

\[
\mathbb{E}\left(\sum_k \frac{1}{\ell(a_k)^2} \mu^2(a_k, a_{k+1} | b)\right) \leq \text{const} \cdot \int_1^\infty \frac{h(z)}{z\ell(z)} \, dz < \infty.
\]
Consequently,

\[
\frac{1}{\ell(a_k)} \mu(a_k, a_{k+1} | b) \to 0 \quad \text{a.s.}
\]
implying

(12)

\[
\lim_{k \to \infty} \frac{\eta(b_k)}{\ell(b_k)} = 0 \quad \text{a.s.}
\]

Secondly, let \( c_k = \exp(k^{1/3}) \), thus \( b_k \approx c_k \exp(3k^{1/3}) \) and \( \nu(b_k, b_{k+1} | c) = O(\exp(3k^{1/3})) \). Now \( \ell\left(\frac{b_{k+1}}{b_k}\right) \sim \frac{1}{3} k^{-2/3} \ell(b_k) \), therefore by (9)

\[
\mathbb{E}\left(\sum_k \frac{1}{\ell(b_k)^2} \mu^2(b_k, b_{k+1} | c)\right) \leq \text{const} \cdot \int_1^\infty \frac{h(z)}{z} \left[ \sum_{k: b_{k+1}/b_k > z} \frac{\ell(\nu(b_k, b_{k+1} | c))^2}{\ell(b_k)^2} \ell\left(\frac{b_{k+1}}{b_k}\right) \right] \, dz + \text{const} \cdot \sum_k \frac{\ell(b_{k+1})^2}{b_k \ell(b_k)^2} \leq \text{const} \cdot \int_1^\infty \frac{h(z)}{z} \left( \sum_{b_{k+1}/b_k > z} \frac{1}{\ell(b_k)} \right) \, dz.
\]
It is not so hard to see that

\[
\sum_{b_{k+1}/b_k > z} \frac{1}{\ell(b_k)} = O\left(\frac{1}{\ell(z)}\right),
\]
hence

$$E\left(\sum_k \frac{1}{\ell(b_k)^2} \mu^2(b_k, b_{k+1} | c)\right) < \infty.$$ 

Together with (12) this implies

$$(13) \quad \lim_{k \to \infty} \frac{\eta(c_k)}{\ell(c_k)} = 0 \quad \text{a.s.}$$

Finally, consider the sequence $N$ of positive integers. Clearly, $\nu(c_k, c_{k+1} | N) = O(c_k)$. This time $c_{k+1}/c_k \to 1$, thus

$$\ell \left(\frac{c_{k+1}}{c_k}\right)^{c_{k+1}/c_k} \int_1 \frac{h(z)}{z} \, dz = O \left(\left(\frac{c_{k+1}}{c_k} - 1\right)^2\right) = O(k^{-4/3})$$

From the second line of (9) it follows that

$$E \left(\sum_k \frac{1}{\ell(c_k)^2} \mu^2(c_k, c_{k+1} | N)\right) \leq \text{const} \cdot \sum_k \left(k^{-4/3} + \frac{1}{c_k}\right) < \infty.$$ 

Combining this with (12) we obtain that $\lim_{n \to \infty} \eta(n)/\ell(n) = 0$ a.s.

The case of uniformly bounded random variables is much simpler. Let $\epsilon$ be an arbitrarily small positive number and $N_k = \exp(e^\epsilon k)$, $k \geq 0$. Since $\sum \ell(N_k)^{-2} < \infty$ and

$$\sum_{N_k > z} \frac{1}{\ell(N_k)} \leq \frac{1}{(1 - e^{-\epsilon})\ell(z)},$$

from (8) we obtain that $\eta(N_k)/\ell(N_k) \to 0$ with probability 1. On the other hand, for every integer $n$ between $N_k$ and $N_{k+1}$ we clearly have

$$\left|\frac{\eta(n)}{\ell(n)}\right| \leq \left|\frac{\eta(n)}{\ell(N_k)}\right| \leq \left|\frac{\eta(n) - \eta(N_k)}{\ell(N_k)}\right| + \left|\frac{\eta(N_k)}{\ell(N_k)}\right|.$$ 

The first term on the right-hand side can be estimated by

$$C \left(\frac{1}{\ell(N_k)} \sum_{N_k < i \leq N_{k+1}} \frac{1}{i}\right),$$

where $P(\sup_n |\xi_n| \leq C) = 1$.

