The sandpile group of trinities and a canonical definition for the planar Bernardi action

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The topic is also connected to

- Laplace matrix,
- polymatroids.

A surprising phenomenon

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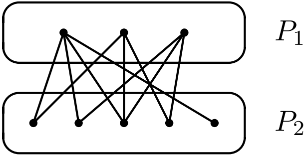

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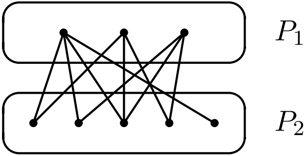

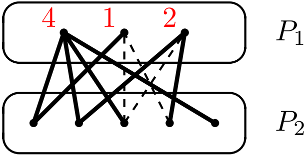

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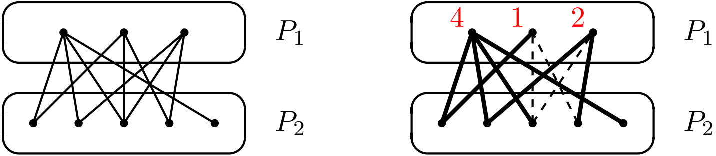

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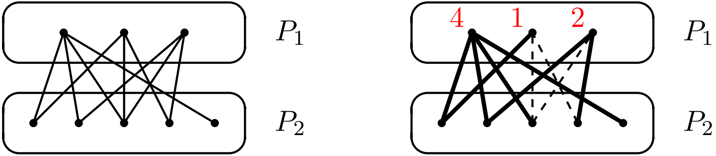

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The proof of Postnikov used polyhedra. Is there some kind of a simple graph-theoretical explanation?

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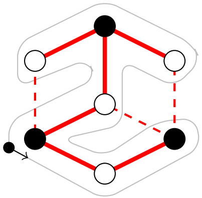
But we need some auxiliary structure to define these trees.

Our auxiliary structure will be an embedding into an orientable surface. That is, a cyclic order of the edges around each vertex. This is called a ribbon structure.

Definition (The tour of a tree (Bernardi))

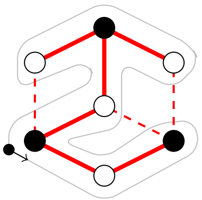
Suppose that a ribbon structure and a base point are given.



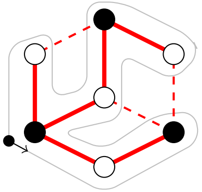
We trace the boundary of the tree following the ribbon structure (as the grey line indicates it on the picture, where the graph is embedded into the plane).

Definition (Jaeger tree)

Let the two partite classes be black and white. Suppose that a ribbon structure and a base point are given. A spanning tree is a black Jaeger tree, if in its Bernardi tour, each non-edge is cut through first at its black endpoint.



The tree on the upper picture is not a black Jaeger tree, since te leftmost vertical non-edge is cut through first at its white endpoint.
The tree on the lower picture is a black Jaeger tree.



Theorem (Kálmán, T.)

For any fixed ribbon structure and base point, the black Jaeger trees represent each black and each white spanning tree degree vector exactly once.

Algorithmically (Kálmán, T.)

For a degree vector on one side, the representing Jaeger tree can be constructed greedily.

The proof of the existence of a Jaeger tree is nontrivial (Kálmán-T.) and it uses a matroid-theoretical lemma.
The uniqueness is a relatively simple argument.

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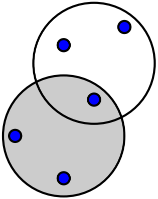
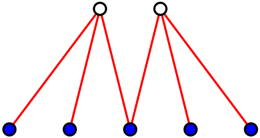
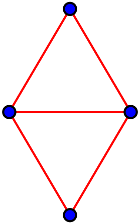
A hypergraph can be encoded by a bipartite graph.





Why are degree vectors interesting?

What corresponds to ordinary graphs?



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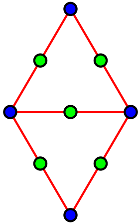

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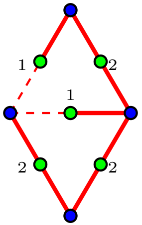

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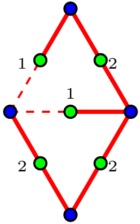

If we subtract one from each coordinate, we get the characteristic vector of a spanning tree.

Let us use the name hypertree for these vectors.

Hypertrees/spanning tree degree vectors on E correspond to spanning trees. **What are hypertrees on V?** They are basically the same as the break divisors from the chip-firing literature.

Why are degree vectors interesting?

Hypertrees have a characterization with submodular functions.

Theorem (Kálmán)

f is a hypertree on V if and only if

(i) $f(S) \leq |\Gamma(S)| - c(S \cup \Gamma(S))$ *for any* $S \subseteq V$,

(ii) $f(V) = |E| - 1$,

where $c(S \cup \Gamma(S))$ *is the number of components of the graph induced by* $S \cup \Gamma(S)$.

Spanning trees are known to be bases of the graphical matroid.
The theorem says that hypertrees are bases of a polymatroid.

We have seen a problem, where there was no topology in the statement, but we needed topological auxiliary data to give a bijection.

Next, we look at a situation where the connection to topology is even more clear.

The sandpile group.

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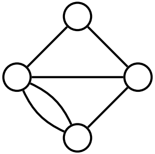
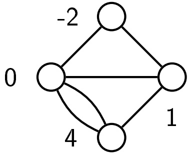


Definition (Chip configuration)

An integer on each vertex of the graph (number of chips).



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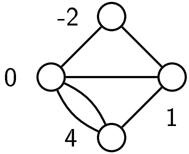

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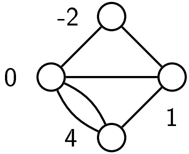

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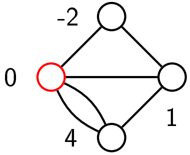

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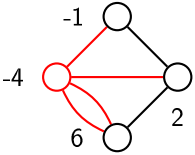

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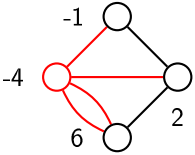

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Firing: The fired vertex passes a chip along each edge incident to it.

This is an equivalence relation. Symmetry: firing each vertex once results in the original configuration.

In vector notation, a firing is the addition of a coloumn of the Laplace matrix. Linear equivalence: $x \sim y$ if and only if $\exists z \in \mathbb{Z}^V$ such that $y = x + L_G z$.

Definition (Sandpile group)

The group of zero-sum chip configurations for coordinatewise addition, factorized by linear equivalence. Notation: $S(G)$.

How is this connected to hypertrees?

Theorem (Baker-Wang)

Hypertrees on V give a system of representatives of chip-firing equivalence classes with $|E|-1$ chips in total.

Corollary (Well-known)

The order of the sandile group is the number of spanning trees.

Remark

There are many papers on bijections between elements of the sandpile group and spanning trees, and also on faithful group actions of the sandpile group on spanning trees.

Sandpile groups and planarity

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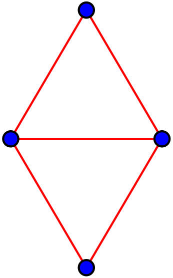
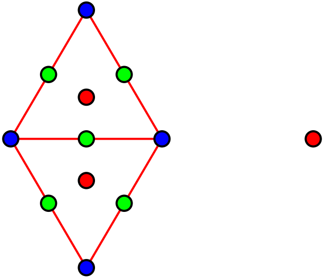
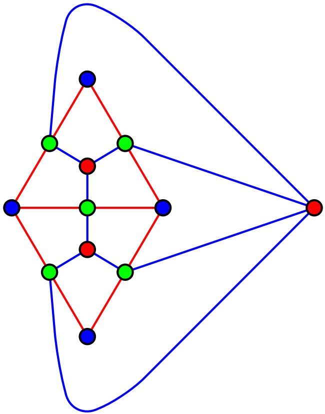
We give a proof using a planar embedding.



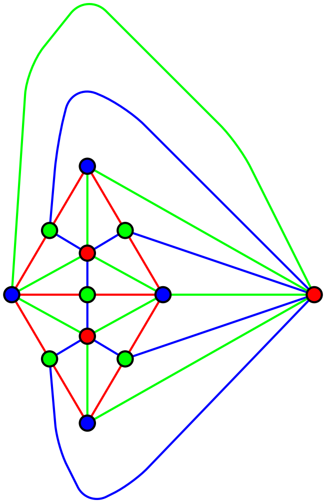
Here is the graph.



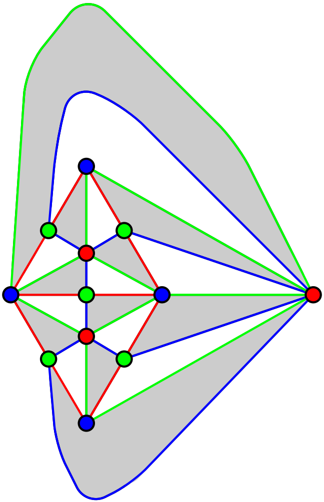
We draw green points for edges and red points for regions.