This converges to $C(e^\epsilon - 1)$, hence $\lim_{n \to \infty} \sup |\eta(n)/\ell(n)| \leq C(e^\epsilon - 1)$ a.s. for every $\epsilon > 0$, thus $\lim_{n \to \infty} \eta(n)/\ell(n) = 0$, as claimed. $\Box$. 
3. Concluding remarks

REMARK 1. The "bounded" part of Theorem 1 can be applied in a.s. central limit theory, as it will be demonstrated by the following example, borrowed from BERKES and DEHLING [1]. In fact, it is a particular case of their Theorem 1.

Let $X_1, X_2, \ldots$ be independent (but not necessarily identically distributed) random variables, $S_n = X_1 + \ldots + X_n$, and $a_n > 0$, $b_n \ (n \geq 1)$ numerical normalizing sequences.

THEOREM 2. Suppose

$$\sup_i E f \left( \left| \frac{S_i - b_i}{a_i} \right| \right) < \infty,$$

where $f > 0$ is a Borel measurable function on $[0, \infty)$ such that both $f(z)$ and $z/f(z)$ are non-decreasing and

$$\int_1^\infty \frac{dz}{z \ell(z)f(z)} < \infty.$$

Assume in addition that

$$a_j/a_i \geq C(j/i)^\gamma \quad (j \geq i)$$

for some positive constants $C$ and $\gamma$. Then for any distribution function $G$ the following statements are equivalent:

(A) For any Borel set $A \subset \mathbb{R}$ with $G(\partial A) = 0$ we have

$$\lim_{n \to \infty} \frac{1}{\log n} \sum_{i=1}^n \frac{1}{i} \mathbb{I} \left( \frac{S_i - b_i}{a_i} \in A \right) = G(A) \quad \text{a.s.,}$$

(B) For any Borel set $A \subset \mathbb{R}$ with $G(\partial A) = 0$ we have

$$\lim_{n \to \infty} \frac{1}{\log n} \sum_{i=1}^n \frac{1}{i} P \left( \frac{S_i - b_i}{a_i} \in A \right) = G(A) \quad \text{a.s.}$$

PROOF. We shall trace the proof of the cited theorem of BERKES and DEHLING. First of all, we can suppose $b_n = 0$, since the additive normalization can be included in the random variables $X_n$. Then we note that the equivalence of (A) and (B) will immediately follow as soon as we show that (4) holds for $\xi_i = g(S_i/a_i) - E g(S_i/a_i)$, where $g(.)$ is an arbitrary bounded Lipschitz-1 function on $\mathbb{R}$ (we can and will suppose
that $0 \leq g \leq 1$). In order to apply our Theorem 1 we need an estimate of $E(\xi_i \xi_j) = \text{cov}(g(S_i/a_i), g(S_j/a_j))$. That was derived in [1] in the following way. Using the independence of the summands $X_i$ and the properties of $g$ we can write

$$\text{cov}(g(S_i/a_i), g(S_j/a_j)) = \text{cov}(g(S_i/a_i), g(S_j/a_j) - g((S_j - S_i)/a_j)) \leq E\left|g(S_j/a_j) - g((S_j - S_i)/a_j)\right| \leq \text{const} \cdot E(\min\{|S_i/a_i|, 1\}).$$

Using the properties of $f$ we have

$$\min\{|S_i/a_i|, 1\} = \frac{\min\{|S_i/a_i|, a_j/a_i\}}{a_j/a_i} \leq \frac{f(\min\{|S_i/a_i|, a_j/a_i\})}{f(a_j/a_i)} \leq \frac{f(|S_i/a_i|)}{f(a_j/a_i)}.$$

Hence by (14) and (16)

$$\text{cov}(g(S_i/a_i), g(S_j/a_j)) \leq \text{const} \cdot E\left[\frac{f(|S_i/a_i|)}{f(a_j/a_i)}\right] \leq \text{const} \cdot \frac{1}{f(C(j/i)\gamma)},$$

i.e., (3) fulfilled with $h(z) = \frac{\text{const}}{f(C^2\gamma)}$. It is easy to see that the integral in (15) is finite if and only if it is finite with $f(cz^2)$ in place of $f(z)$. Thus Theorem 1 can be applied to complete the proof. \(\square\)

For further applications see [3].

Remark 2. Dealing with weighted sums, why do we concentrate on logarithmic weighting instead of extending our results for a much wider class of weighted averages, although the method itself could be applied more generally? This question can also be asked in connection with the a.s. central limit theorem. However, the logarithmic weighting is intrinsic, as it has been pointed out to me by P. MAJOR.