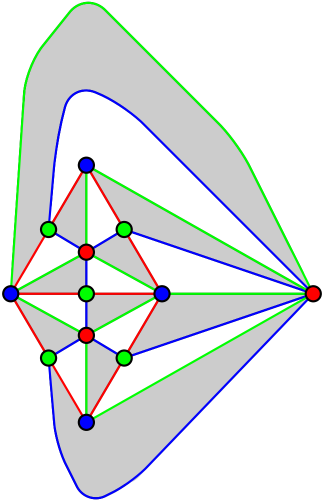
We draw the dual graph.



And also the medial graph (we connect regions with vertices lying on their boundary).



Then we two-color the triangles.



Notations

V : set of vertices of G,

E : set of edges of G and G^* (green vertices on the figure),

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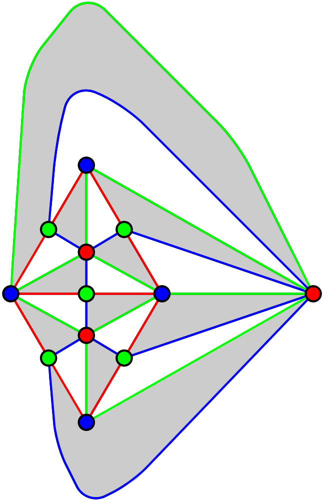

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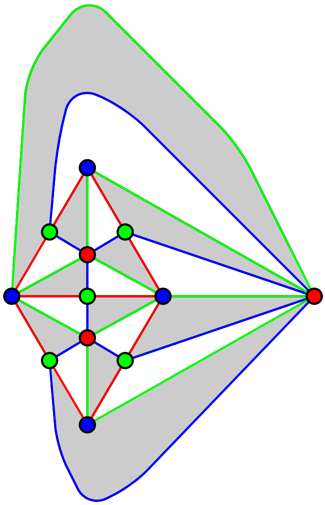

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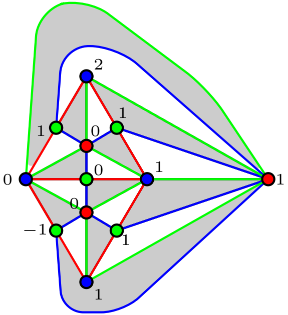
Let us embed these vectors in a common space!



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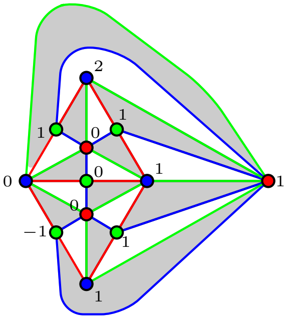

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We factorize $\mathbb{Z}^{V \cup E \cup R}$ by white triangle equivalence.

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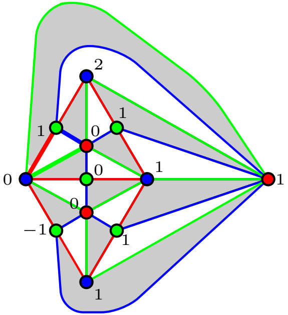
$(x_V, x_E, x_R) \approx_W (y_V, y_E, y_R)$ if $(x_V, x_E, x_R) - (y_V, y_E, y_R)$ can be written as an integer linear combination of characteristic vectors of white triangles.



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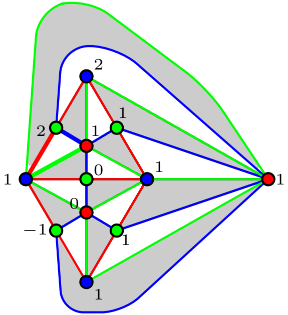

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Theorem (Kálmán, Lee, T.)

Those equivalence classes of the trinity sandpile group that contain at least one element of the form $(x_V, 0, 0)$ with $|x_V| = 0$ form a group isomorphic to $S(G)$.

Theorem (Kálmán, Lee, T.)

Those equivalence classes of the trinity sandpile group that contain at least one element of the form $(0, 0, x_R)$ with $|x_R| = 0$ form a group isomorphic to $S(G^*)$.

Lemma (Kálmán, Lee, T.)

An equivalence class contains an element of the form $(x_V, 0, 0)$ with $|x_V| = 0$ if and only it contains an element of the form $(0, 0, x_R)$ with $|x_R| = 0$.

Using the same technique, one can also give a canonical definition for the planar Bernardi and rotor-router actions.

Thank you for your attention!