For the sake of simplicity let us consider the example from the Introduction. Let $X_1, X_2, \ldots$ be i.i.d random variables with mean 0 and variance 1 and $S_n = X_1 + \ldots + X_n$. By the invariance principle $S_i \approx W(i)$, where $W(t)$ denotes a standard Wiener process. Hence

$$\frac{1}{\log n} \sum_{i=1}^{n} \frac{1}{i} \mathbf{1}(S_i/\sqrt{i} < x) \approx \frac{1}{\log n} \int_{1}^{n} \mathbf{1}(W(t)/\sqrt{t} < x) \frac{dt}{t}.$$

Substituting $t = e^u$ into the right-hand side we obtain

$$\frac{1}{\log n} \int_{0}^{\log n} \mathbf{1}(e^{-u/2}W(e^u) < x) du.$$

Since the integrand is stationary, ergodic theory can help. Summarizing what has happened we can see that the logarithmic weighting corresponds
to a time-transform $t \mapsto e^t$ which turns the process $W(t)/\sqrt{t}$ stationary. This also explains why $j/i$ appears on the right-hand side of (3): for $j = e^t$ and $i = e^s$ we get $j/i$ as a function of $t - s$ (see also Remark 3 below).

PELIGRAD and RÉVÉSZ [4] has also investigated what other weighting can replace the logarithmic one so that the a.s. central limit theorem be still preserved. It turned out that essential improvement cannot be achieved: though $i^{-1}$ can be multiplied by some logarithmic terms, it cannot be replaced by $i^{-1-\varepsilon}$, say.

**Remark 3.** Since $j/i \leq j - i + 1$ for $1 \leq i \leq j$, condition (3) of Theorem 1 can be replaced with the more familiar

$$(3') \quad \mathbb{E}(\xi_i \xi_j) \leq h(|j - i|) \quad \text{for all } i, j.$$

**Remark 4.** For those interested in the a.s. convergence of the arithmetic mean of random variables the following assertion can be deduced from Theorem 1.

**Theorem 3.** Let $\xi_1, \xi_2, \ldots$ be arbitrary random variables with finite variances. Suppose there exist a positive non-increasing function $h : [0, +\infty) \to \mathbb{R}$ and a positive integer $m$ such that

$$(17) \quad \int_1^\infty \frac{h(z)}{z} \ell_m(z) dz < \infty$$

and

$$|\mathbb{E}(\xi_i \xi_j)| \leq h(|j - i|) \quad \text{for all } i, j.$$

Then

$$\lim_{n \to \infty} n^{-1} \sum_{i=1}^n \xi_i = 0 \quad \text{a.s.}$$

If, in addition, the random variables $\{\xi_n, n \geq 1\}$ are uniformly bounded, (14) can be weakened to require that

$$(17') \quad \int_1^\infty \frac{h(z)}{z} dz < \infty.$$

**Proof.** For $e^{k-1} \leq i < e^k$ define $\xi'_i = \xi_k$. Then for every $i \leq j$, $e^{k-1} \leq i < e^k$, $e^{n-1} \leq j < e^n$ we have

$$|\mathbb{E}(\xi'_i \xi'_j)| = |\mathbb{E}(\xi_k \xi_n)| \leq h(n - k) \leq h'(j/i),$$

where $h'(z) = h(\ell(z) - 1)$. Hence (17) resp. (17') imply

$$\int_{\ell(z)}^{\infty} h'(z) \frac{\ell_{m+1}(z)^2}{z} dz = \int_{\ell(z)}^{\infty} \frac{h(\ell(z) - 1)}{\ell(z)} \ell_m(\ell(z))^2 \frac{dz}{z} = \int_{1}^{\infty} \frac{h(t-1)}{t} \ell_m(t)^2 dt < \infty$$

and

$$\int_{\ell(z)}^{\infty} \frac{h'(z)}{z \ell(z)} dz \leq \int_{1}^{\infty} \frac{h(t-1)}{t} dt < \infty,$$

resp. Since

$$\frac{1}{\ell(e^k)} \sum_{i=1}^{k} \frac{1}{i} \xi_i' \sim \frac{1}{k} \sum_{i=1}^{k} \xi_i,$$

application of Theorem 1 to variables $\xi_i'$ immediately completes the proof. \(\square\)

This theorem slightly improves Theorem 3.7.4 of [7] on weakly stationary sequences, because the conditions of the latter imply (17) with $m = 1$.

**REMARK 5.** Is it true that (2') and (17') are sufficient even in the non-bounded case? The method applied in the proof of Theorem 1 does not seem to be suitable for this “small” improvement.

**References**


